Majorization, correlating catalysts, and the one-shot interpretation of entropic quantities

Markus P. Müller[1,2,4], Matteo Lostaglio[3],
Michele Pastena[4], and Jakob Scharlau[4]

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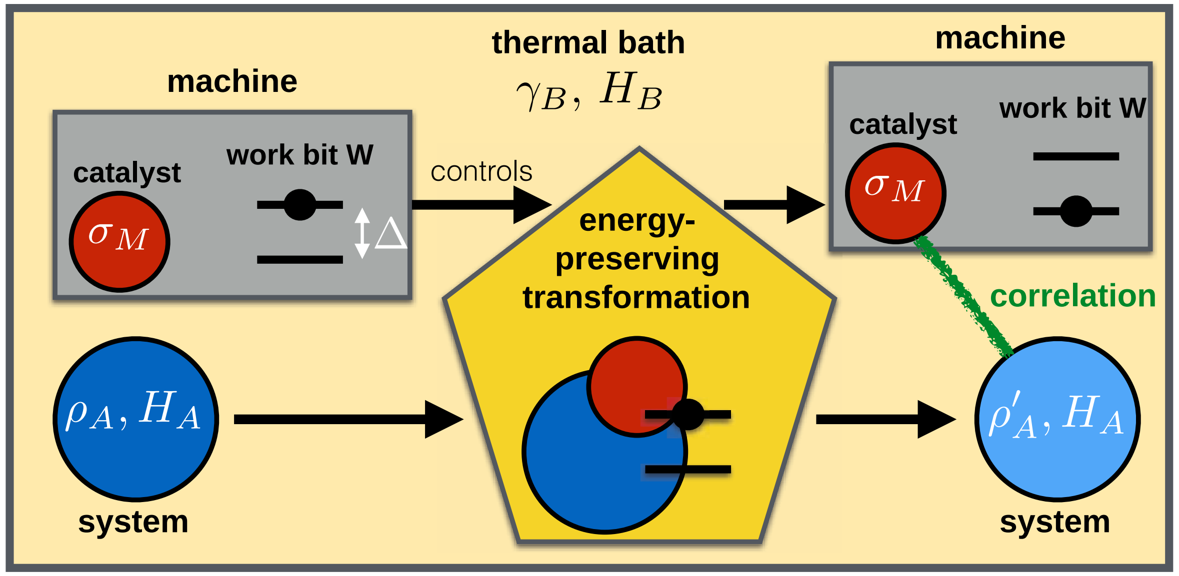
Outline

1. Intro: Majorization in quantum information

2. Main mathematical results

3. Implications for quantum thermodynamics



4. Implications for quantum information (in progress)

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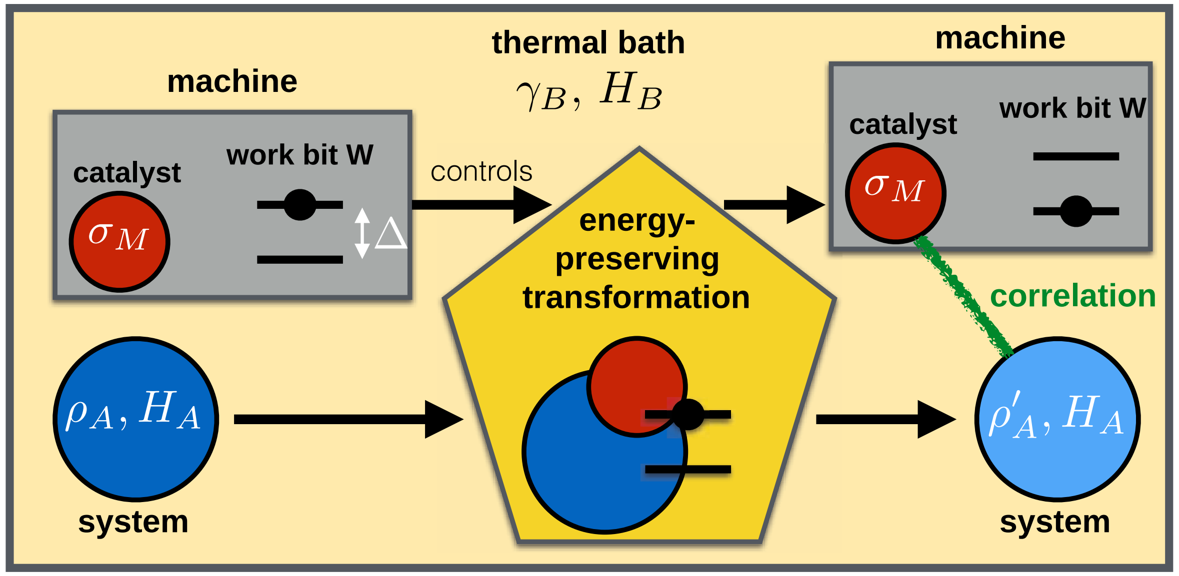

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Intro: Majorization in quantum information

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$$|\psi\rangle = \sum_i \sqrt{\lambda_\psi^{(i)}}\,|i\rangle \otimes |i\rangle$$

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Majorization: prob. vectors $p = (p_1, \ldots, p_n), \quad q = (q_1, \ldots, q_n)$

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e.g. $(1,0,0) \succ (.7,.2,.1) \succ \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

(Resource theory of) noisy operations

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M.+P. Horodecki, and J. Oppenheim, *Reversible transformations from pure to mixed states, and the unique measure of information*, Phys. Rev. A **67**, 062104 (2003).

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Write $\rho_A \xrightarrow{\text{noisy}} \rho'_A$ if there is a finite-dim. B, a unitary U_{AB} and maximally mixed state $\mu_B = \mathbf{1}_B/d_B$ such that

$$\rho'_A = \mathrm{Tr}_B \left[U_{AB} \left(\rho_A \otimes \mu_B \right) U_{AB}^\dagger \right].$$

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Blueprint of the resource theory of q. thermodynamics…

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Theorem. For all $\varepsilon > 0$ there is $\rho'_A(\varepsilon)$ and

$$\rho_A \xrightarrow{\text{noisy}} \rho'_A(\varepsilon), \qquad \|\rho'_A - \rho'_A(\varepsilon)\| < \varepsilon$$

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$\rho_A \succ \rho'_A$

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Need large d_B for small ε.

Catalysis

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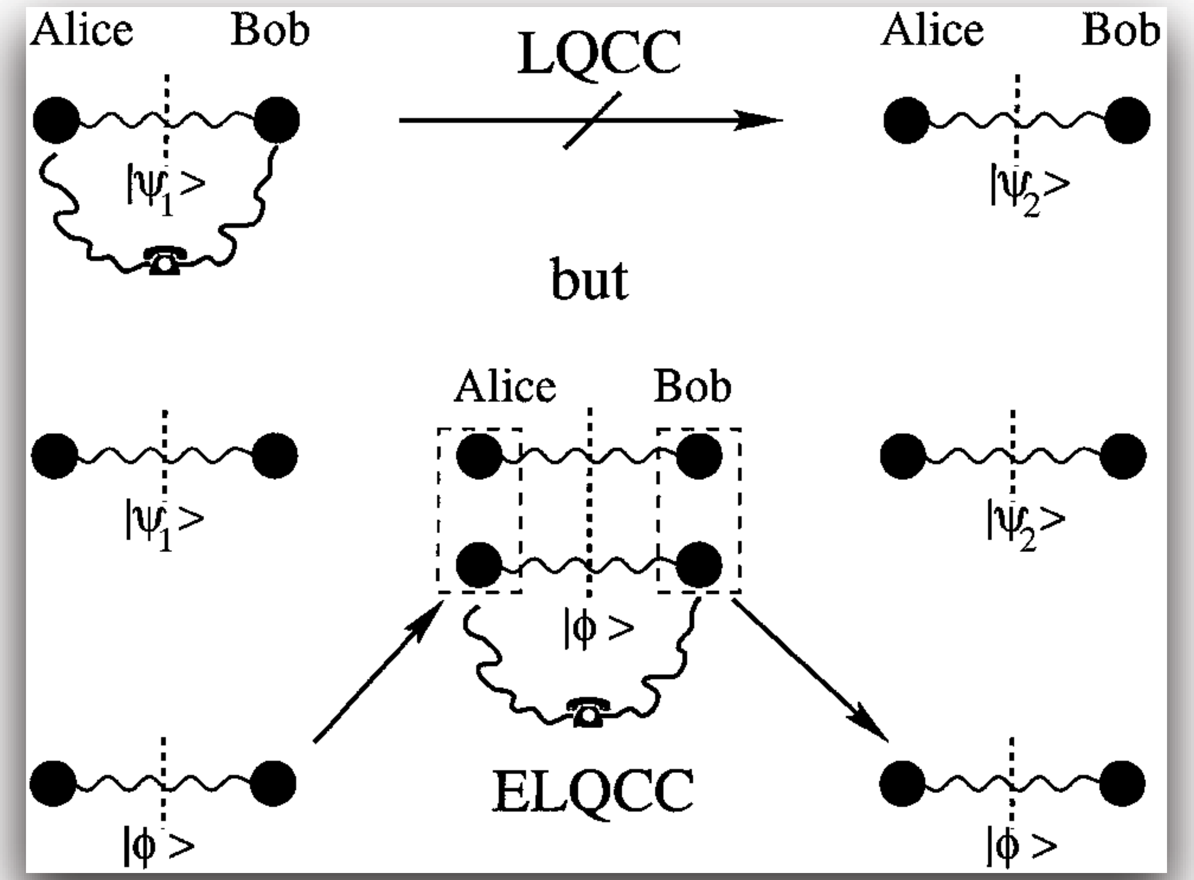

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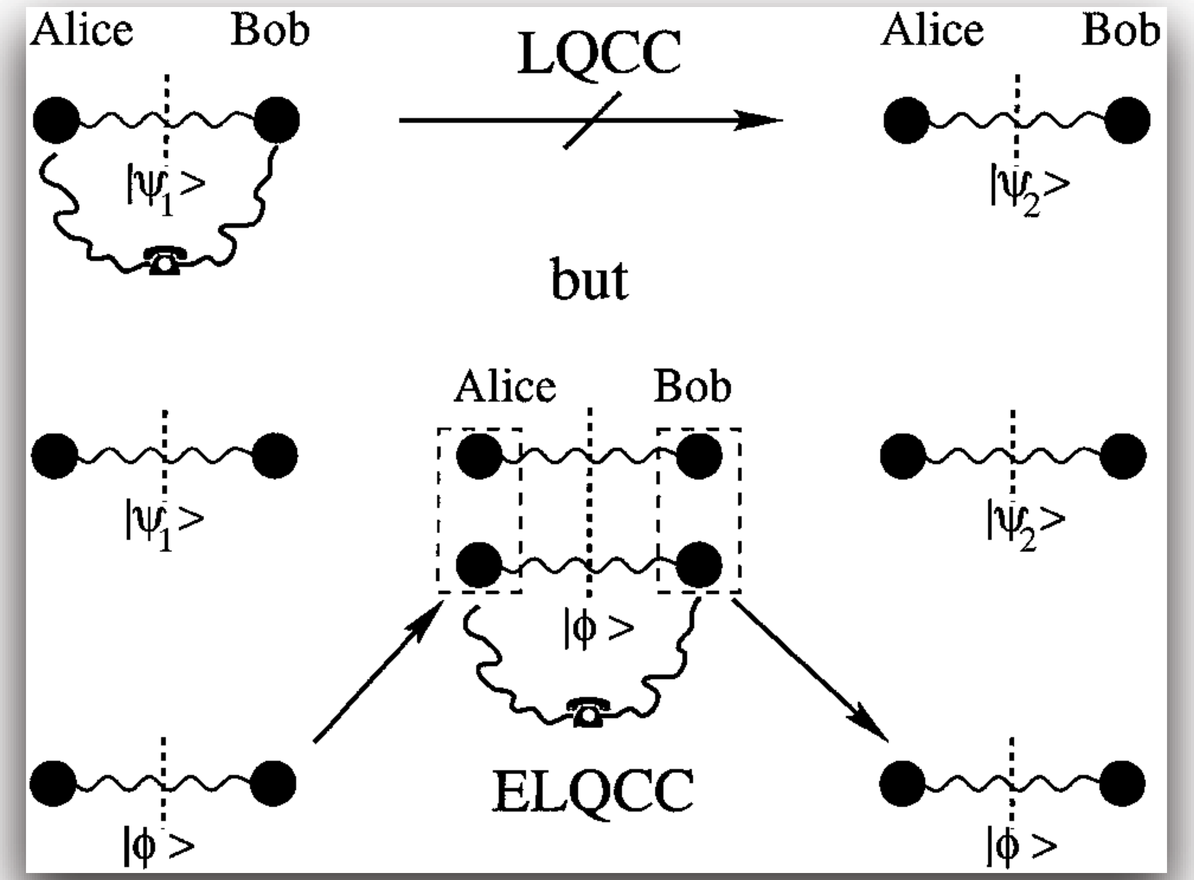

There are states with

$$|\psi_1\rangle \overset{\text{LOCC}}{\not\longrightarrow} |\psi_2\rangle \quad \textbf{but}$$

$$|\psi_1\rangle \otimes |\varphi\rangle \overset{\text{LOCC}}{\longrightarrow} |\psi_2\rangle \otimes |\varphi\rangle$$

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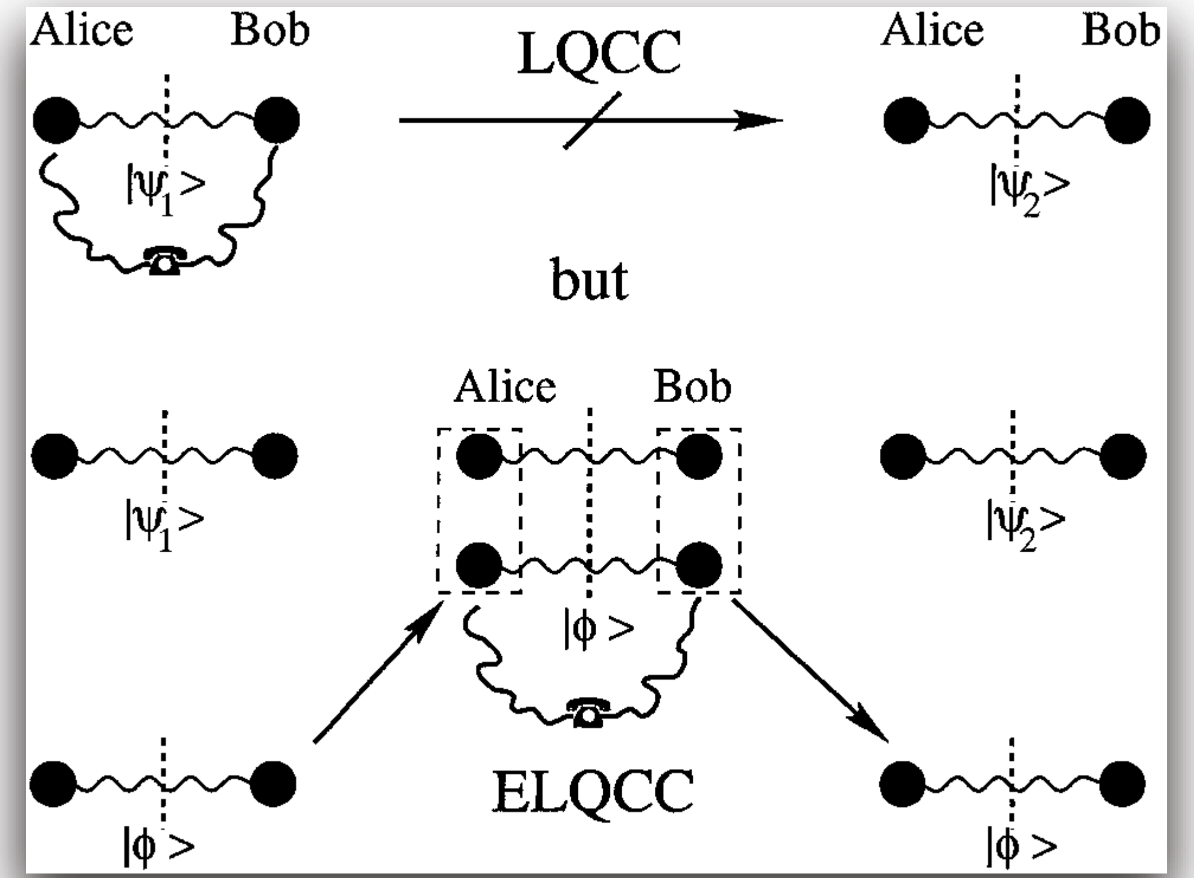

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because there are prob. vectors p, q, c with $p \not\succ q$

but $p \otimes c \succ q \otimes c.$

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Given $p, q,$ when is there c such that $p \otimes c \succ q \otimes c$?

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Lemma (Klimesh; Turgut 2007):

Assuming $p^\downarrow \neq q^\downarrow$, there is such a c if and only if

$$H_\alpha(p) < H_\alpha(q) \text{ for all } \alpha \in \mathbb{R} \setminus \{0\} \text{ and}$$
$$H_{\mathrm{Burg}}(p) < H_{\mathrm{Burg}}(q).$$

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$$H_\alpha(p) = \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \log \sum_{i=1}^{n} p_i^\alpha, \qquad H_{\mathrm{Burg}}(p) = \frac{1}{n} \sum_{i=1}^{n} \log p_i .$$

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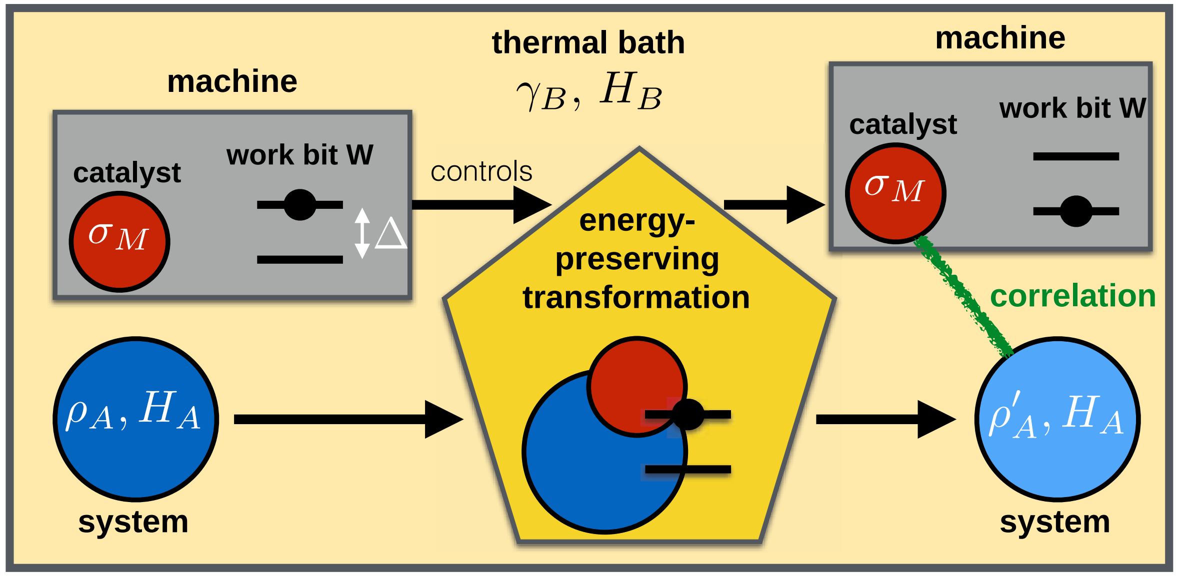

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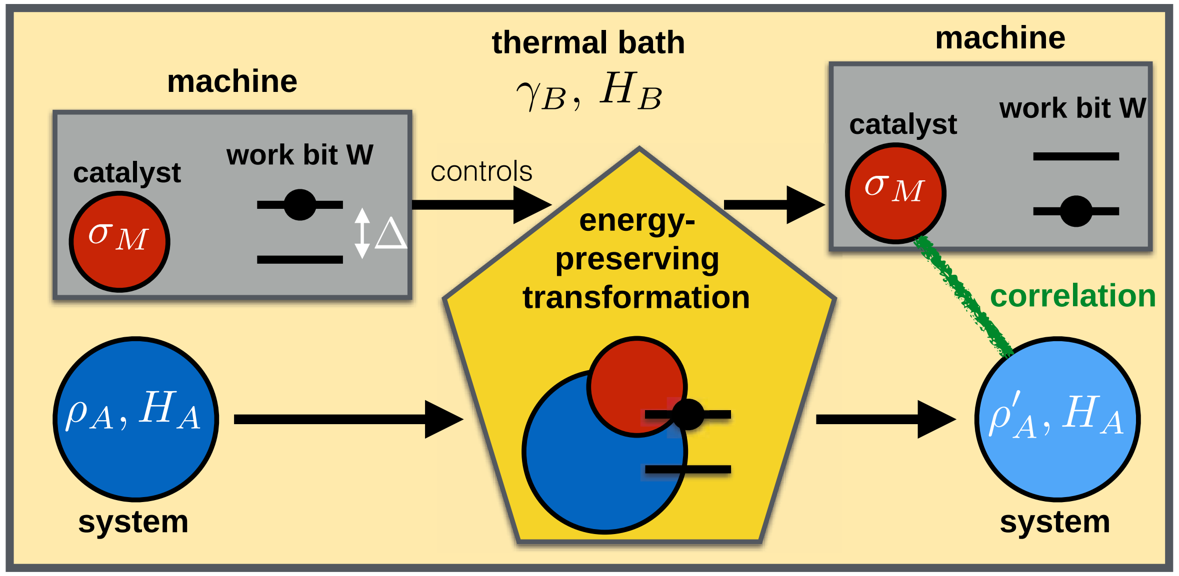

4. Implications for quantum information (in progress)

Main mathematical results

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Theorem 3 (Ref. [2]). *Let ρ and ρ' be quantum states on A such that $\rho \succ \rho'$, and let B be a copy of A. Then there exists a unitary U_{AB} such that*

$$\rho'_A = \mathrm{Tr}_B \left[U_{AB}(\rho_A \otimes \mu_B) U_{AB}^\dagger \right],$$

that is, the noisy transition from ρ to ρ' can be achieved exactly *with an auxiliary system that is of the same size as A. Moreover, U_{AB} can be chosen to leave the maximally mixed state μ_B on B invariant.*

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Answers a question by Bengtsson and Życzkowski.

Main mathematical results

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$$p_X \otimes p'_Y \succ p'_{XY} \tag{1}$$

if and only if $H_0(p) \leq H_0(p')$ and $H(p) < H(p')$. Moreover, for every $\varepsilon > 0$, we can choose Y and p'_{XY} such that the mutual information is $I(X:Y) \equiv S(p'_{XY} \| p'_X \otimes p'_Y) < \varepsilon$.

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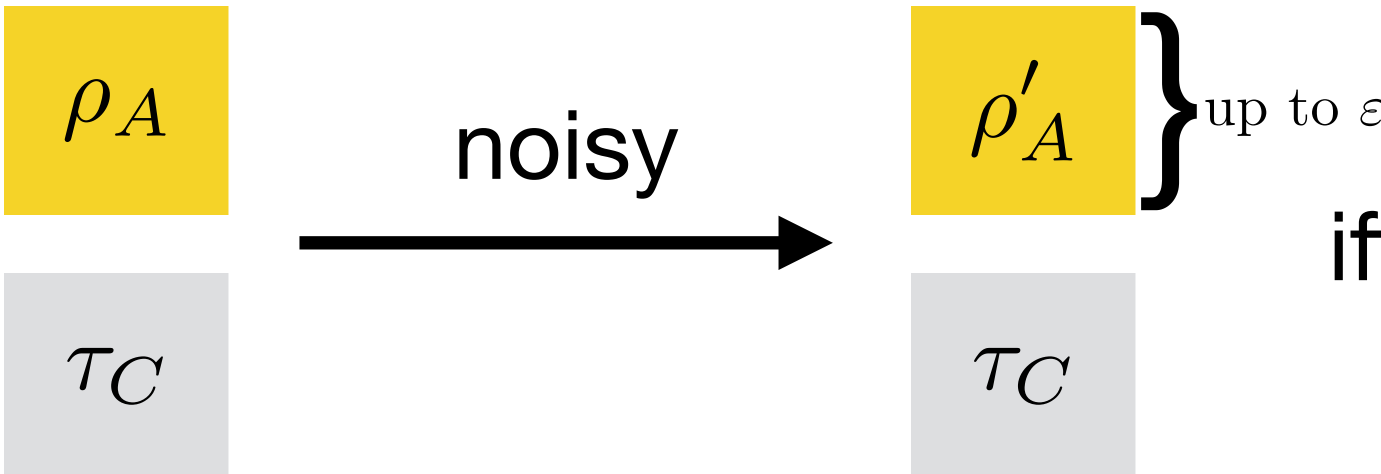
$$H_0(p) = \log \#\{i : p_i \neq 0\}, \qquad H(p) = -\sum_i p_i \log p_i .$$

Catalytic noisy operations

Klimesh/Turgut's 2007 catalysis result implies:



iff $H_\alpha(\rho_A) \leq H_\alpha(\rho'_A)\ \forall \alpha.$

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$$\begin{matrix}\rho_A \\ \tau_C\end{matrix} \xrightarrow{\text{noisy}} \begin{matrix}\left.\rho'_A\right\}\text{up to } \varepsilon \\ \tau_C\end{matrix}$$

iff $H_\alpha(\rho_A) \leq H_\alpha(\rho'_A) \ \forall \alpha.$

$$\mathrm{Tr}_B\left[U_{ACB}\left(\rho_A \otimes \tau_C \otimes \mu_B\right) U_{ACB}^\dagger\right]$$

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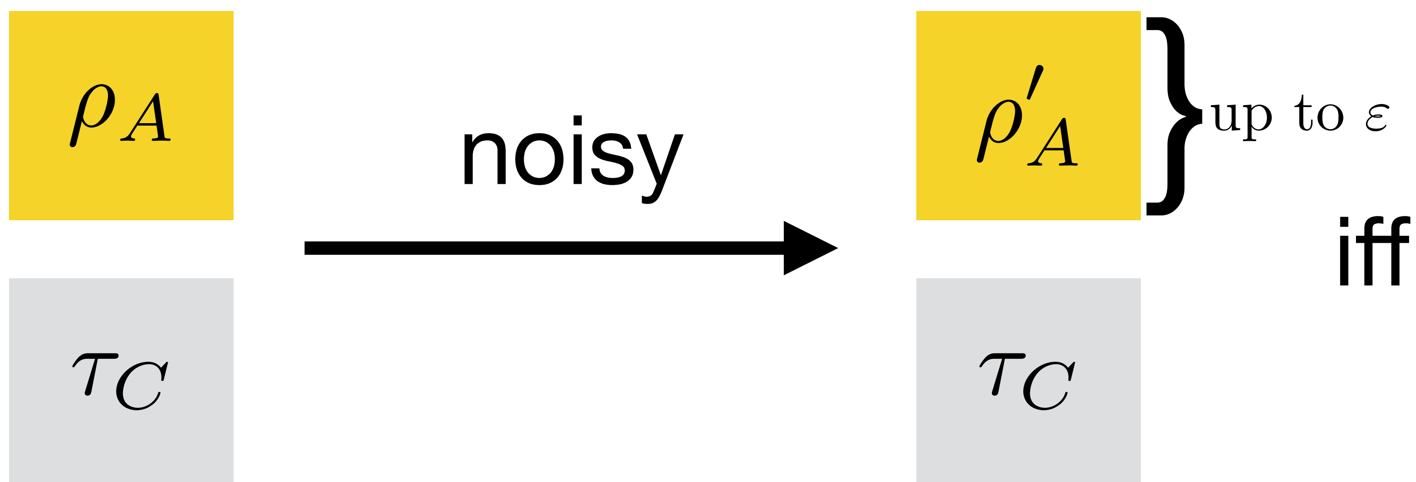

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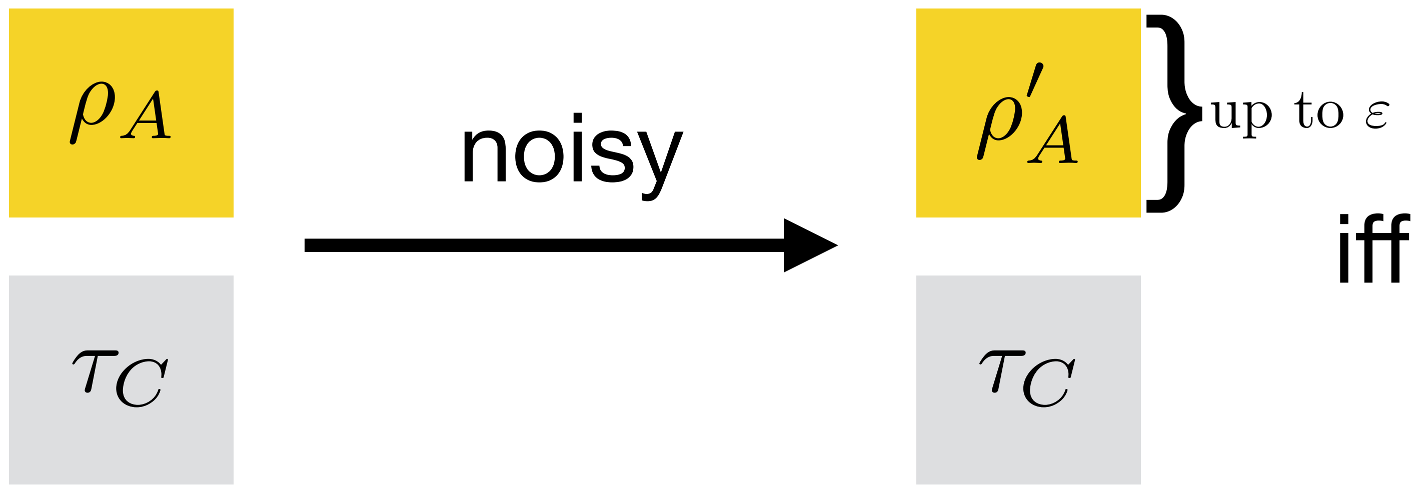
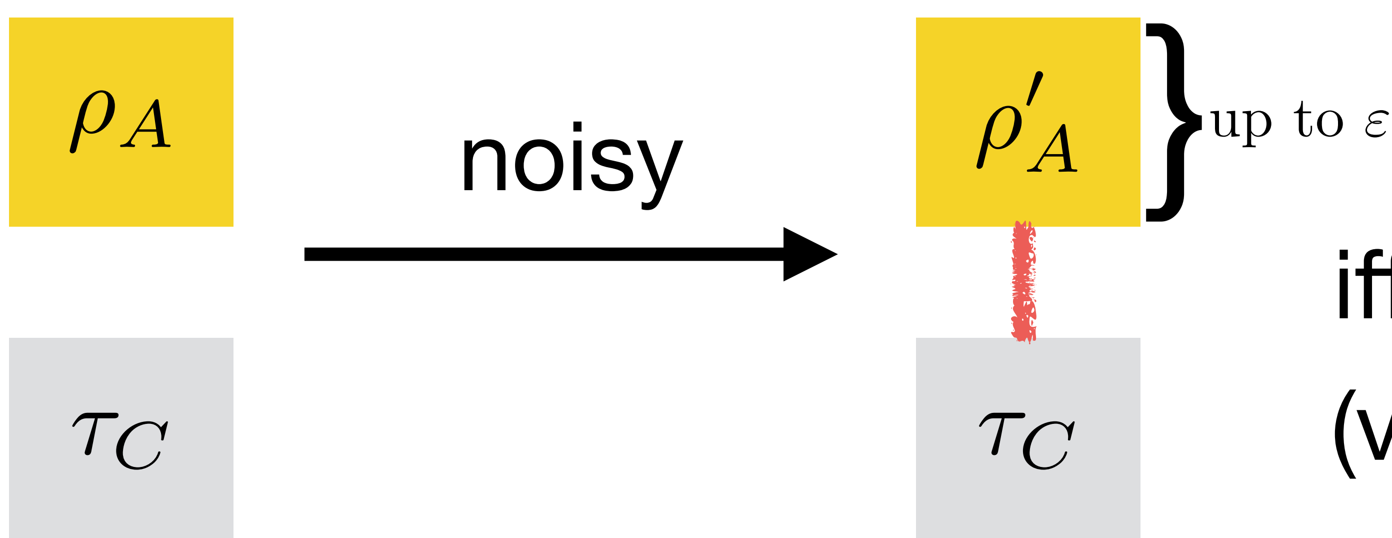


iff $H_\alpha(\rho_A) \leq H_\alpha(\rho'_A) \;\; \forall \alpha.$

Our result implies:



iff $H(\rho_A) \leq H(\rho'_A)$

(von Neumann entropy!)

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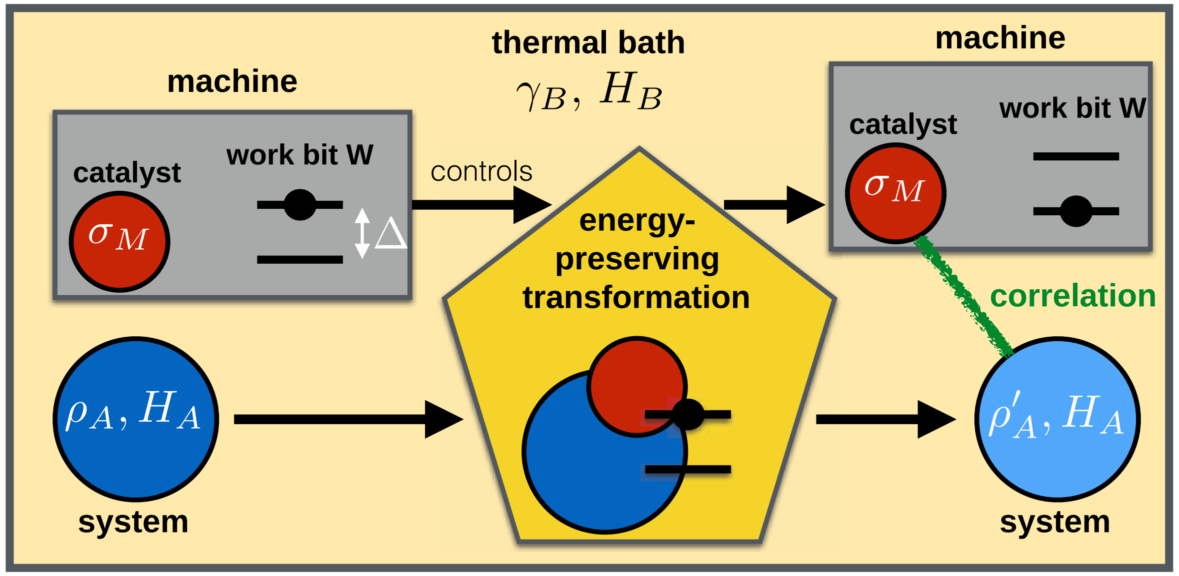

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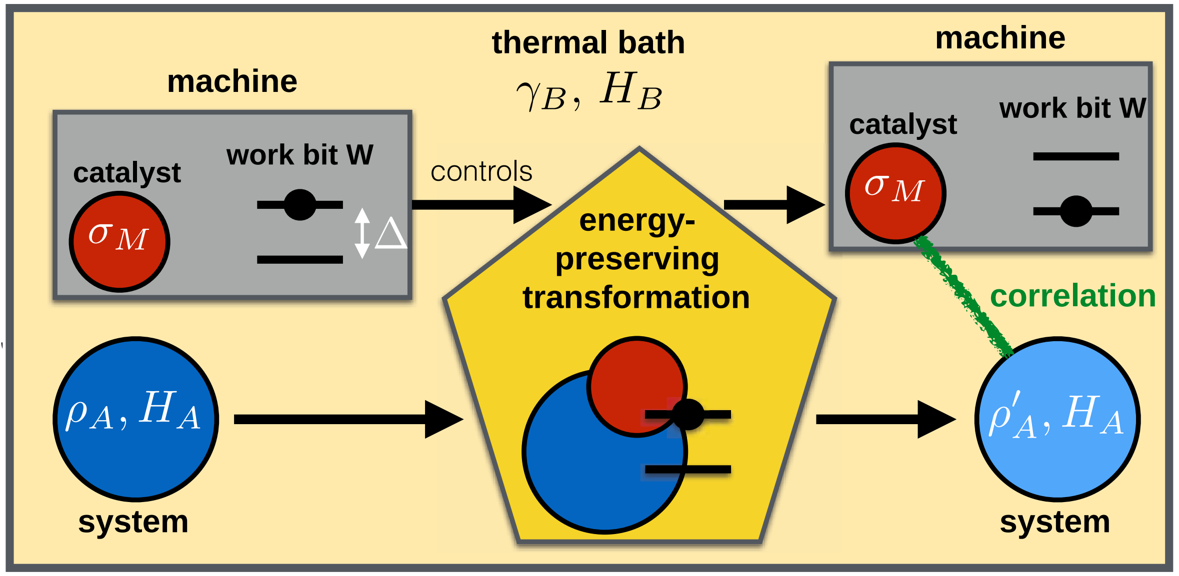

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Implications for quantum thermodynamics

The second laws of quantum thermodynamics

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Quantum thermo for small&strongly correlated systems: formulate as a **resource theory**.

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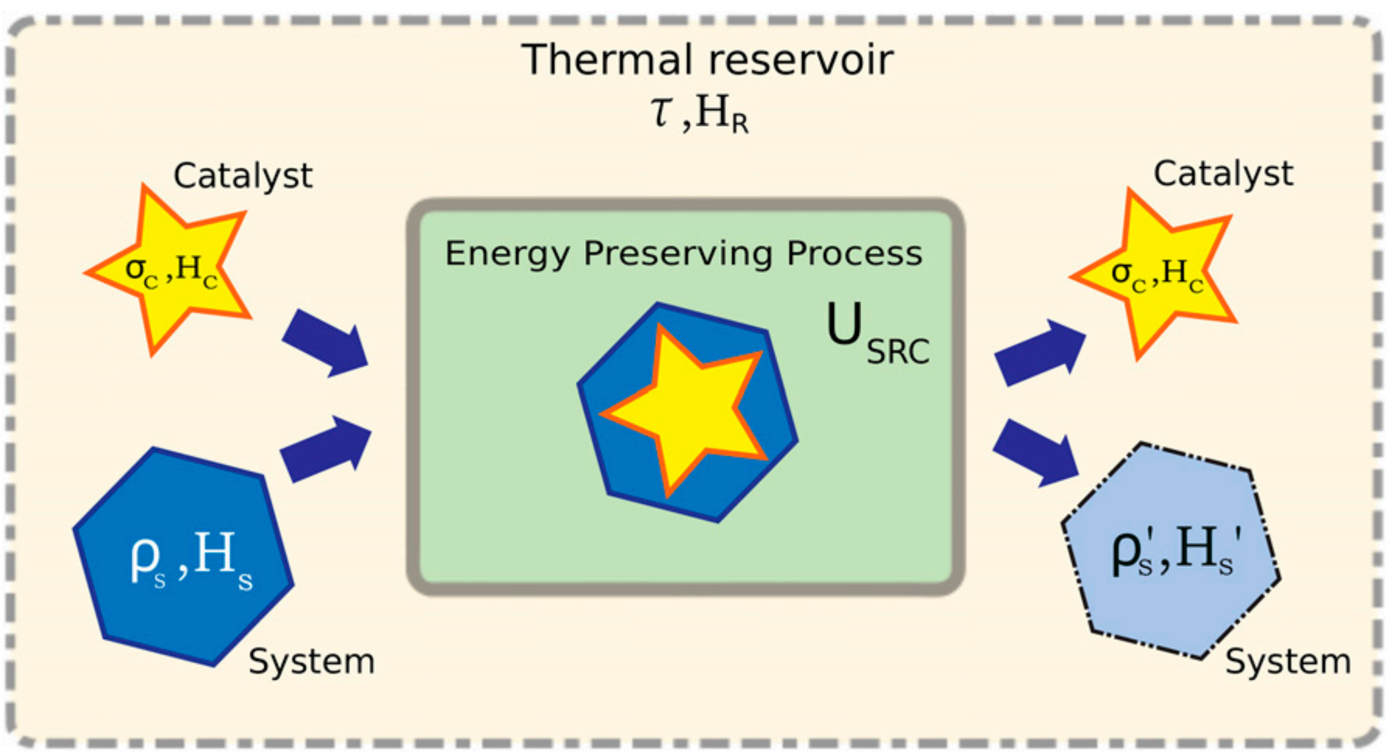

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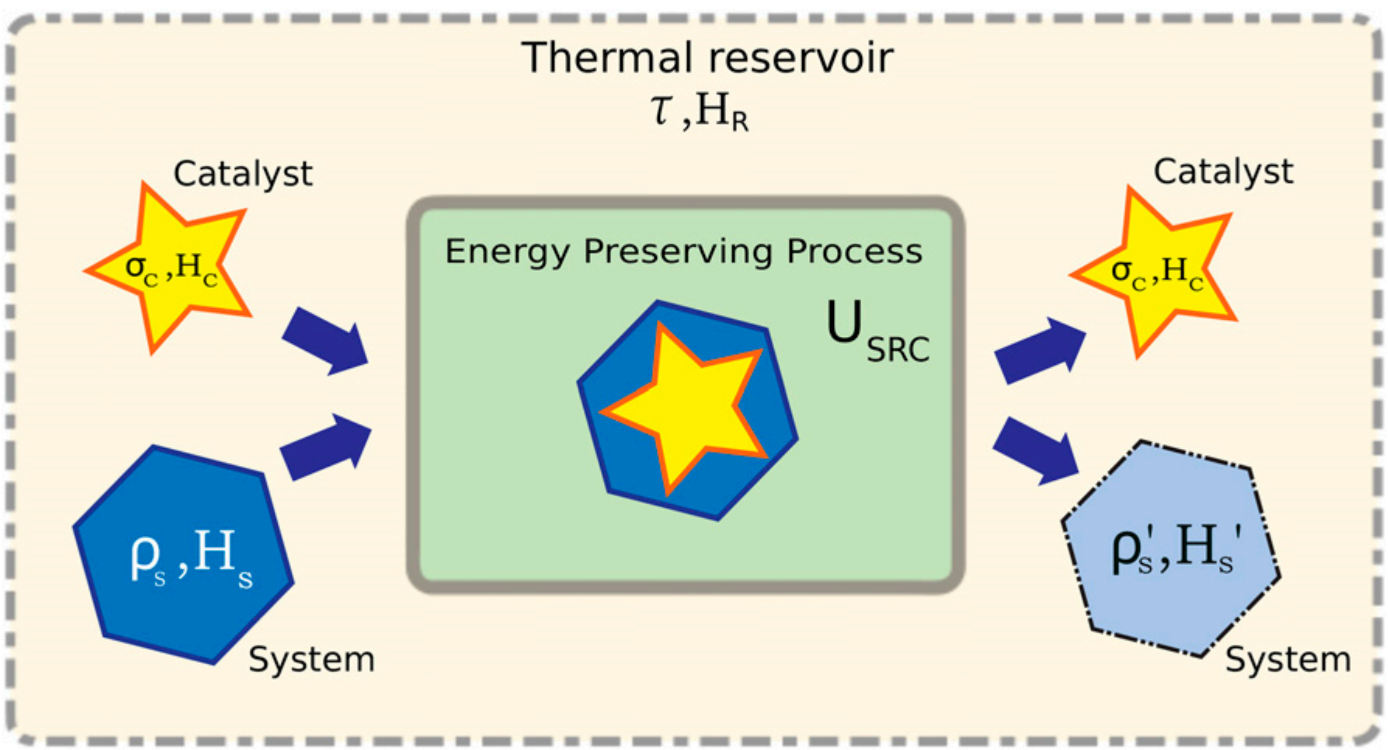

$$\mathrm{Tr}_R\left[U_{SRC}(\rho_S \otimes \sigma_C \otimes \gamma_R)U_{SRC}^\dagger\right] = \rho'_S \otimes \sigma_C$$

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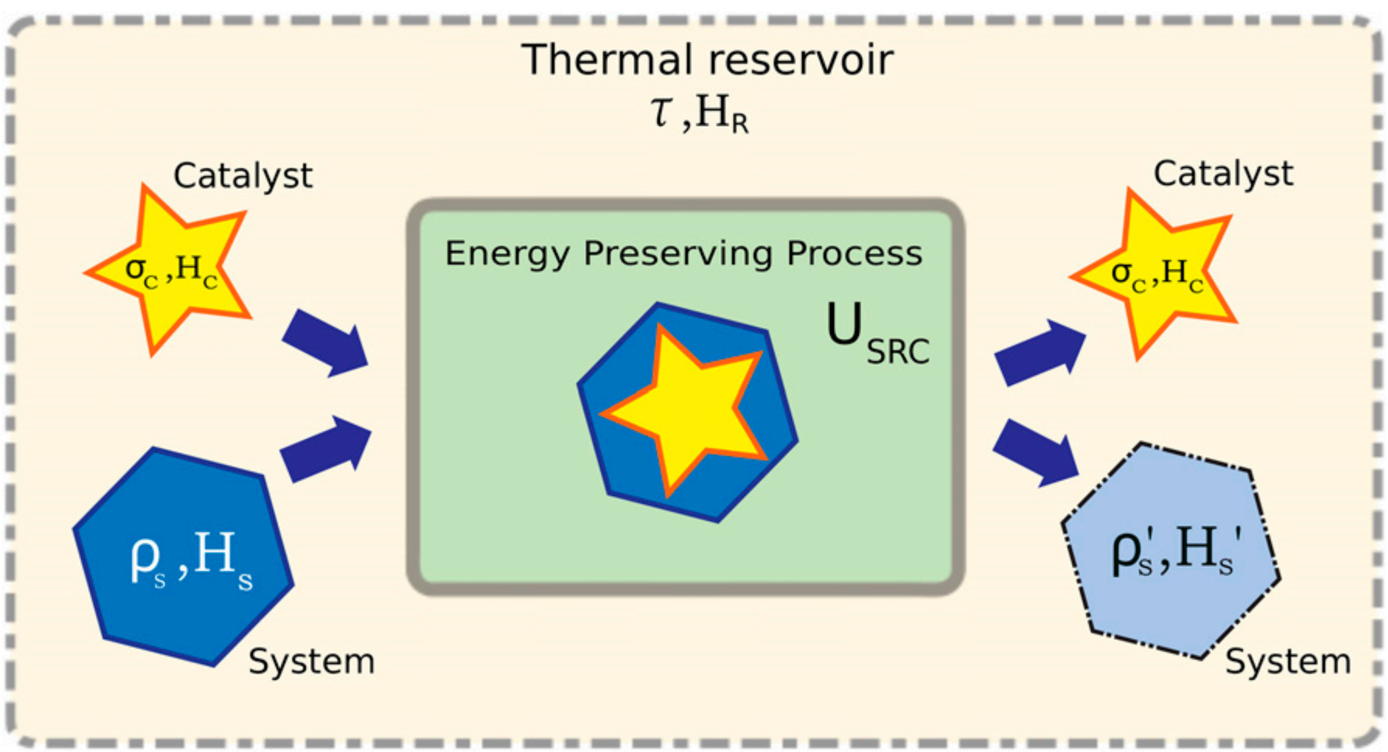
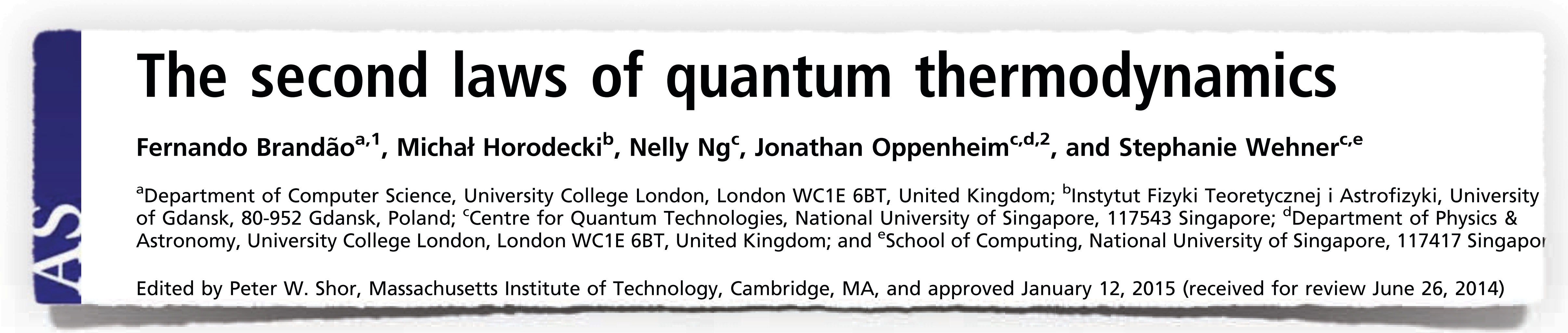


$$\mathrm{Tr}_R\left[U_{SRC}(\rho_S\otimes\sigma_C\otimes\gamma_R)U_{SRC}^\dagger\right]$$
$$=\rho_S'\otimes\sigma_C$$

thermal reservoir

Implications for quantum thermodynamics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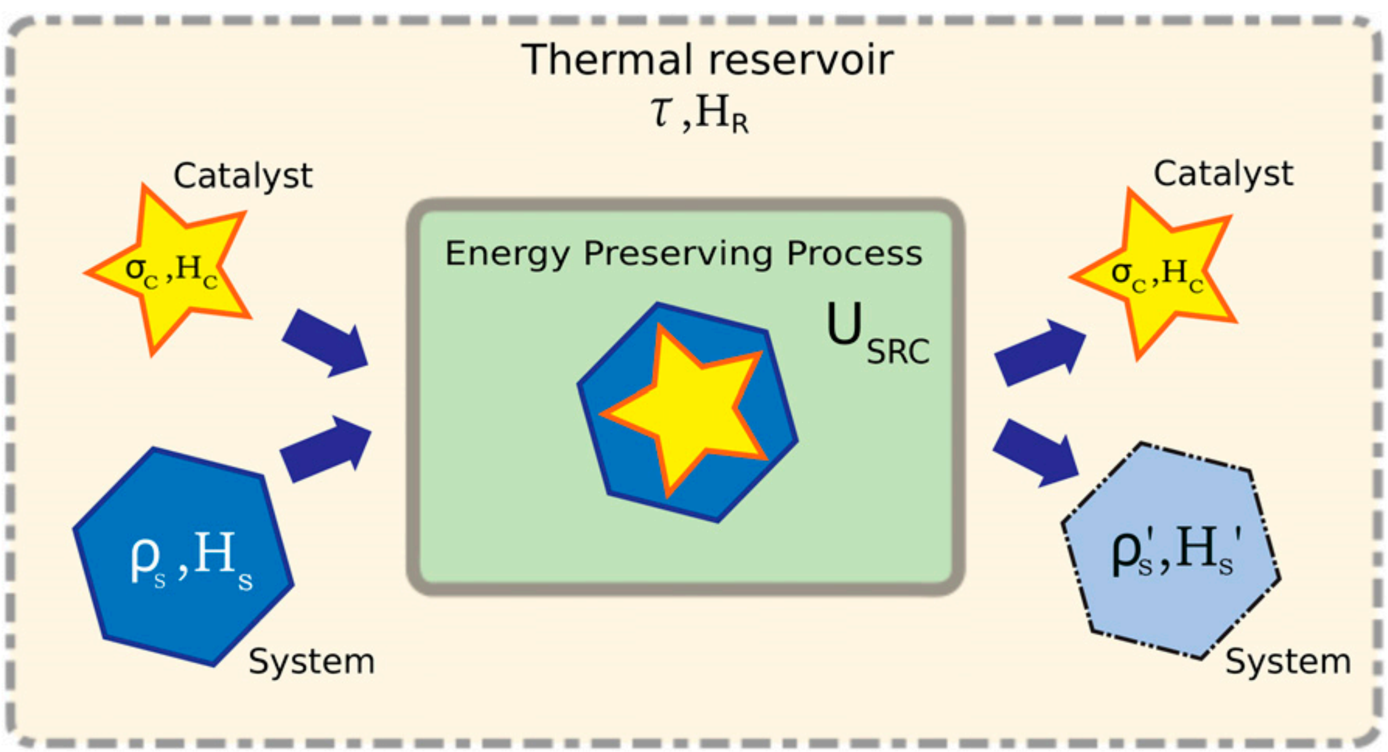

$$\mathrm{Tr}_R\left[U_{SRC}(\rho_S\otimes\sigma_C\otimes\gamma_R)U_{SRC}^\dagger\right]$$
$$=\rho'_S\otimes\sigma_C$$

thermal reservoir

$$[U_{SRC},H_S+H_R+H_C]=0$$

(energy strictly preserved)

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Theorem: $\rho \to \rho'$ is possible (for block-diagonal states) iff

$F_\alpha(\rho) \geq F_\alpha(\rho') \ \ \forall\alpha$ ("free energies").

The second laws of quantum thermodynamics

Fernando Brandão[a,1], Michał Horodecki[b], Nelly Ng[c], Jonathan Oppenheim[c,d,2], and Stephanie Wehner[c,e]

[a]Department of Computer Science, University College London, London WC1E 6BT, United Kingdom; [b]Instytut Fizyki Teoretycznej i Astrofizyki, University of Gdansk, 80-952 Gdansk, Poland; [c]Centre for Quantum Technologies, National University of Singapore, 117543 Singapore; [d]Department of Physics & Astronomy, University College London, London WC1E 6BT, United Kingdom; and [e]School of Computing, National University of Singapore, 117417 Singapor

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("free energies").

$$F(\rho_A) \equiv F_1(\rho) = \mathrm{tr}(\rho_A H_A) - k_B T S(\rho_A),$$

$$F_\alpha(\rho) = k_B T S_\alpha(\rho \| \gamma) + F_\alpha(\gamma).$$

Rényi divergence

Implications for quantum thermodynamics

MM, *Correlating thermal machines and the second law at the nanoscale*, arXiv:1707.03451

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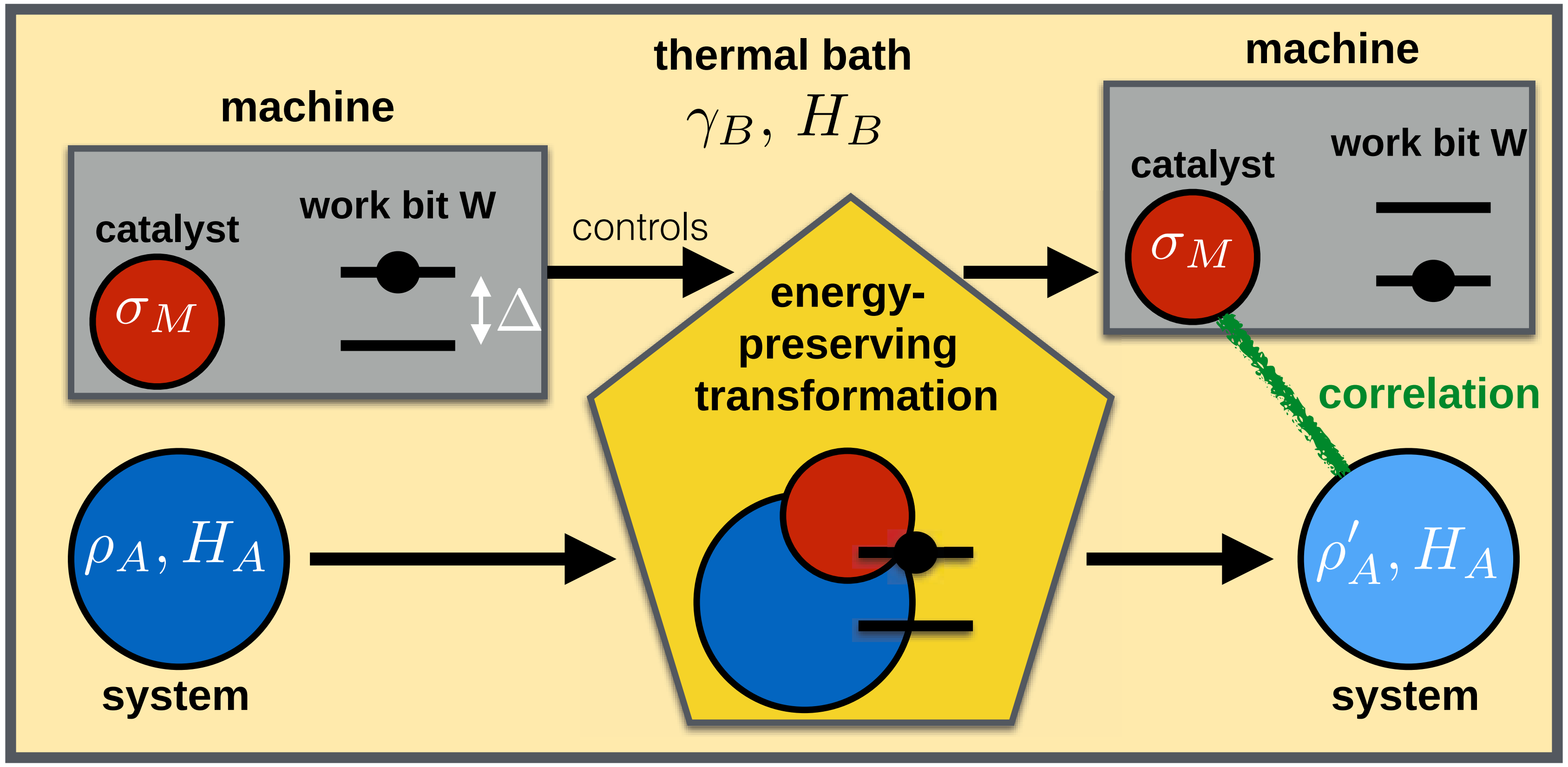

Preserve catalyst exactly, but allow **correlations** to build up.

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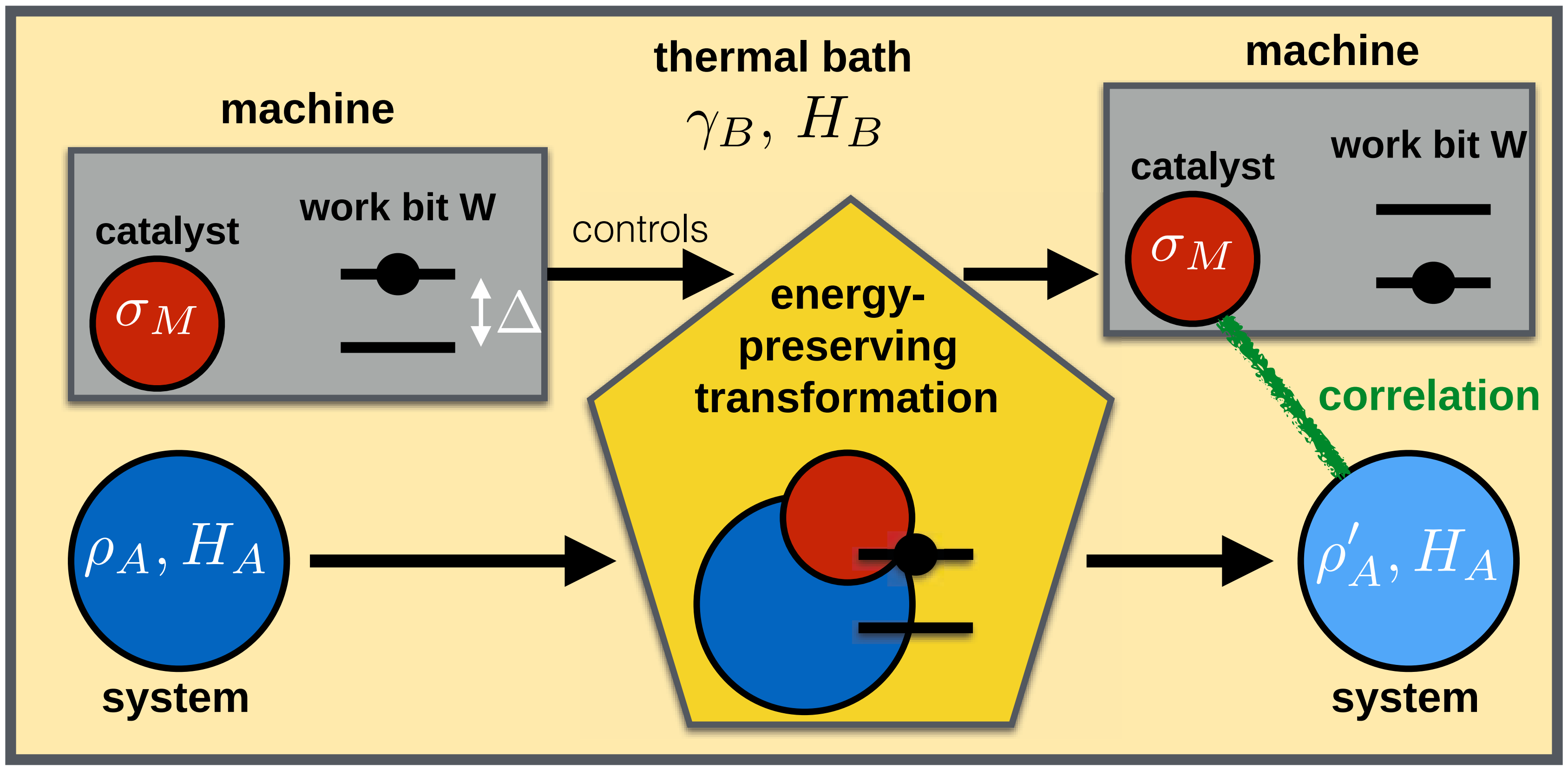

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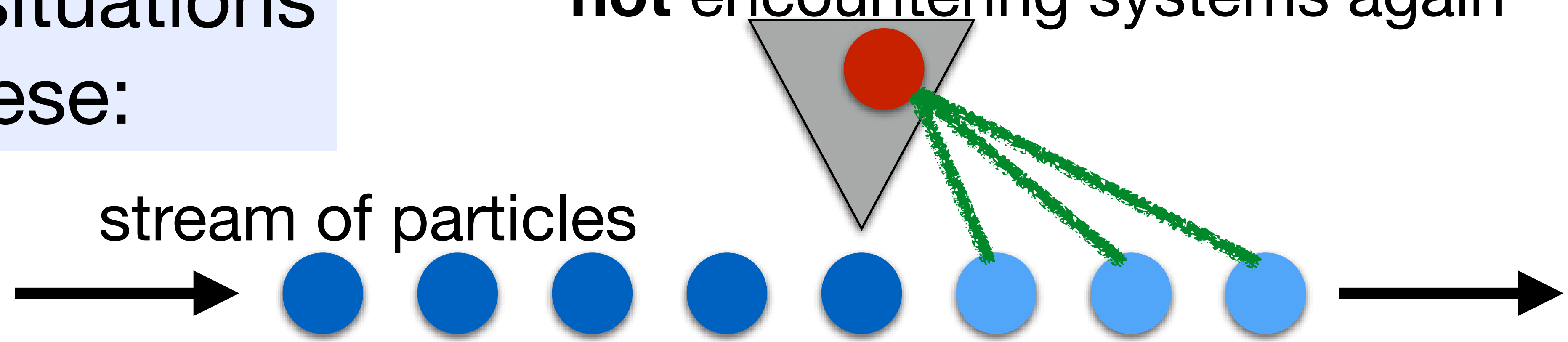
Applies to situations like these:

thermal machine,
acting single-shot,
not encountering systems again



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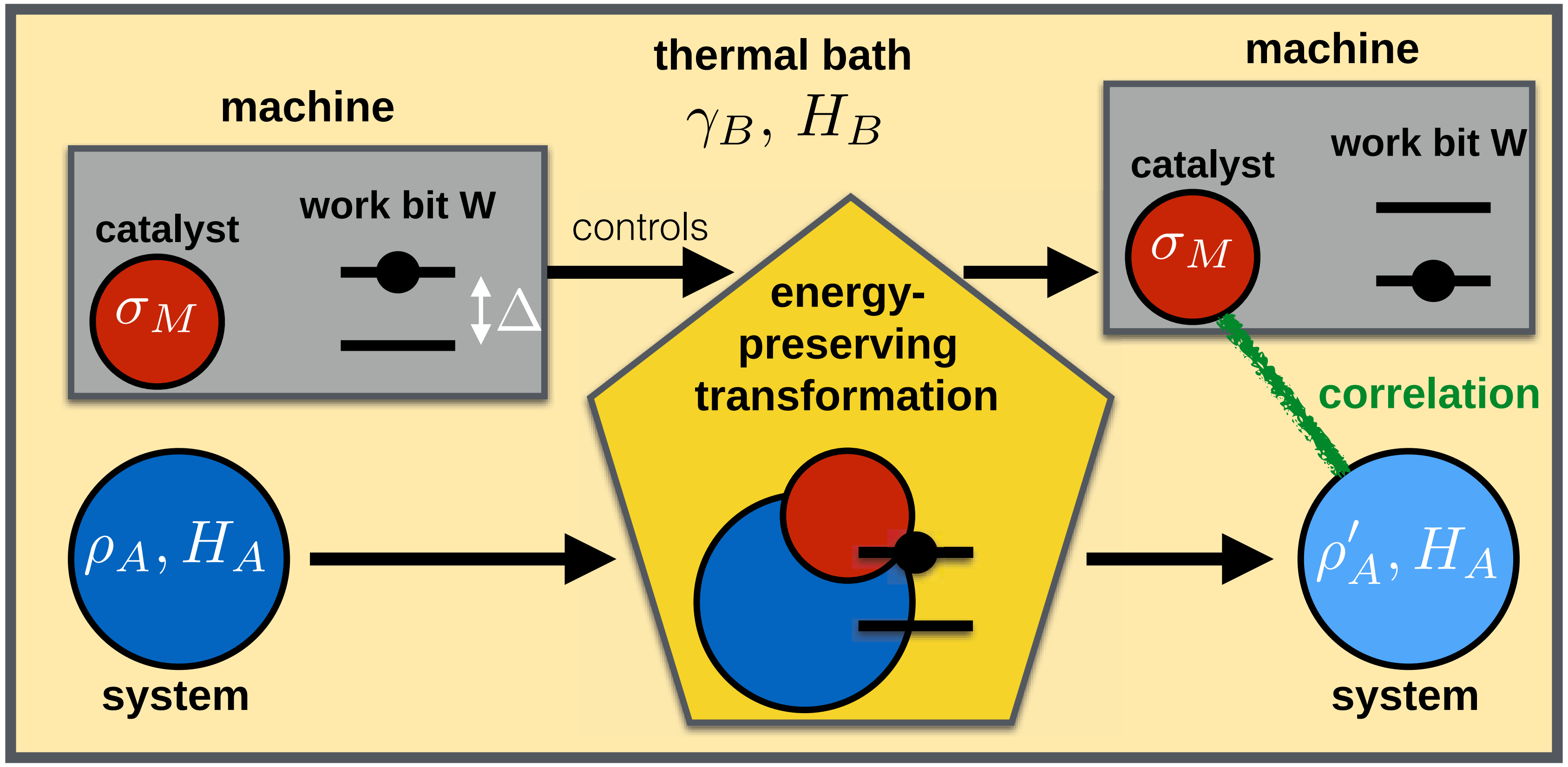

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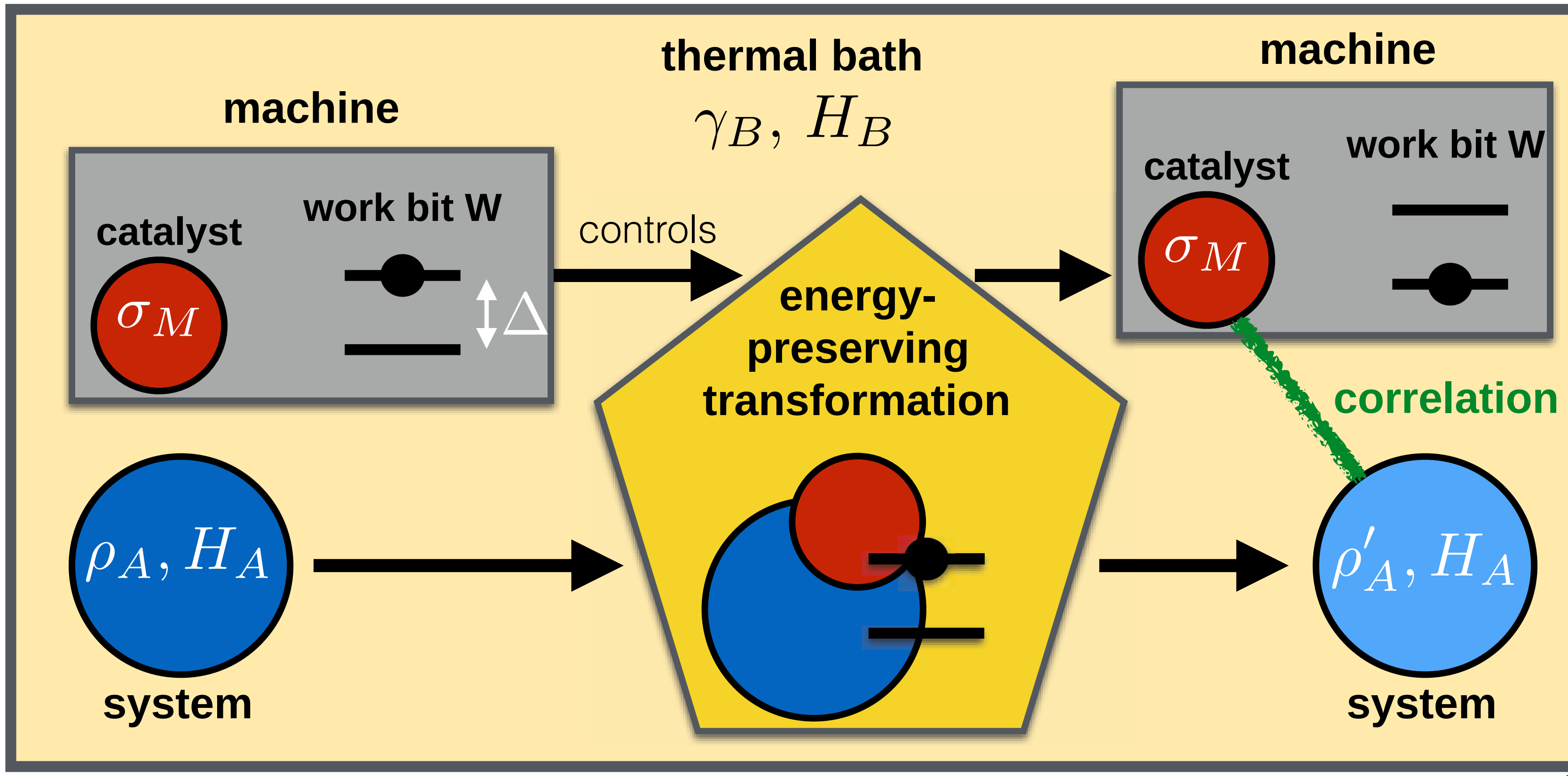

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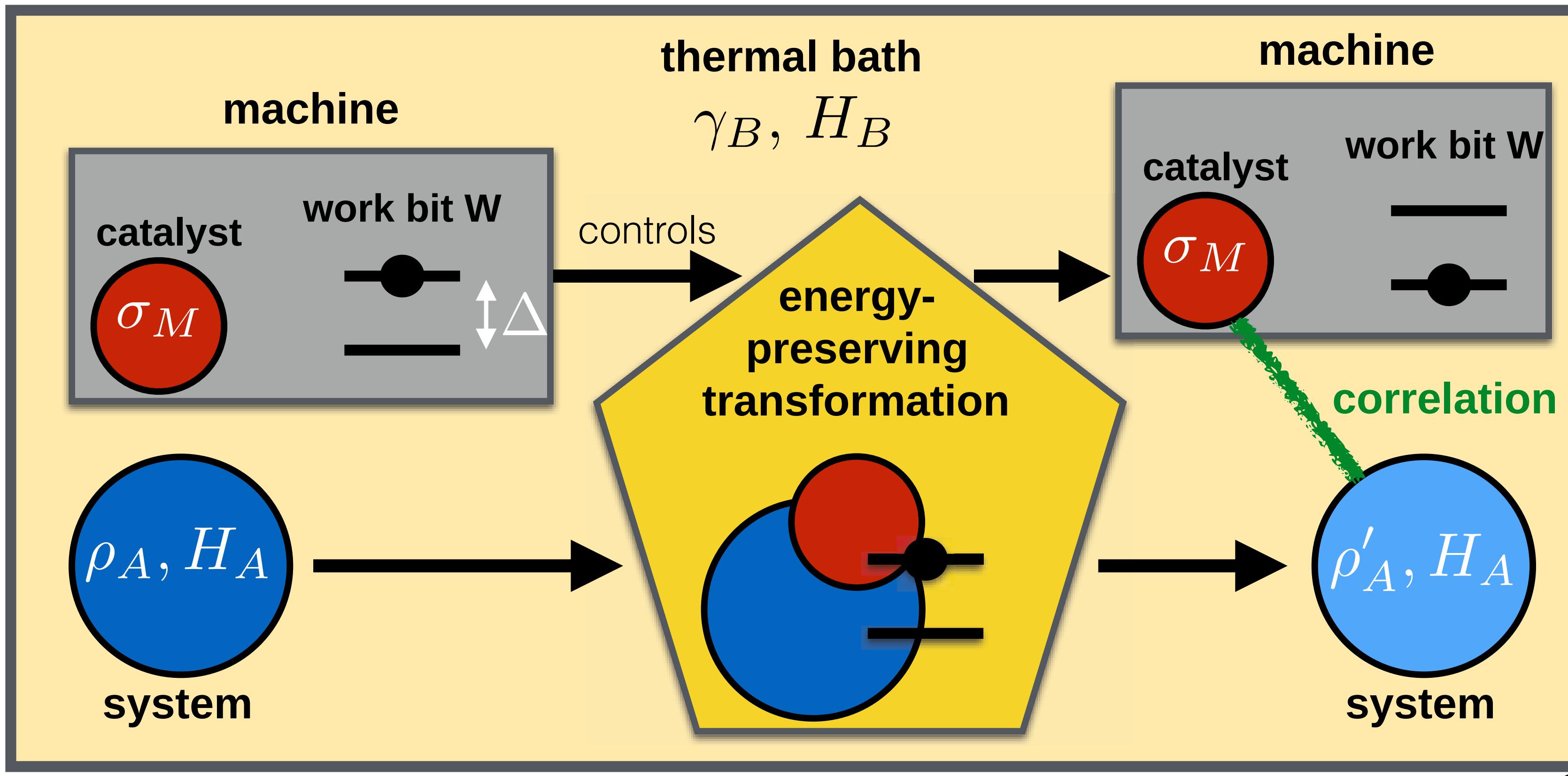

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One-shot operational interpretation of Helmholtz free energy!

Second laws ⟶ second law!

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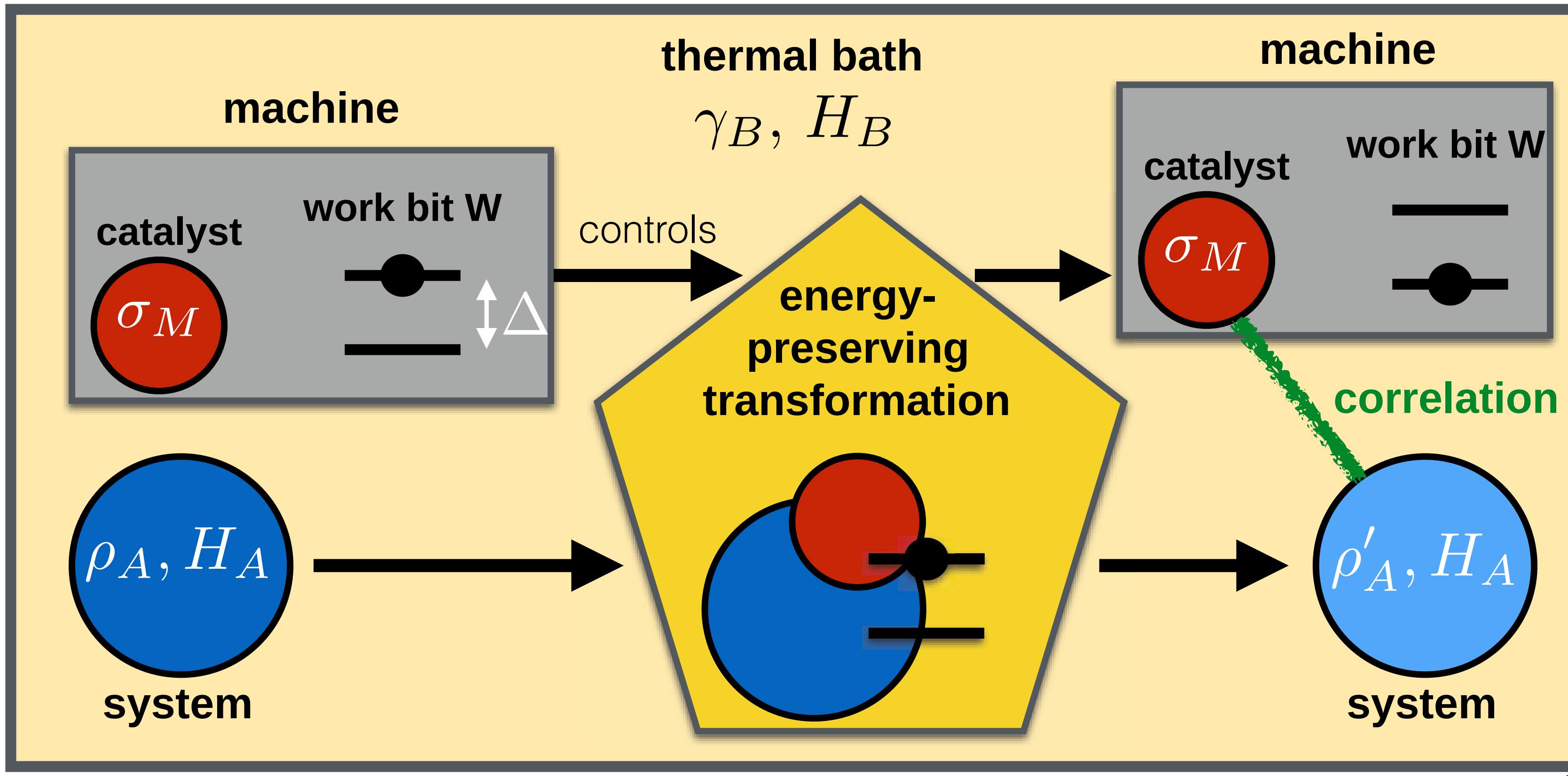

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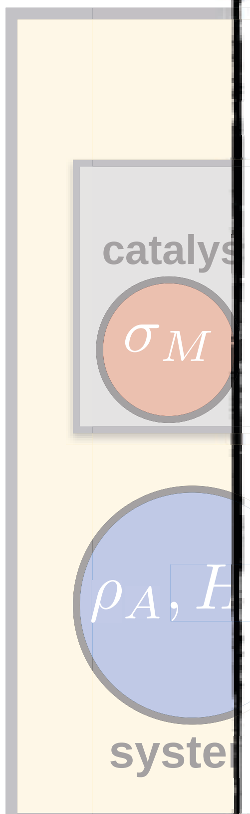
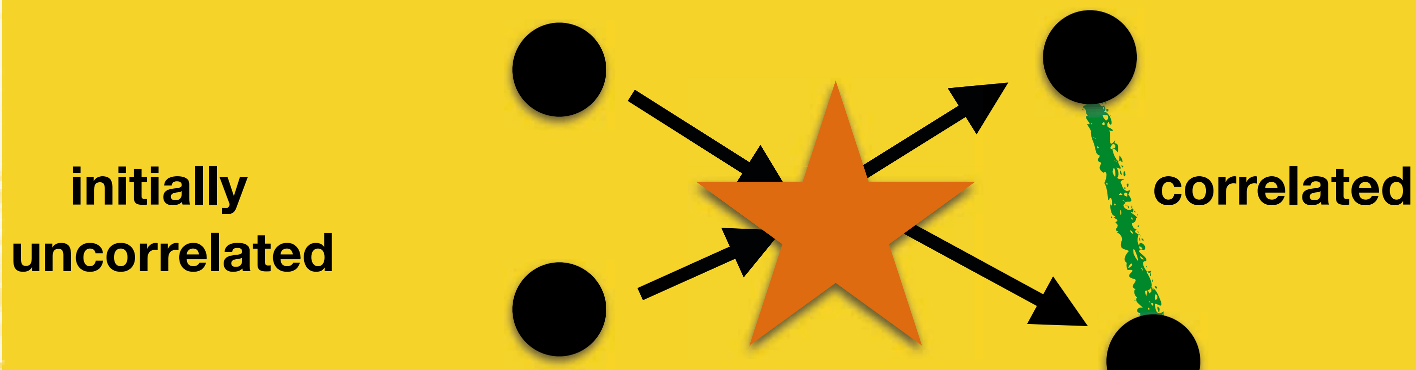


In spirit somewhat close to Boltzmann's original **Stoßzahlansatz** (1896):



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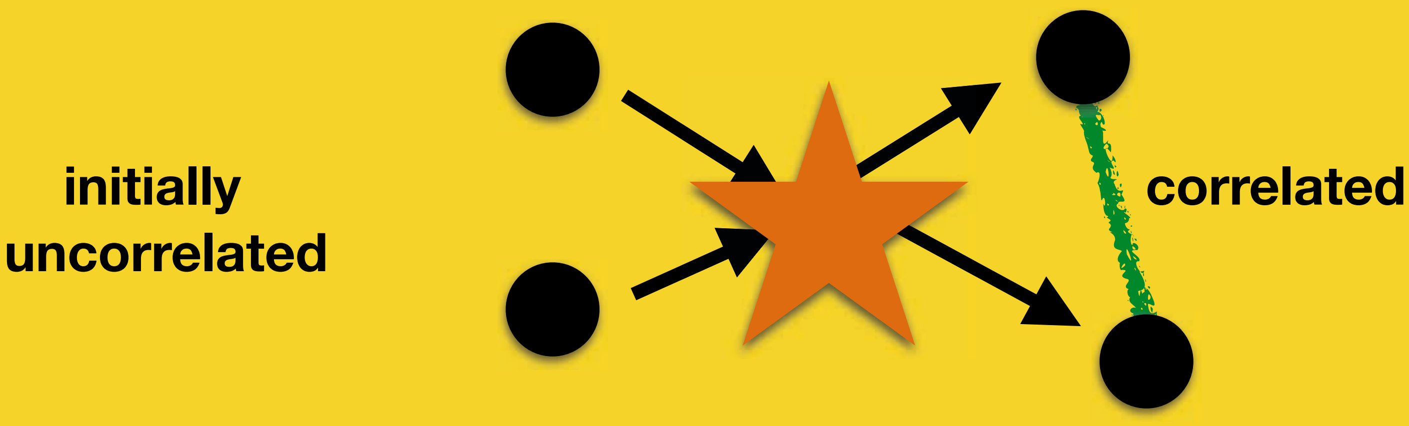

Conjecture. Result also true in the presence of quantum coherence.

Extractable work / work of formation…

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… are both in general random variables.

ARTICLE

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$$\lim_{n\to\infty}\frac{1}{n}F^{\varepsilon}_{0/\infty}(\rho^{\otimes n}) = F(\rho).$$

Work characterized by *F* only in the thermodynamic limit.

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Work of formation (when allowing correlations)…

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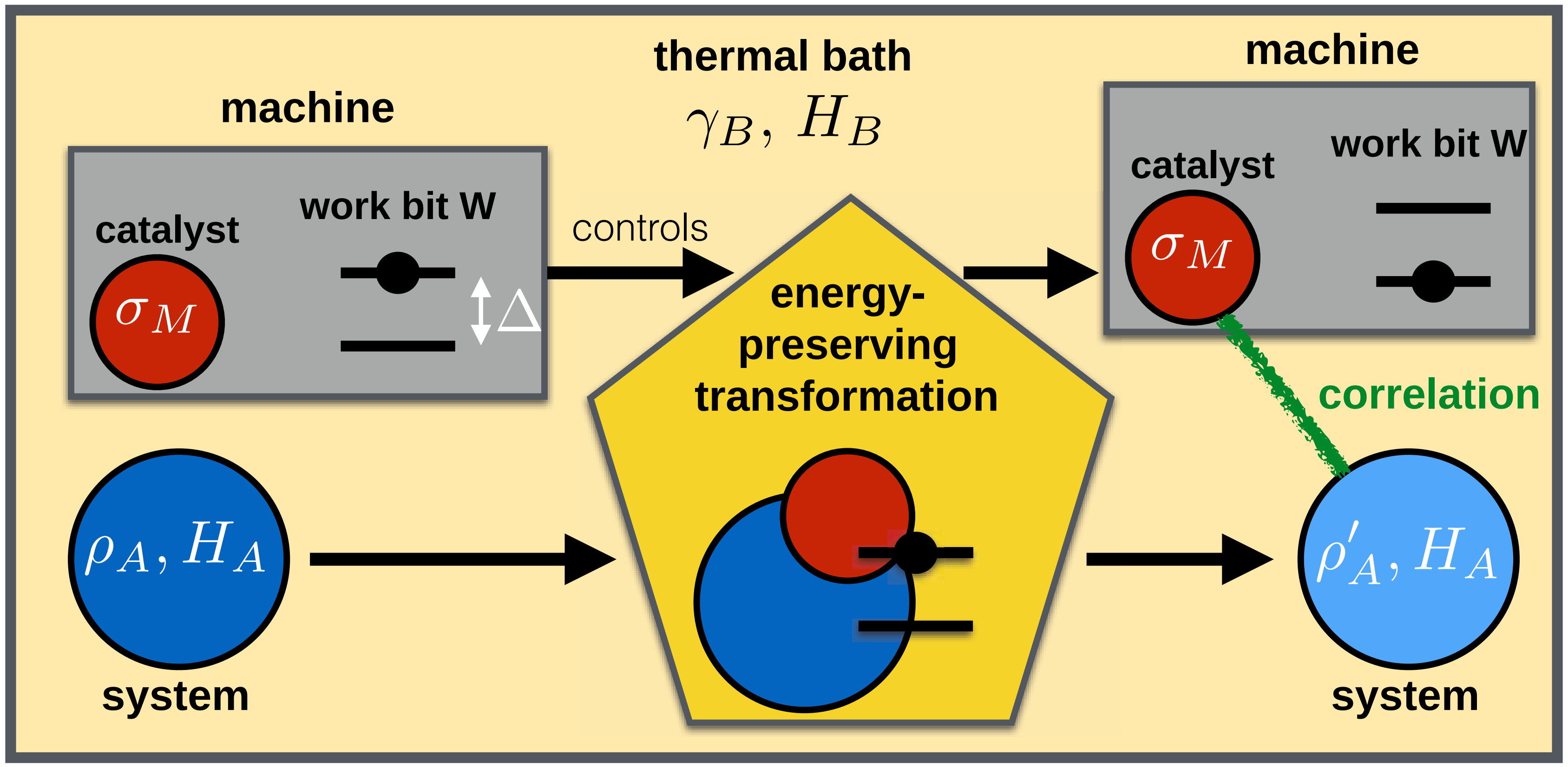

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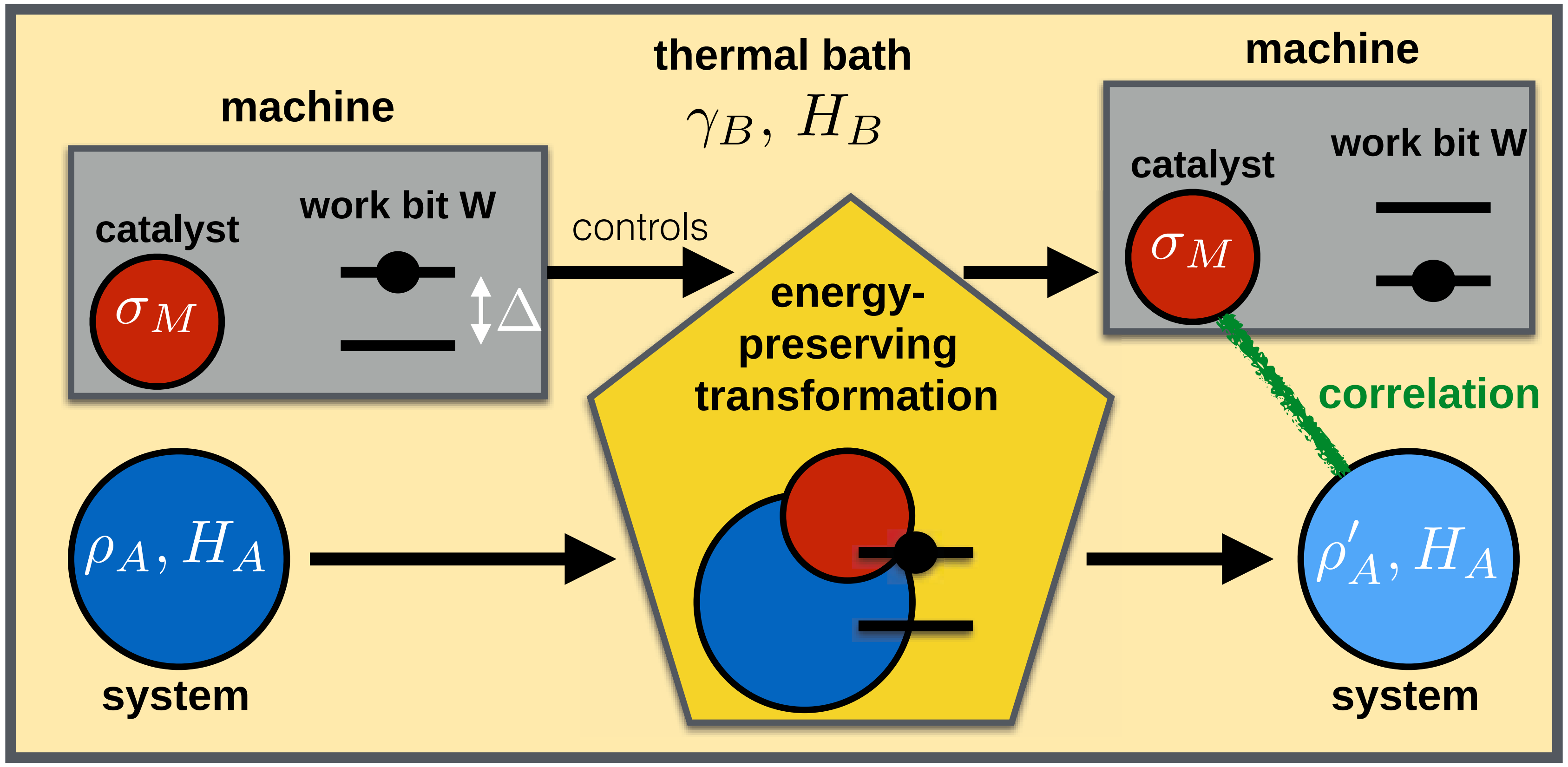

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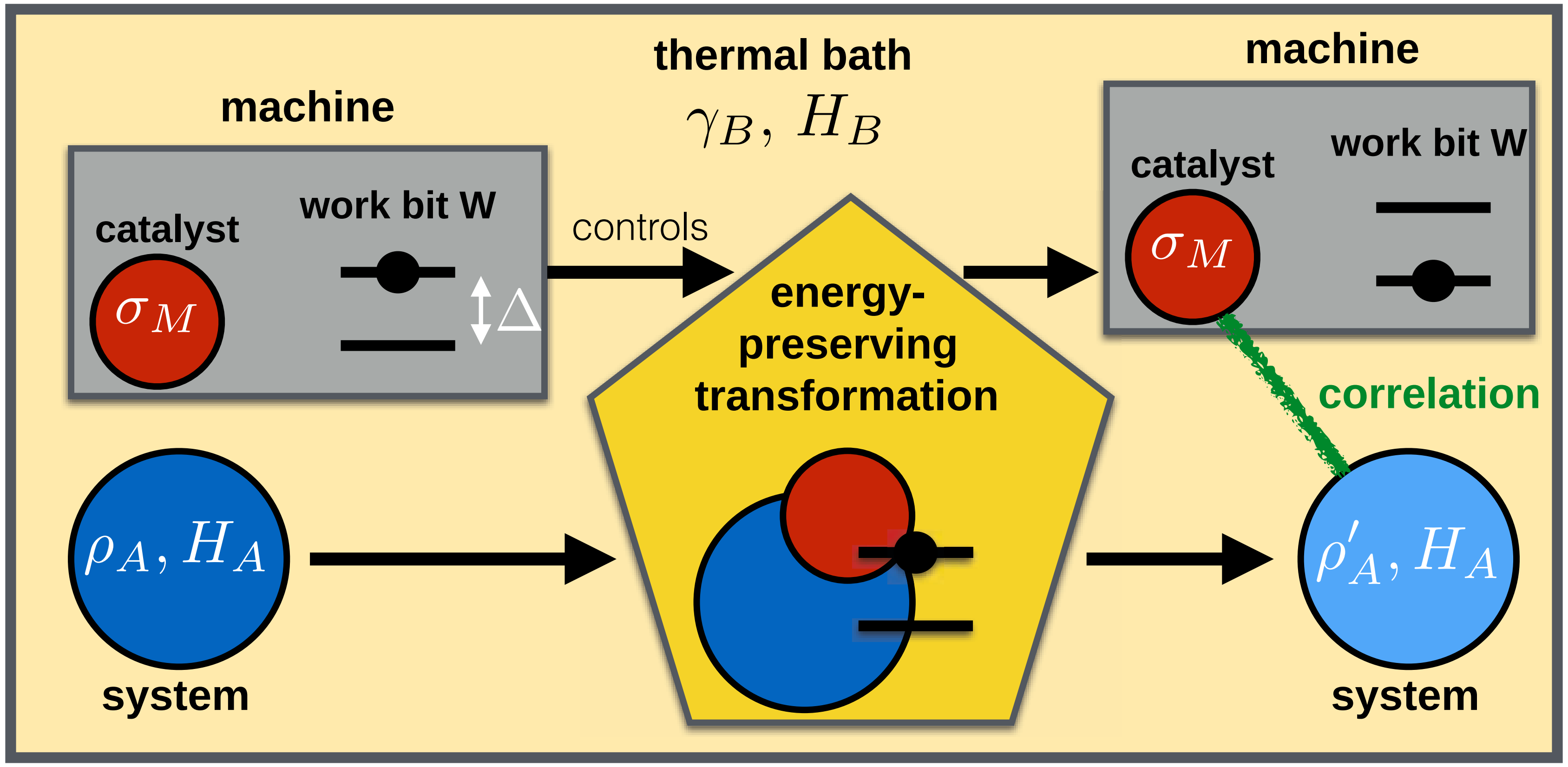

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$$\rho_A \otimes \sigma_M \otimes |e\rangle\langle e|_W \mapsto \sigma_{AM} \otimes |g\rangle\langle g|_W$$

can be achieved by a thermal operation, where $\sigma_A := \mathrm{Tr}_M \sigma_{AM}$ is arbitrarily close to ρ'_A.

The state σ_M is exactly identical before and after the transformation, M is finite-dimensional, and the resulting correlations between A and M can be made arbitrarily small.

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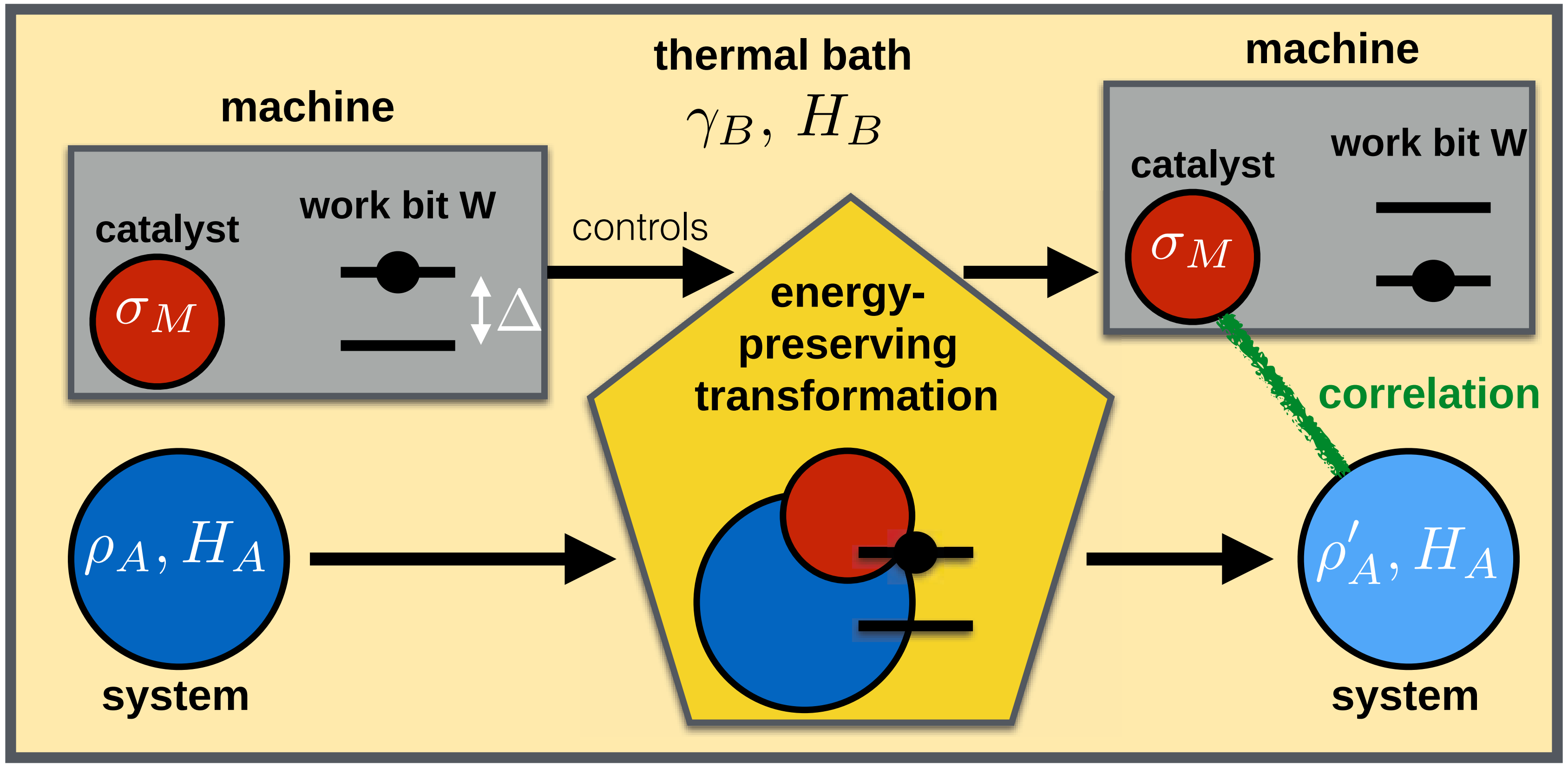

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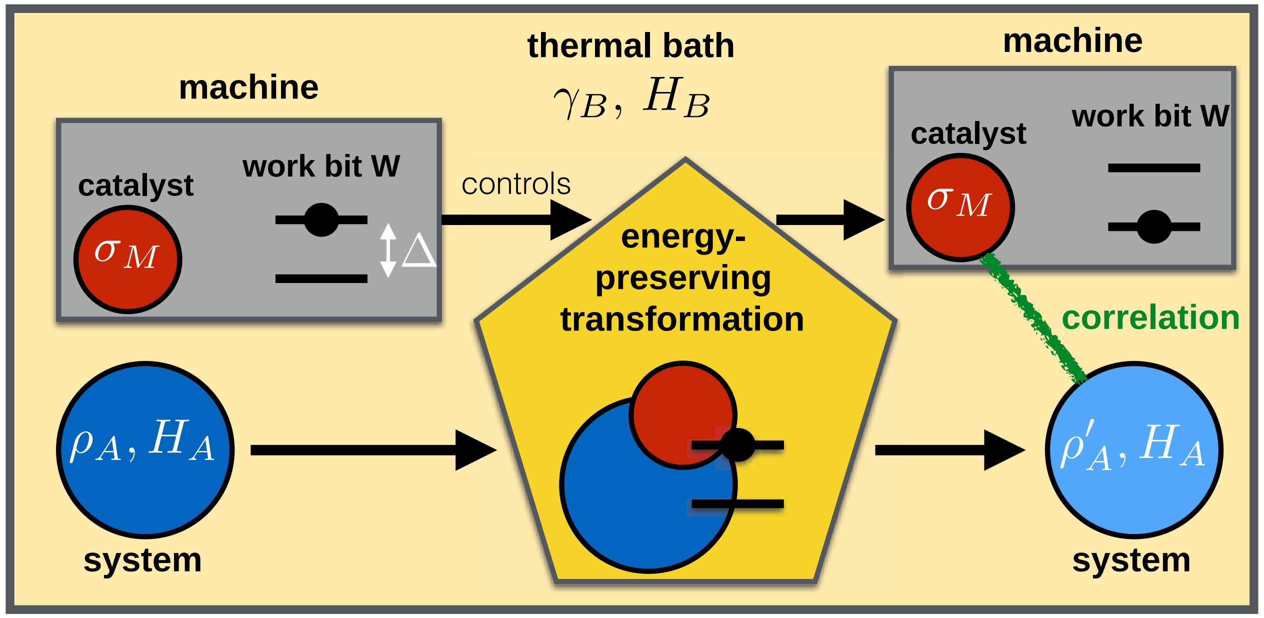

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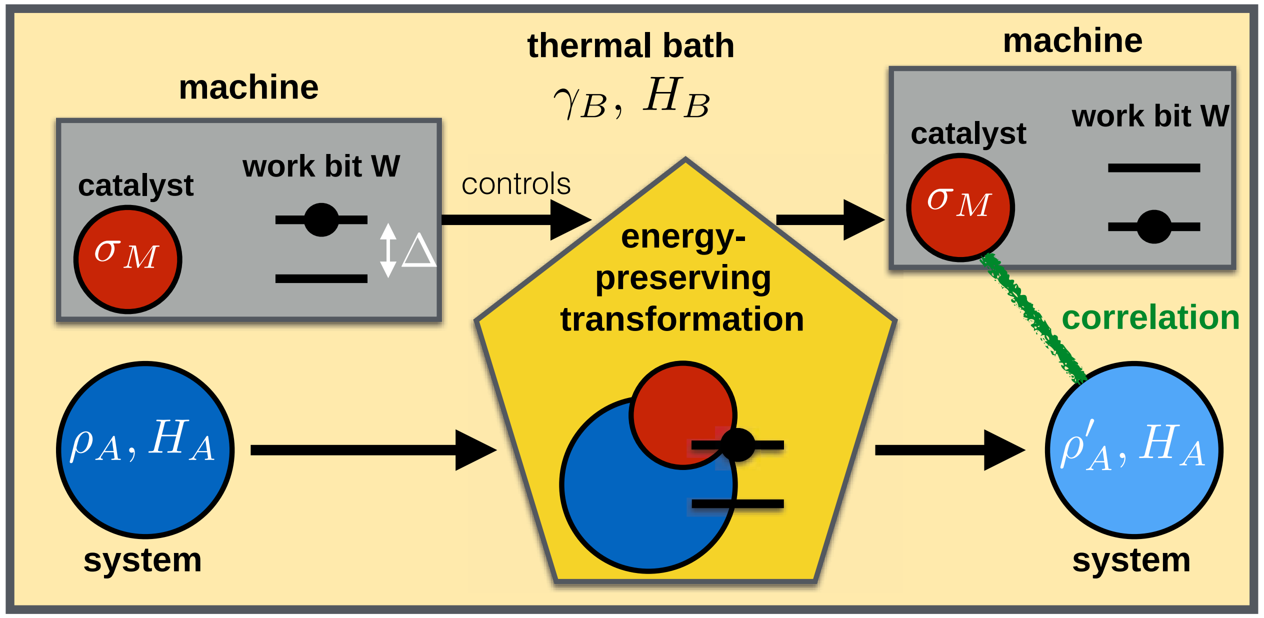

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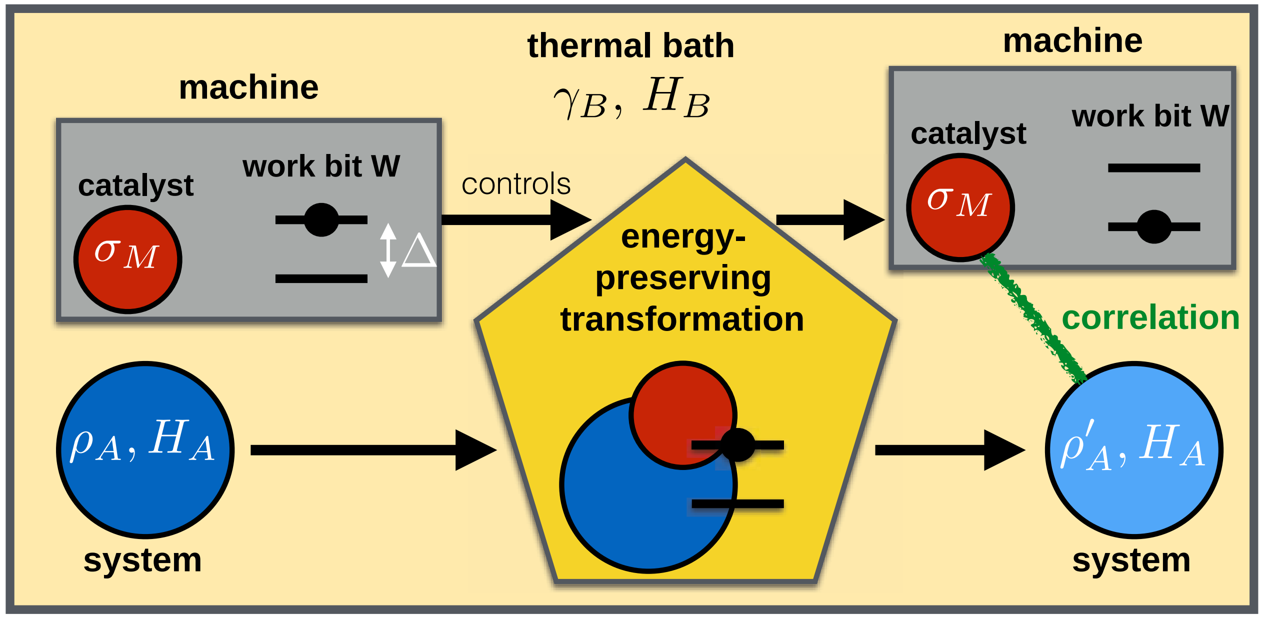

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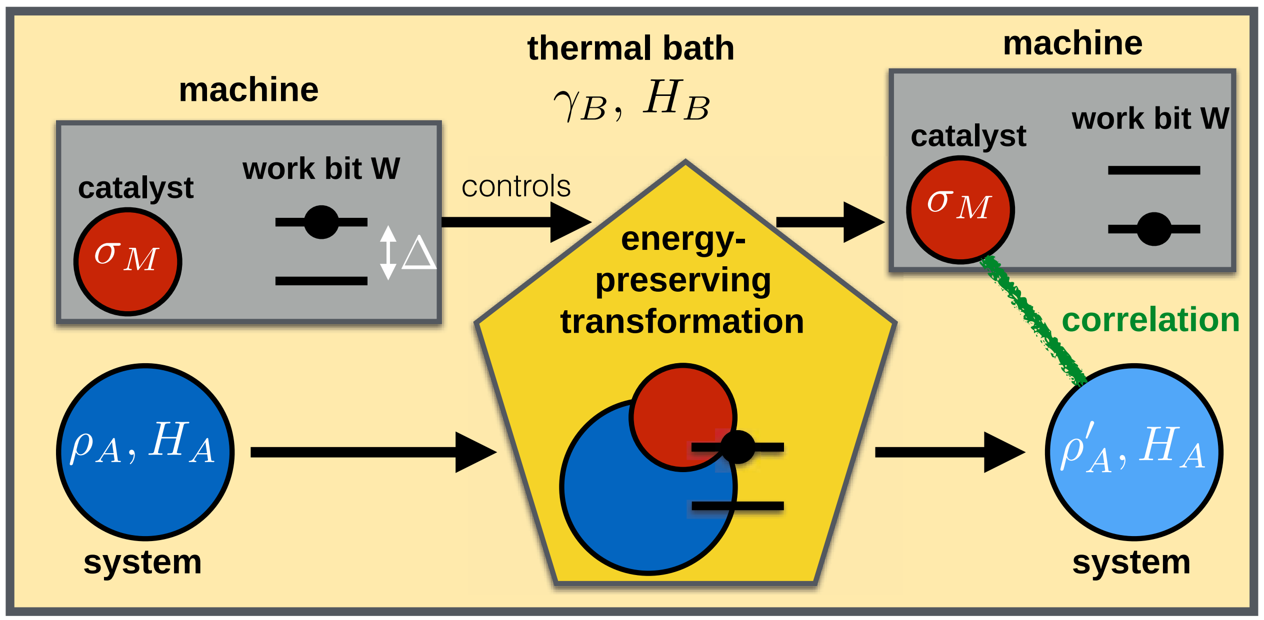

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Here $\sigma_M = \mathrm{Tr}_{AS}\sigma_{AMS}$ *remains identical during the transformation,* $\sigma_S = \tau_S^{(m,n,\varepsilon)}$*, and* σ_A *is as close to* ρ'_A *as we like. This can be achieved for any choice of* $\varepsilon > 0$*, as long as* n/m *is large enough.*

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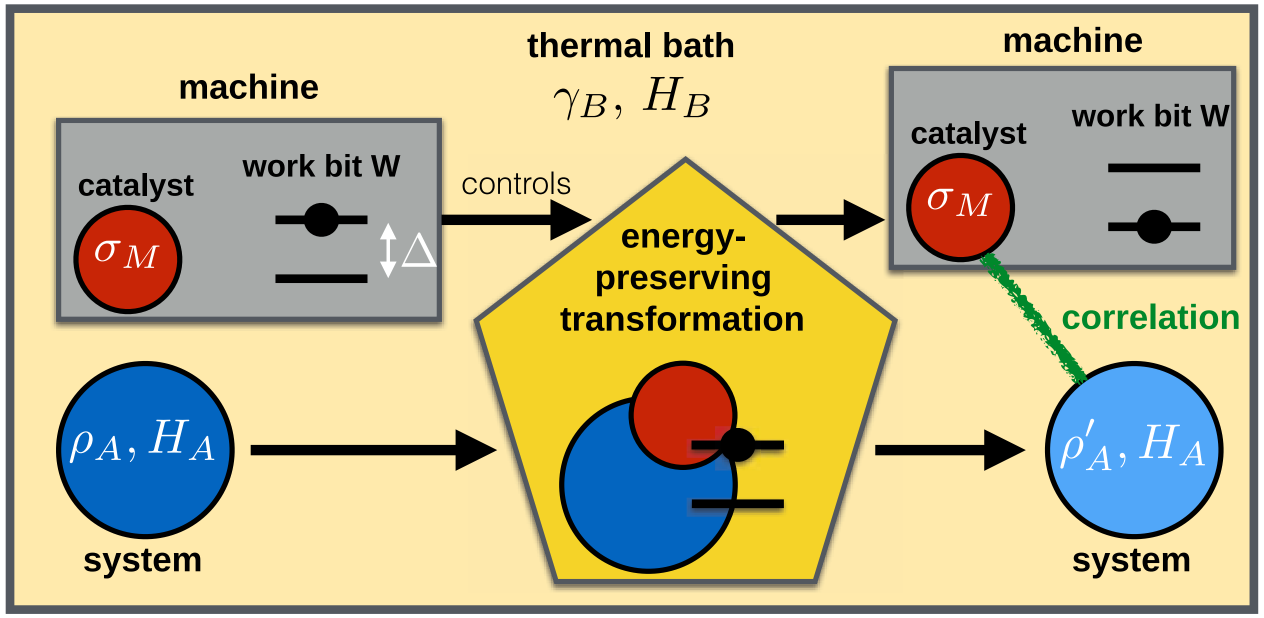

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I.e. can make fluctuations arbitrarily small (but not zero).

Stochastic independence as a resource

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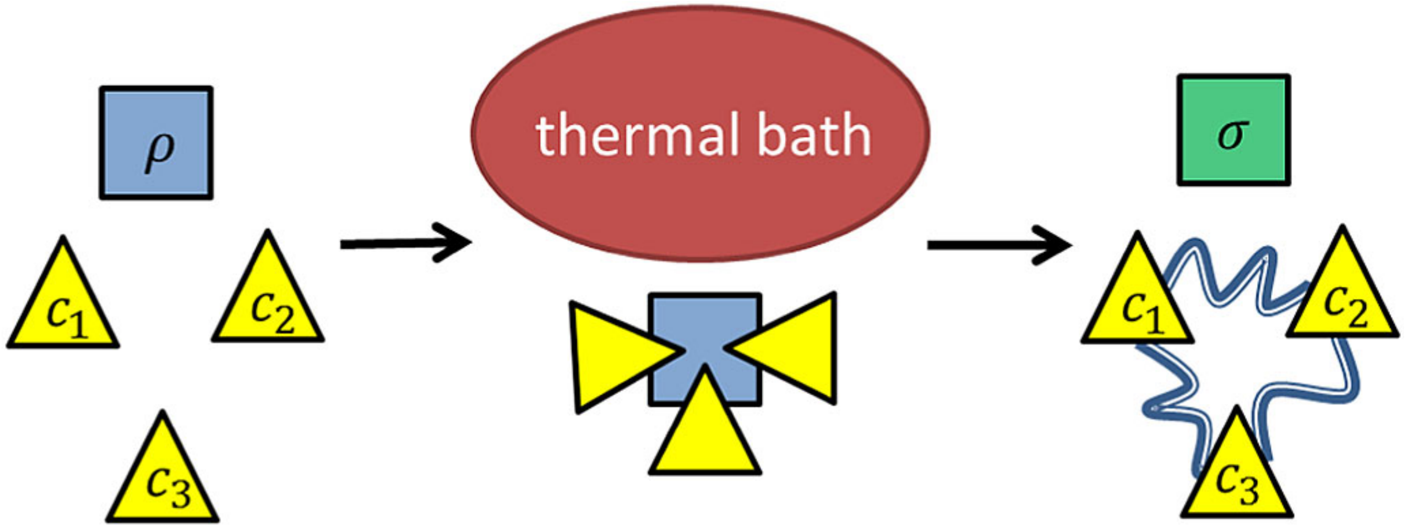

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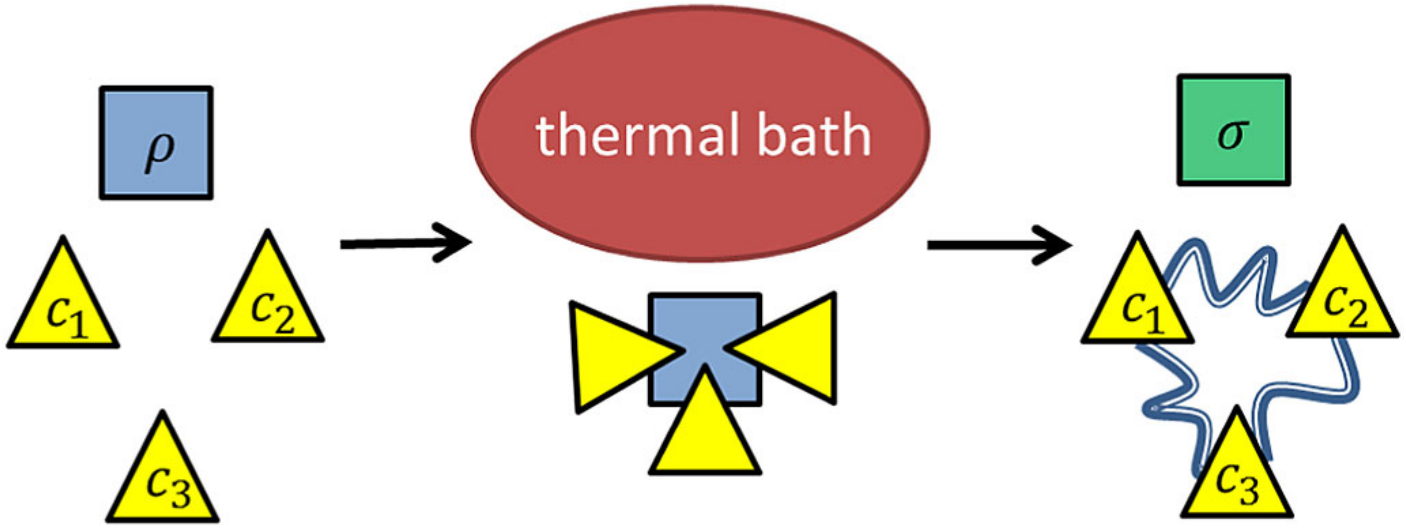

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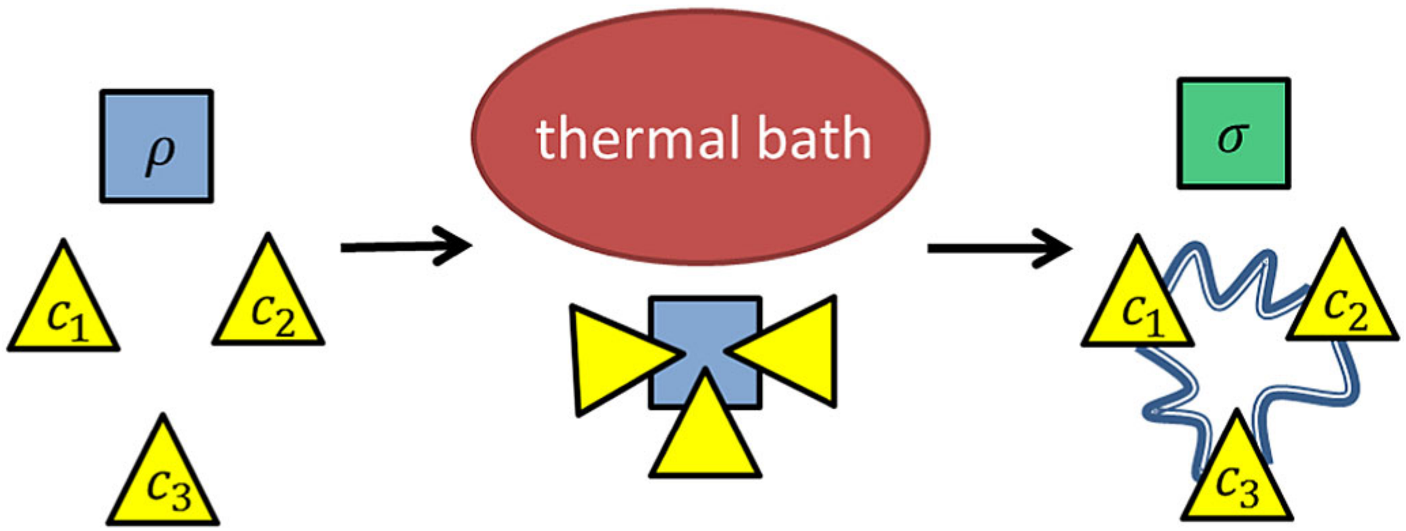

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The exact opposite of what one would expect from standard thermodynamics!

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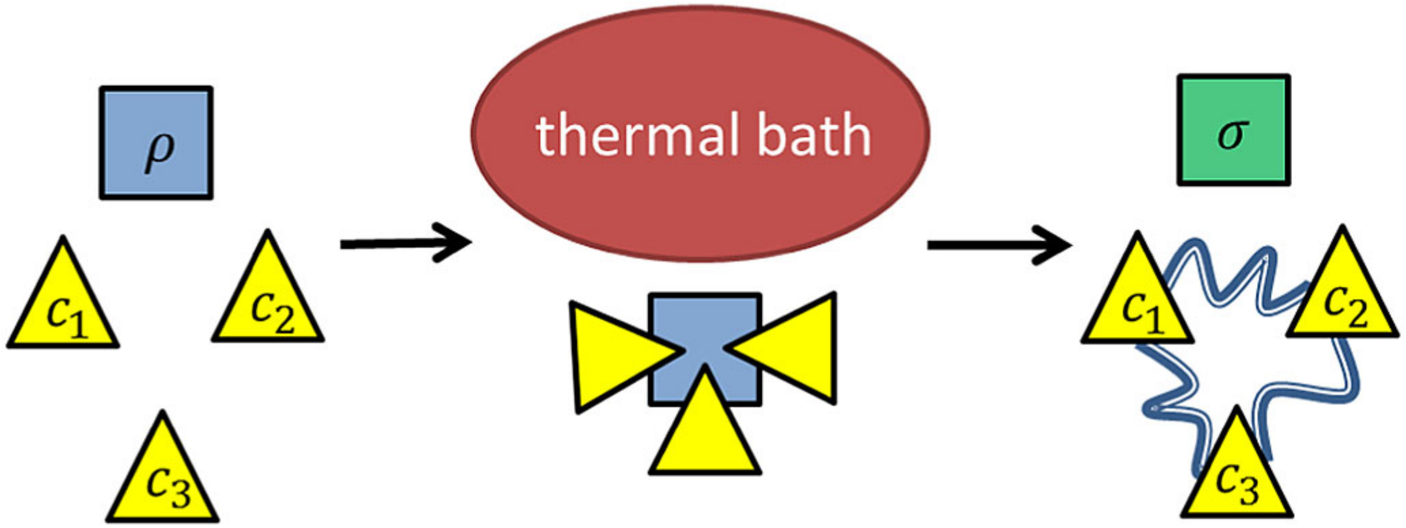

Correlating external systems can allow otherwise impossible state transitions. **"Trade fluctuations for correlations."**

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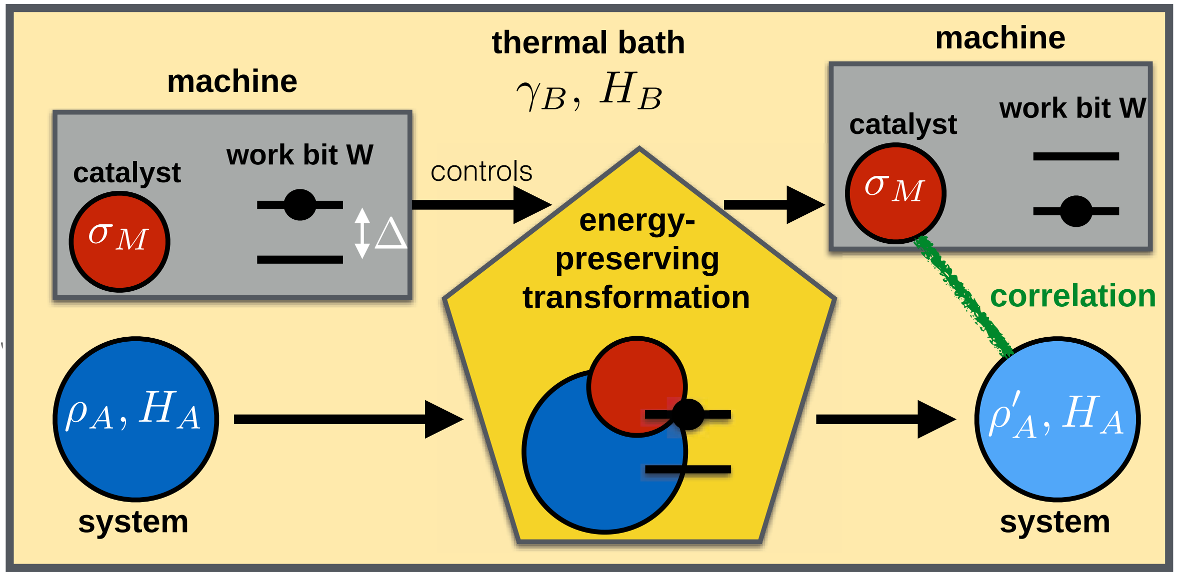
Outline

1. Intro: Majorization in quantum information

2. Main mathematical results

3. **Implications for quantum thermodynamics**



4. Implications for quantum information (in progress)

Outline

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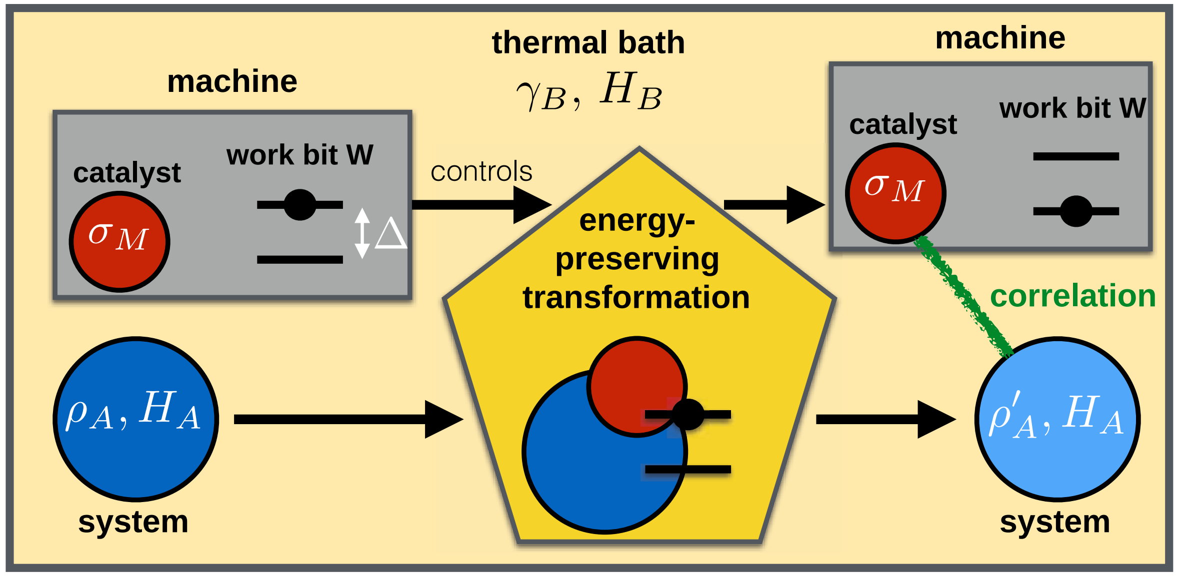

4. **Implications for quantum information (in progres**

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PRL **118,** 080503 (2017) PHYSICAL REVIEW LETTERS week ending 24 FEBRUARY 2017

Catalytic Decoupling of Quantum Information

Christian Majenz,[1,*] Mario Berta,[2] Frédéric Dupuis,[3] Renato Renner,[4] and Matthias Christandl[1]

[1]*Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø*
[2]*Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, California 91125, USA*
[3]*Faculty of Informatics, Masaryk University, Brno, Czech Republic*
[4]*Institute for Theoretical Physics, ETH Zurich, 8093 Zürich, Switzerland*

(Received 24 May 2016; published 23 February 2017)

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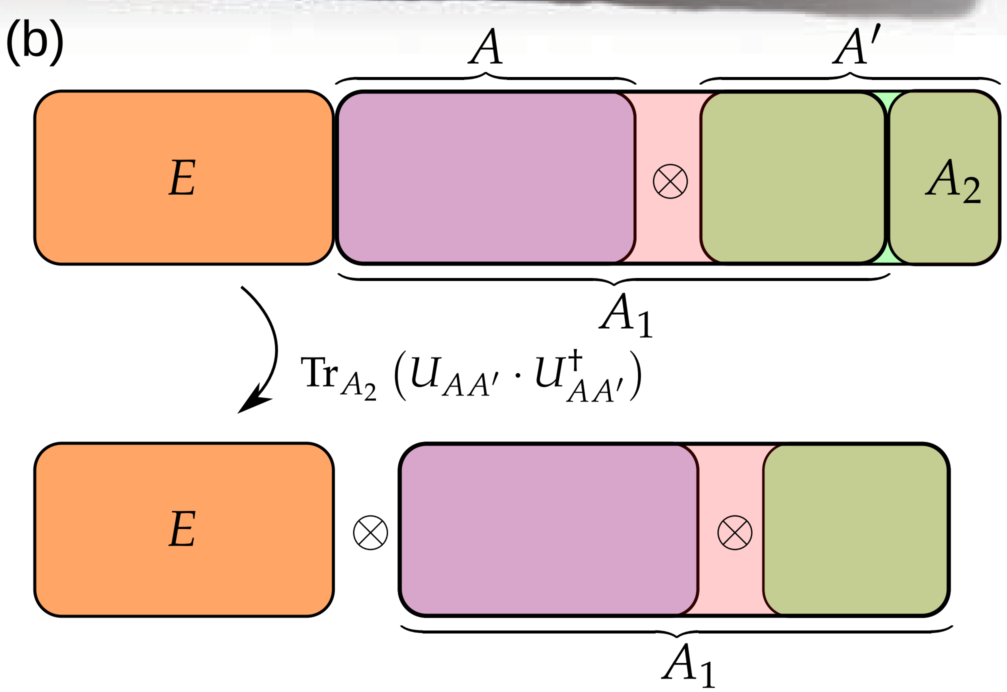
[3]*Faculty of Informatics, Masaryk University, Brno, Czech Republic*
[4]*Institute for Theoretical Physics, ETH Zurich, 8093 Zürich, Switzerland*
(Received 24 May 2016; published 23 February 2017)

Theorem 1: (Catalytic decoupling) For any bipartite quantum state ϱ_{AE} and $0 < \delta \leq \varepsilon \leq 1$, we have:

$$R_c^\varepsilon(A;E)_\varrho \lesssim \frac{1}{2} I_{\max}^{\varepsilon-\delta}(E;A)_\varrho, \tag{11}$$

where $\lesssim$ stands for smaller or equal up to terms $\mathcal{O}(\log\log|A| + \log(1/\delta))$. We also have the converse

$$R_c^\varepsilon(A;E)_\varrho \geq \frac{1}{2} I_{\max}^{\varepsilon}(E{:}A)_\varrho. \tag{12}$$



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Interesting, for example, because standard entropies have **dual spacetime interpretations**:

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S. Ryu and T. Takayanagi, *Holographic Derivation of Entanglement Entropy from AdS/CFT*, Phys. Rev. Lett. **96**, 181602 (2006).

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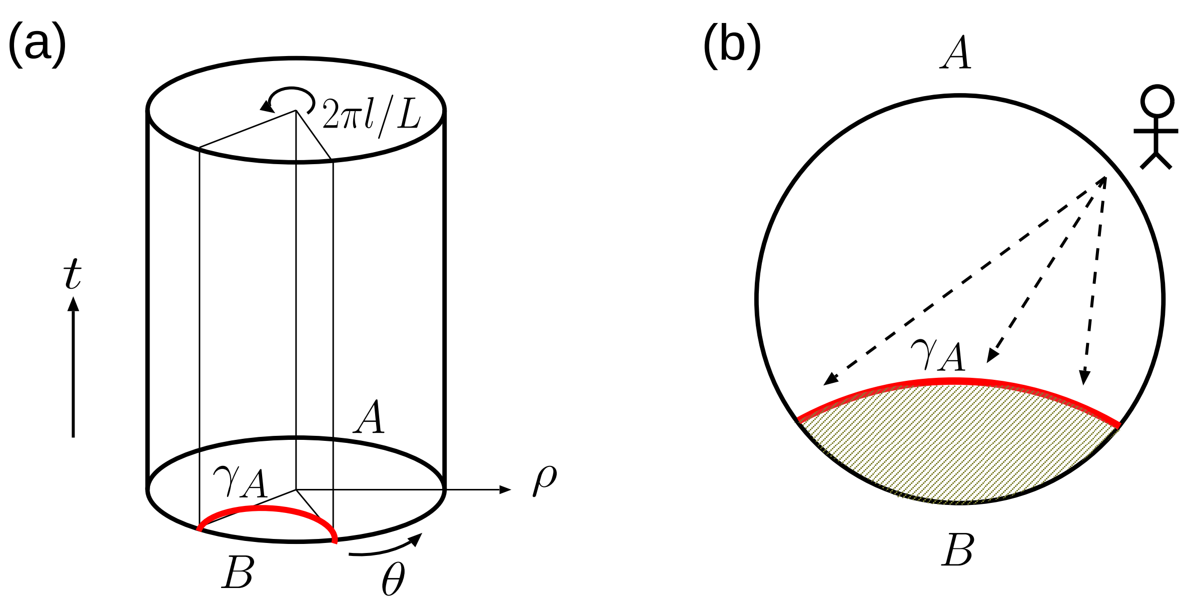

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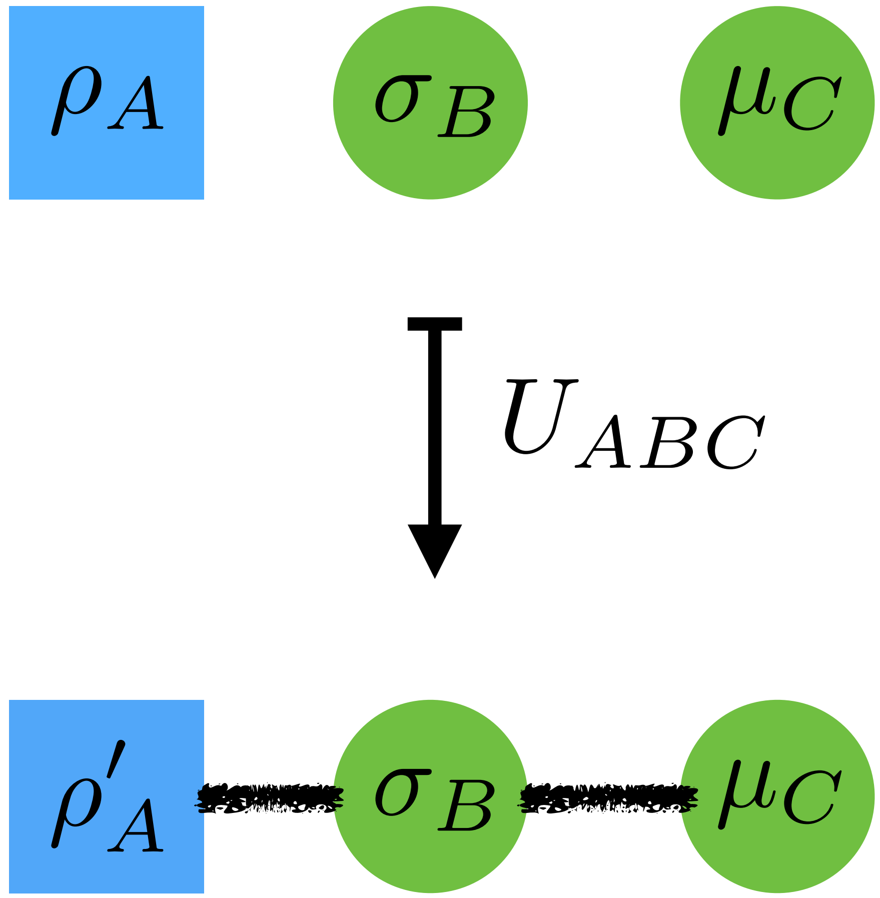

Implications for quantum information

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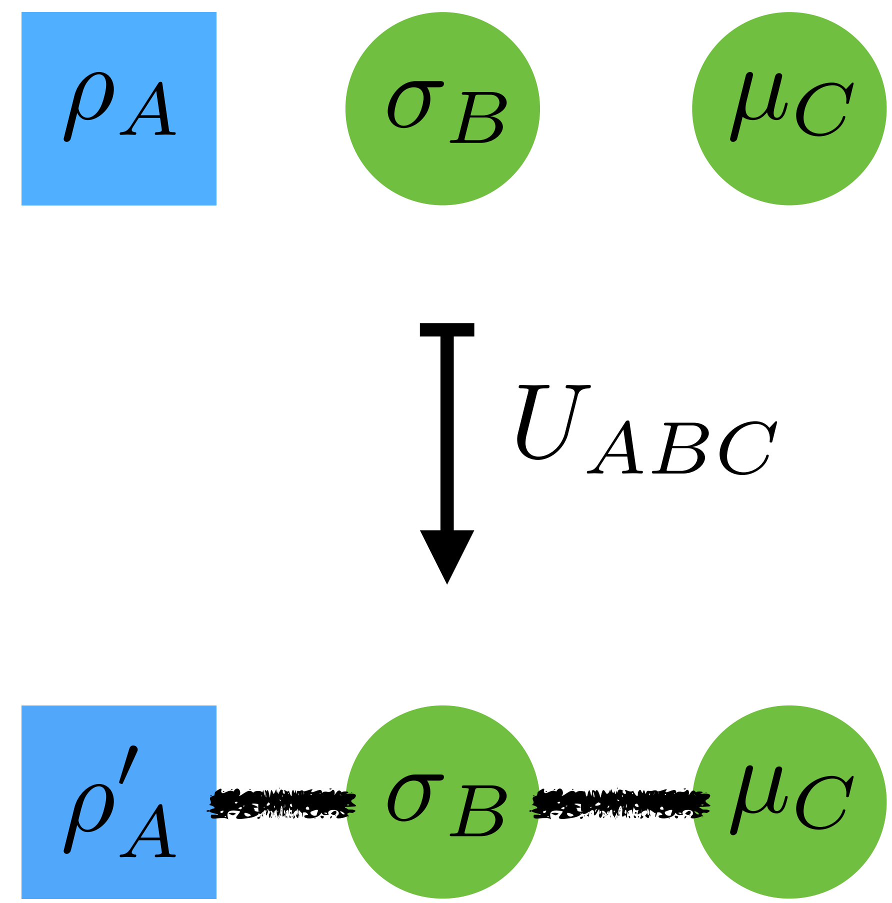
But here's an example, following from the above:



$$S(\rho_A)+S(\sigma_B)+S(\mu_C)=S(\rho'_{ABC})\leq S(\rho'_A)+S(\sigma_B)+S(\mu_C).$$

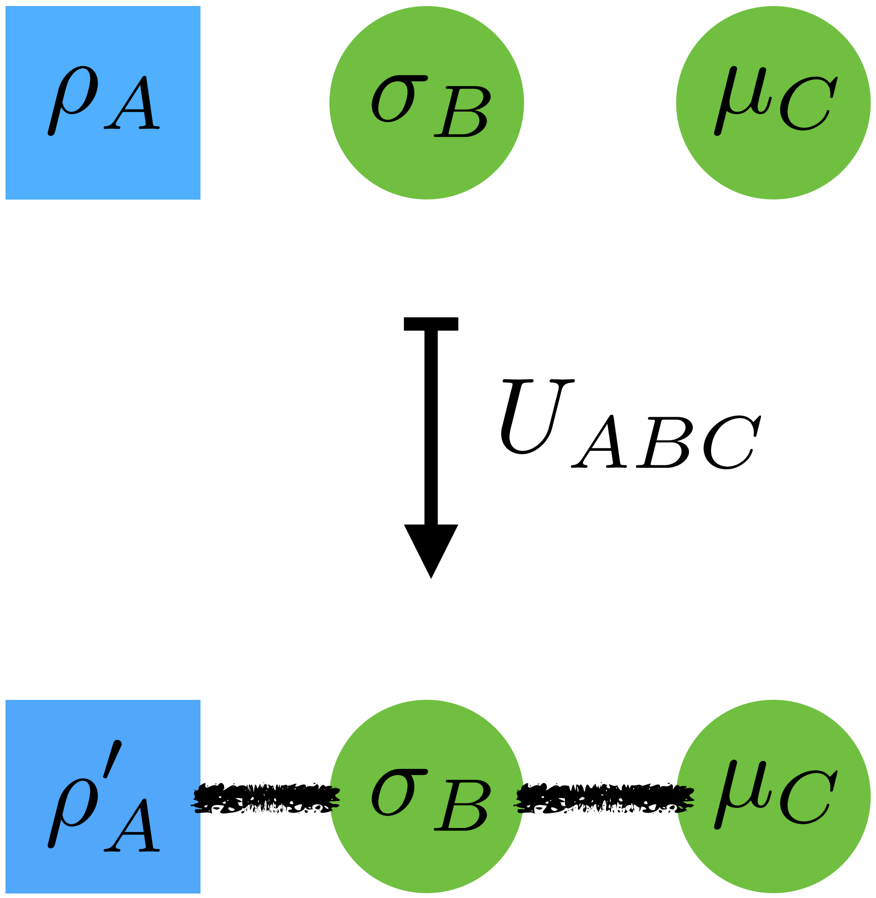
Implications for quantum information

What's possible here? Don't know (yet).
But here's an example, following from the above:

Theorem 5. *Let ρ_A and ρ'_A be quantum states with full rank which are not unitarily equivalent, i.e. do not have the exact same set of eigenvalues. Then there exists a finite auxiliary system B, a quantum state σ_B, and a copy C of AB with maximally mixed state μ_C as well a unitary U_{ABC} such that*

$$U_{ABC}(\rho_A \otimes \sigma_B \otimes \mu_C)U^\dagger_{ABC} = \rho'_{ABC}$$

with marginals ρ'_A on A, $\rho'_B = \sigma_B$ and $\rho'_C = \mu_C$ if and only if $\underline{S(\rho_A) < S(\rho'_A)}$ for the von Neumann entropy S.



$$\underline{S(\rho_A)} + S(\sigma_B) + S(\mu_C) = S(\rho'_{ABC}) \leq \underline{S(\rho'_A)} + S(\sigma_B) + S(\mu_C).$$

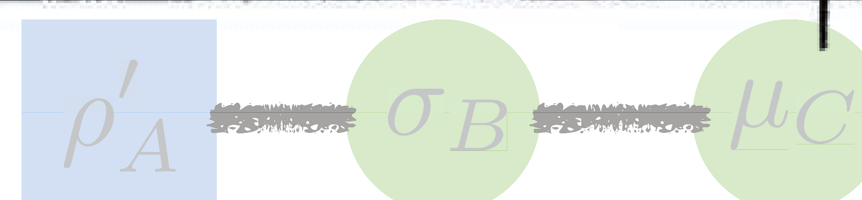
Implications for quantum information

What's possible here? Don't know (yet).
But here's an example, following from the above:

Open Questions. Can we do without the C system? Or recycle BC? And do the same if A is correlated with some other system (decoupling)?

Relation to versions of the quantum marginal problem.

with marginals ρ'_A on A, $\rho'_B = \sigma_B$ and $\rho'_C = \mu_C$ if and only if $S(\rho_A) < S(\rho'_A)$ for the von Neumann entropy S.



$$S(\rho_A) + S(\sigma_B) + S(\mu_C) = S(\rho'_{ABC}) \leq S(\rho'_A) + S(\sigma_B) + S(\mu_C).$$

Conclusions

- **Majorization**: some new results
  In particular $\quad p_X \otimes p'_Y \succ p'_{XY} \Leftrightarrow H(X) \leq H(X')$

MM, **arXiv:1707.03451** (+refs)
Further with M. Lostaglio, M. Pastena, J. Scharlau, see http://mpmueller.net

- **Quantum thermodynamics**: standard 2nd law; natural one-shot interpretation of free energy *F*
- **Quantum info**: one-shot int. of standard entropies (?)



Thank you!