LOCAL DECODERS AND THRESHOLDS OF TOPOLOGICAL QUANTUM CODES

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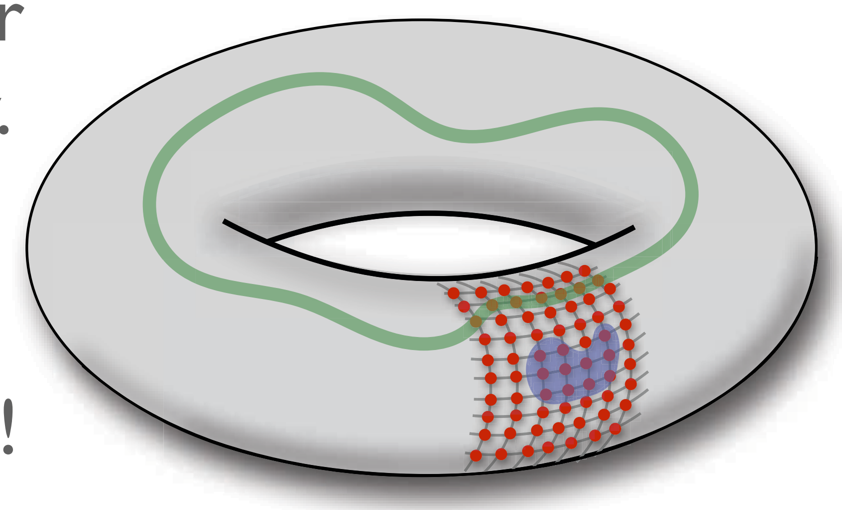
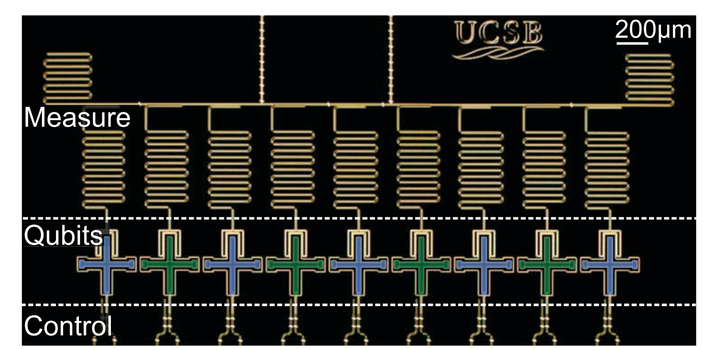
TOPOLOGICAL STABILIZER CODES

- Qubits on a manifold, (geometrically) local stabilizer generators, logical information encoded non-locally.
- Well-known models: toric and color codes.
- Can be built in the lab: 2D and local measurements!
- Desired properties:
  - fault-tolerant logical gates,
  - efficient decoders,
  - high threshold.





Kelly et al., Nature 519, (2015)

- **Decoder:** algorithm finding correction from stabilizer measurements.
- **Threshold** p_{th} = max error rate the code & decoder can tolerate.

OUTLINE

This talk: **local** (in space/time) decoders w/ **provable** thresholds.

Many toric/color code decoders: non-local, local but heuristic (*Harrington, Dennis, Fowler, Breuckmann, Herold, Duclos-Cianci, Haah, Hastings, Brown,…*).

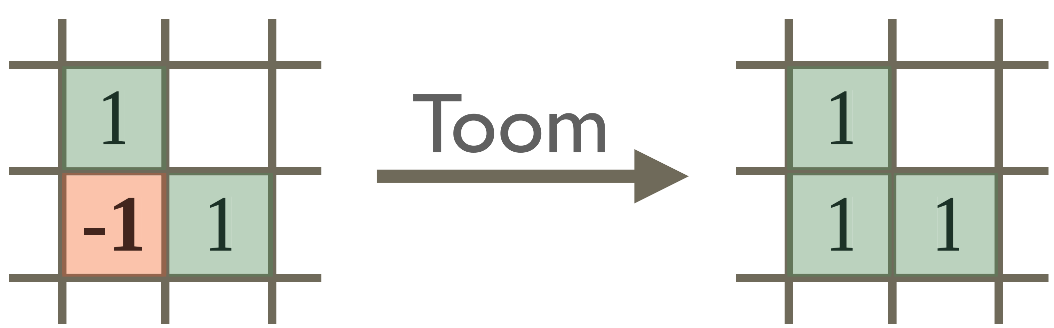
1. Generalization of Toom's rule to any lattice.
2. Local TC decoder w/ non-zero threshold.
3. Reduction of CC decoding to TC decoding.
4. 3D CC thresholds via stat-mech mappings (arXiv: **1708.07131**)

NEED FOR (LOCAL) ERROR CORRECTION

- Errors can accumulate! To prevent that — diagnose and correct errors.
- **Example:** classical memory protecting one bit ±1
  - repetition code,
  - decoder — majority vote.
- Noise flips some bits. Collecting (global) information takes time — new errors can appear!
- **Goal:** suppress/remove errors by local operations.

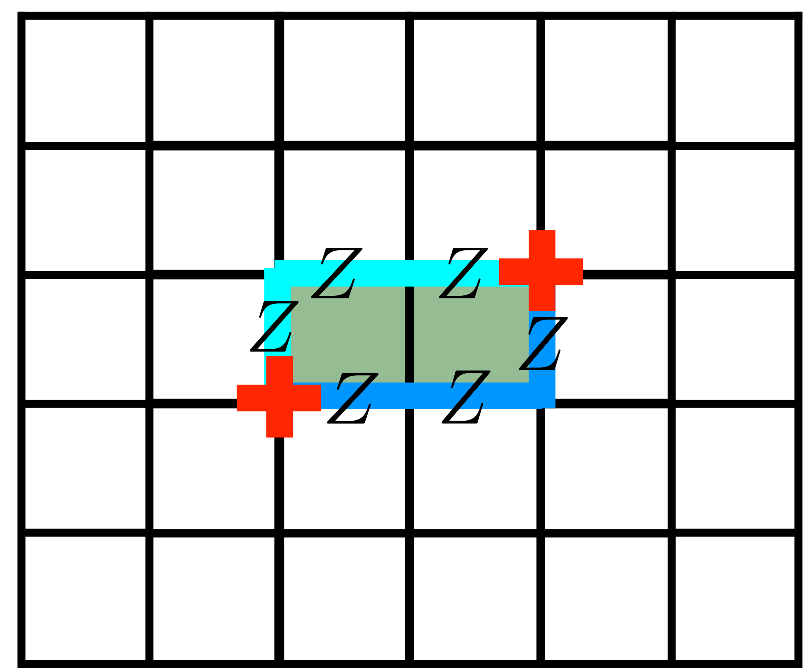
1	1	1	1	1
1	-1	-1	1	1
1	1	-1	1	1
1	1	1	1	1
-1	1	1	1	-1

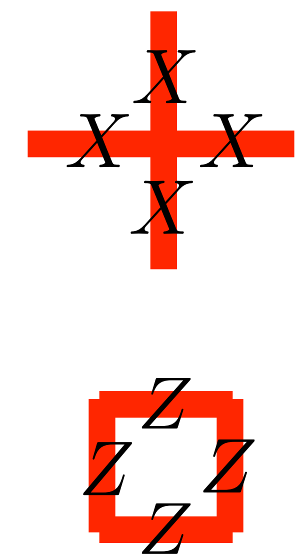
- **Toom's rule:** flip bit (face) if it differs from both N and E neighbors.



DECODING PROBLEM

- Unlike classical bits, quantum information can't be accessed directly.
- **Stabilizer (CSS) codes:** measure X/Z-stabilizers and correct Z/X-errors separately. We consider ideal measurements.
- **Decoding:** find position of errors from violated stabilizers (excitations).
- **2D toric code** (*Kitaev*):
  - qubits = edges,
  - stabilizers = Z-faces & X-vertices,
  - Z-errors = edges,
  - excitations = vertices.

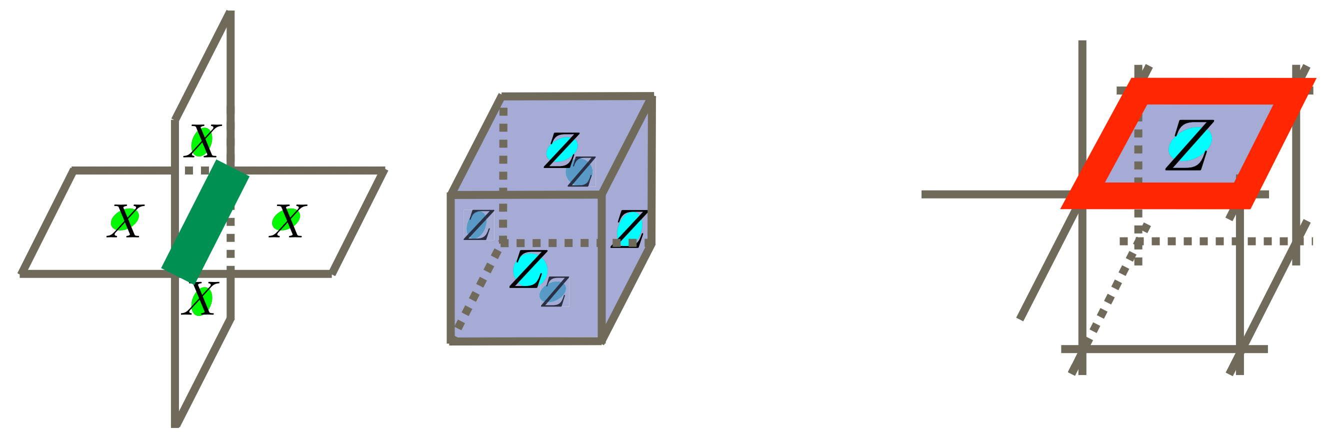




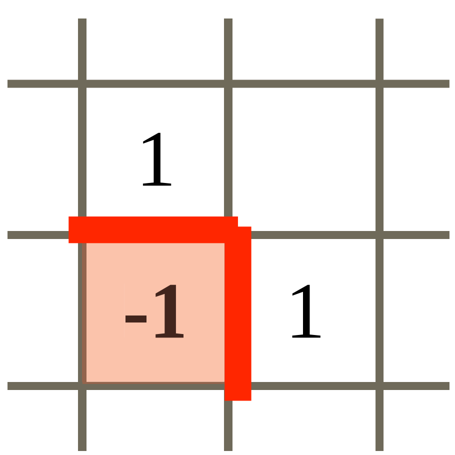
- Decoding successful if error and correction differ by stabilizer.

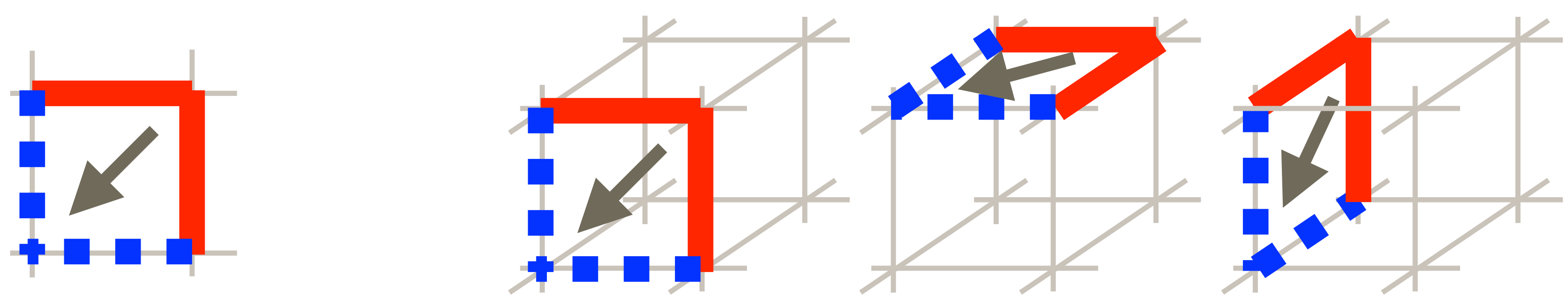
TOOM'S RULE AS DECODER

- Toric/color code in d dim w/ (k-1)- & (d-k-1)-dim excitations, k=1,…,d-1.
- **3D toric code:**
  - qubits = faces,
  - stabilizers = X-edges, Z-cubes.



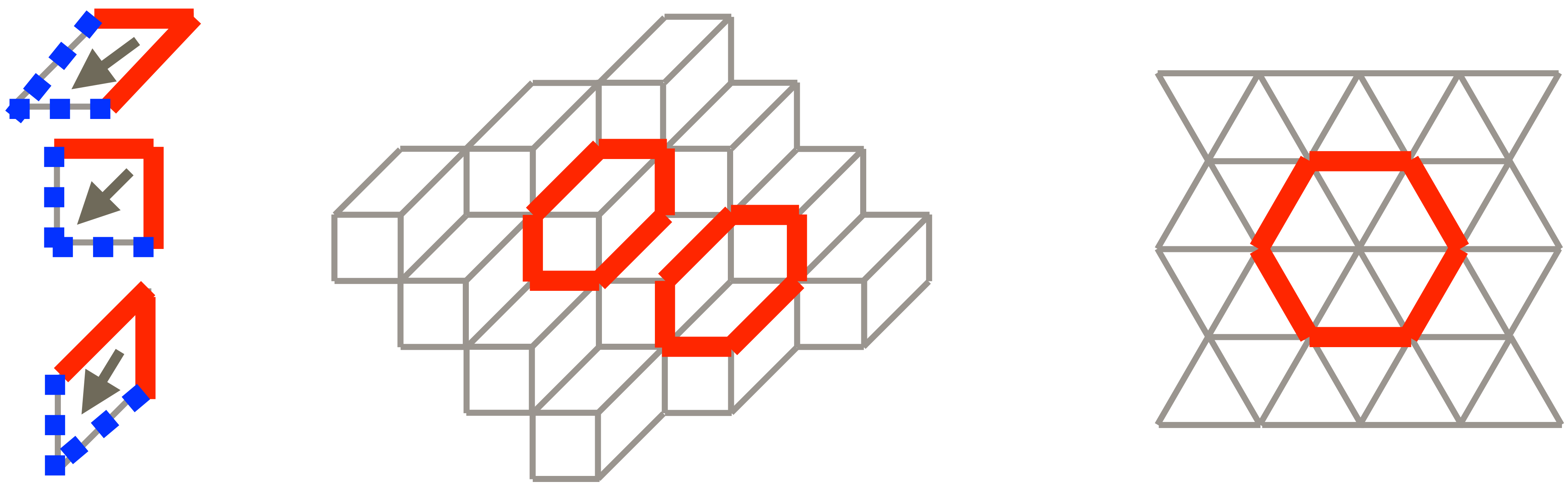
- Decode Z errors = flip faces w/ boundary matching loop-like excitations.
- Toom's rule — a rule for (re)moving domain walls, i.e. "move NE corners".





PROBLEMS W/ GENERALIZATION

- Is there a rule à la Toom (to move domain walls) on any lattices?
- Not obvious how to generalize beyond the square/cubic lattices. Simple rules fail, e.g. "move NE corners".



- We want a deterministic rule — simpler to analyze!
- We focus on triangulated lattices.

LOCAL EFFICIENT DECODERS: TORIC AND COLOR CODES

- **Questions:**
  - Toom's rule on any lattice?
  - does decoding w/ Toom's rule work?

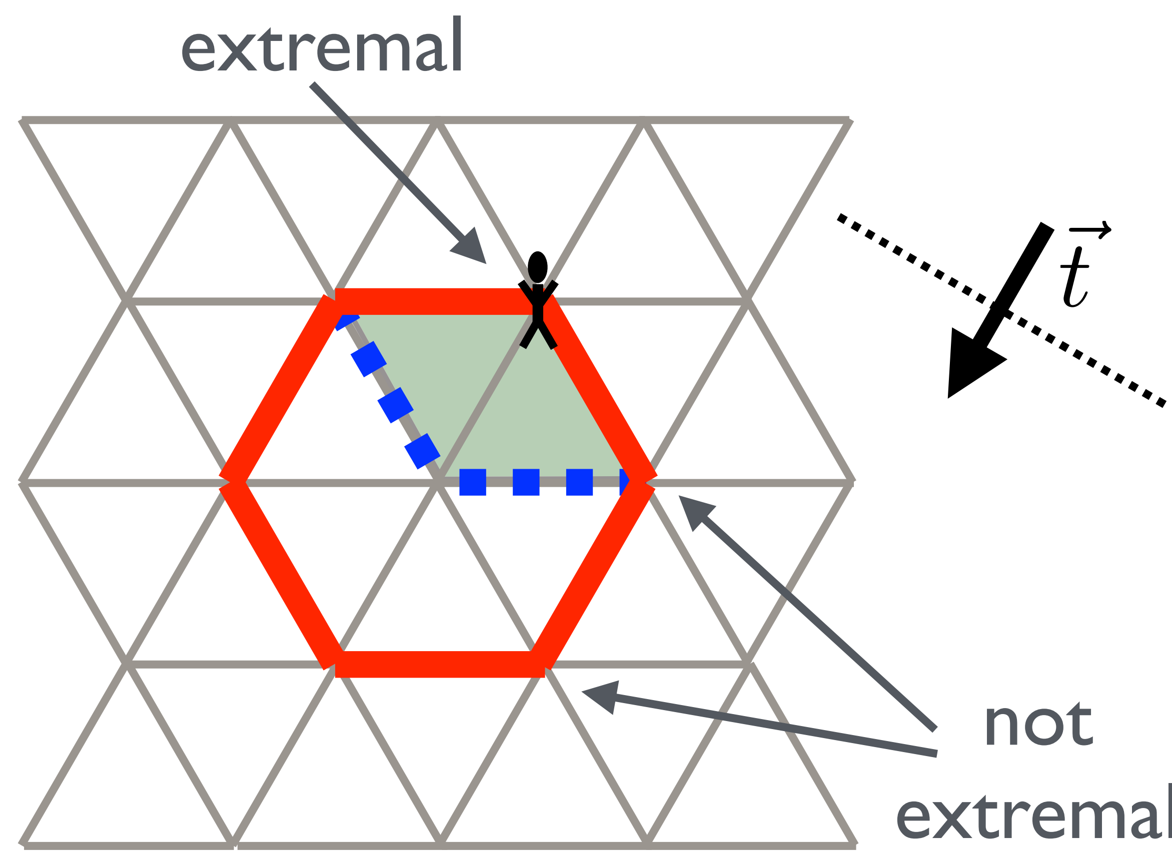
- **Sweep Rule** — a generalization of Toom's rule to any d-dim lattice and k-dim domain walls for k=1,…,d-1.

- **Threshold** for local toric code decoders based on the Sweep Rule.

- **Local color code decoders** in d$\geq$3 dim by using any toric code decoder.

SWEEP RULE

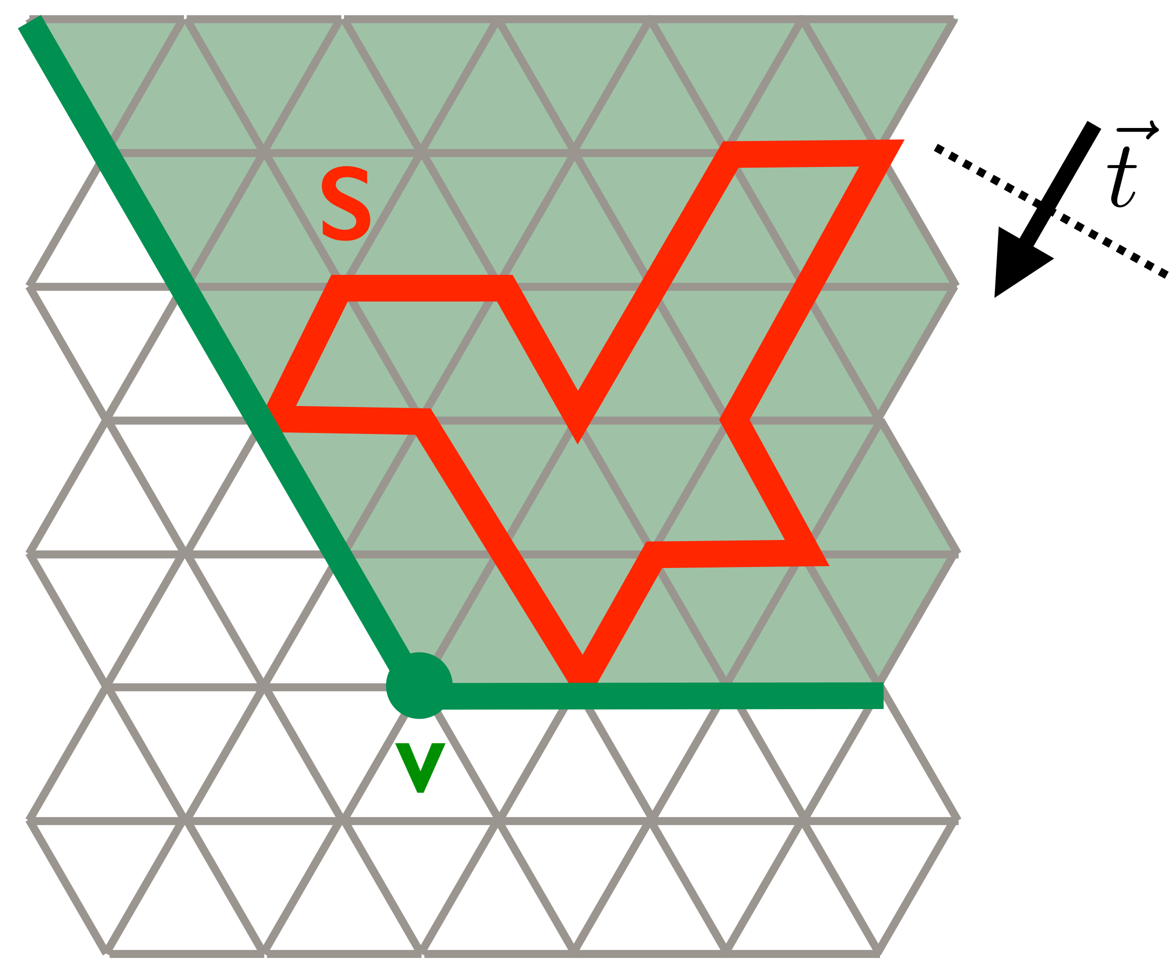
- Change of perspective:
  not faces but vertices!
- Introduce the sweep direction.
- **Extremal vertex v:**
  local restriction of the domain wall
  is in the sweep direction from v.



- **Sweep Rule:** if vertex extremal, flip faces in the sweep direction.
- Sweep Rule in d≥2 dim defined similarly. Important to flip right cells!

PROPERTIES OF SWEEP RULE

- The sweep direction induces a partial order ≤ over the set of vertices. Alternative picture: vertices in spacetime, path & causal path.
- Two notions:
  - cone(v) = {vertices u | u≤v}
  - sup(S) = least lower bound of S
- **Correction region:**
  domain wall S stays within cone(sup(S)).
- **Monotone:**
  max length of the causal path between sup(S) and any vertex of domain wall S.

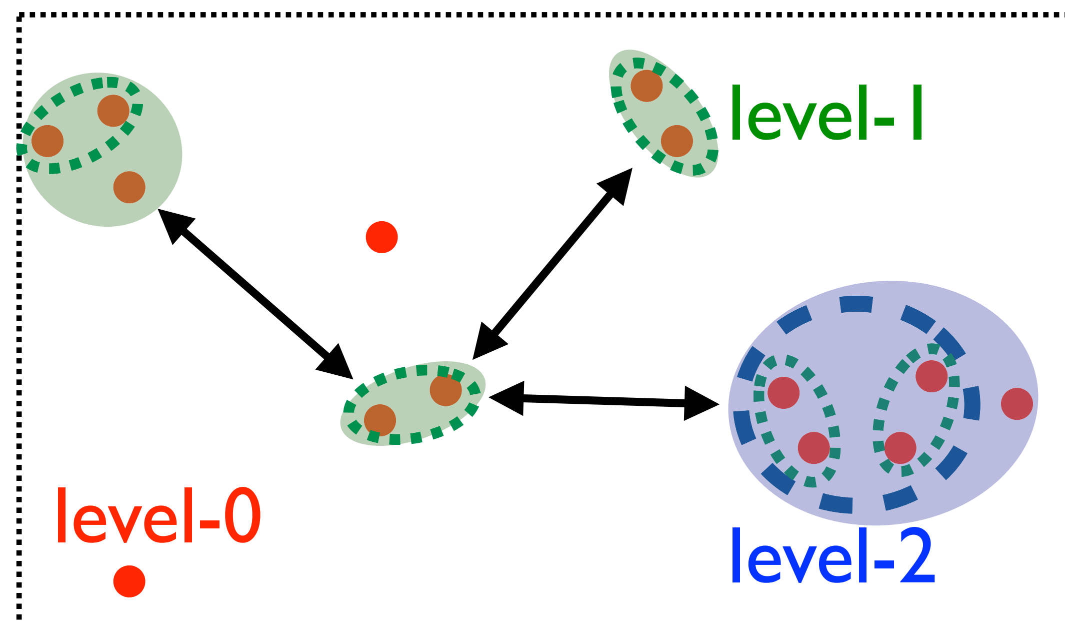


SWEEP DECODER

- Toric code in $d \geq 3$ dim w/ k-dim excitations for $k=1,\ldots,d-2$.
- **Sweep Decoder**
  1. repeat M times: simultaneously apply the Sweep Rule for every vertex v,
  2. correction = flipped faces.
- Decoder can fail because:
  (a) domain walls have not been removed in M time steps,
  (b) correction introduced logical error.
- <u>**Our result:**</u> Sweep Decoder has **non-zero threshold p_c**:
  if error rate $p \leq p_c$ then pr(success)$\longrightarrow 1$ in the limit of lattice size $L \longrightarrow \infty$.

KEY LEMMAS

- We use ideas of Gacs, Harrington, Bravyi&Haah.
- Level-0 chunk = single error, level-1 chunk = nearby pair of errors, …, level-n chunk = two disjoint level-(n-1) chunks & diameter $\leq Q^n/2$.
- **Lemma 1:** for sufficiently small p the probability of having a level-n chunk is suppressed doubly exponentially in n.



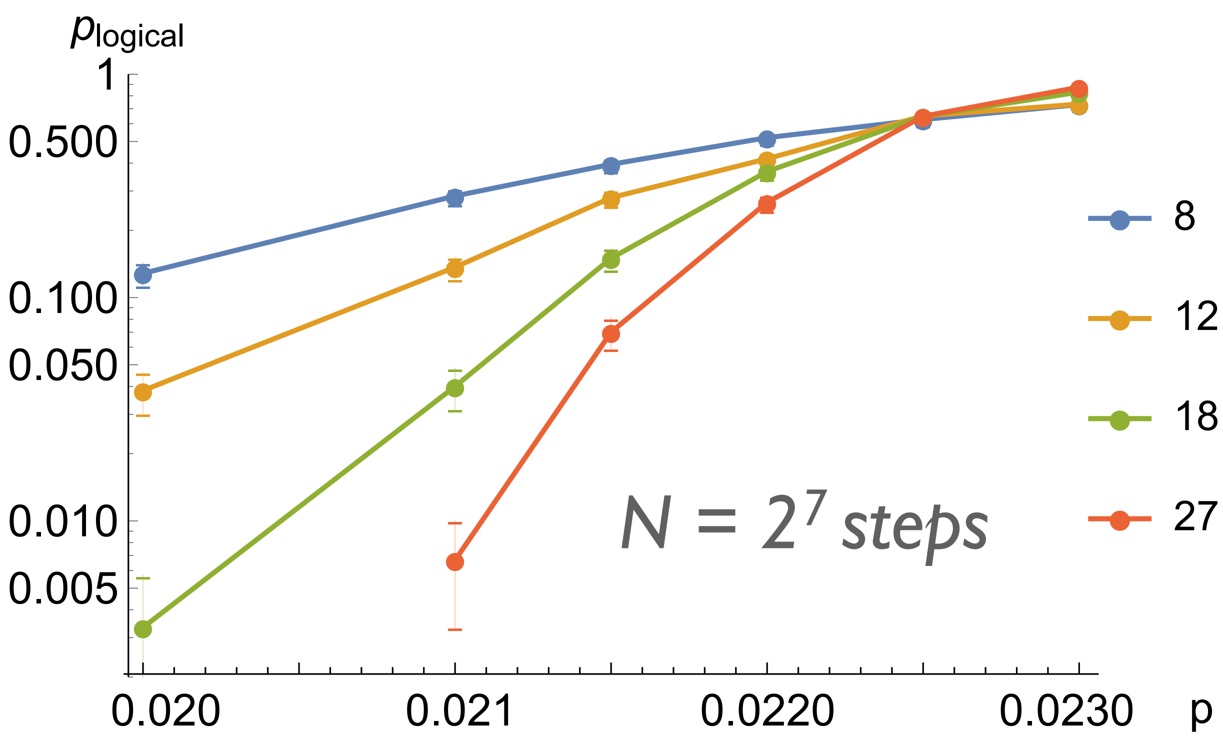
- Level-n error E_n = union of level-n chunks.
- Disjoint decomposition of errors: $E = (E_0\text{-}E_1) + (E_1\text{-}E_2) + \ldots + (E_{m-1}\text{-}E_m) + E_m$.
- **Lemma 2:** if C is a level-i cluster of errors in $(E_i\text{-}E_{i+1})$, then C is "**not too big**" ($diam(C) \leq Q^i$) and "**far from other errors**" ($d(M, (E_i\text{-}M)) \geq Q^{i+1}/3$).

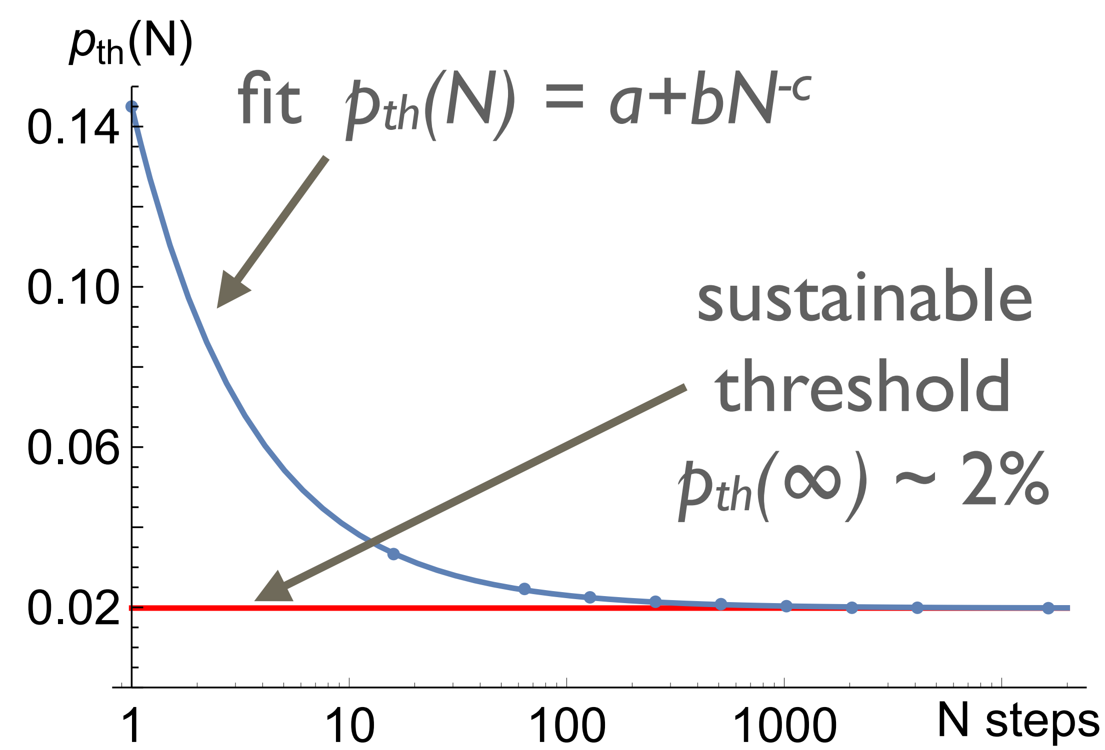
PUTTING THINGS TOGETHER

- Assumptions on the lattice: (locally) Euclidean, …
- Isolated error removed in 1 step, level-1 cluster in ~ Q steps, …, level-i cluster C in time ~ Q^i (use: C "not too big" & monotone).
- Removal of level-i cluster C unaffected by other level-j clusters for all $j \geq i$ (use: C "far from other errors" & correction region).
- Correction of C inside the cone of its boundary, which for low-level clusters ($i \leq \log L$) is a correctible region — no logical error!
- Run local updates for total time ~ Q^i, where i ~ log L. Higher-level clusters might not be removed, but they are very unlikely!
- Failure of the decoder due to presence of high-level clusters: pr(fail) $\leq$ poly(L) exp(-cL) $\longrightarrow$ 0 as L $\longrightarrow \infty$.

NOISY MEASUREMENTS: 3D TORIC CODE NUMERICS

- Realistic setting — noisy measurements w/ prob = p.
- **Iterate N times:** add new errors, imperfectly measure stabilizers, apply one round of local correction everywhere.
- **After N iterations:** measure perfectly, decode, check for logical errors.
- We can find threshold $p_{th}(N)$ and analyze its behavior in the limit N—›∞.





OUTLINE

1. Generalization of Toom's rule.

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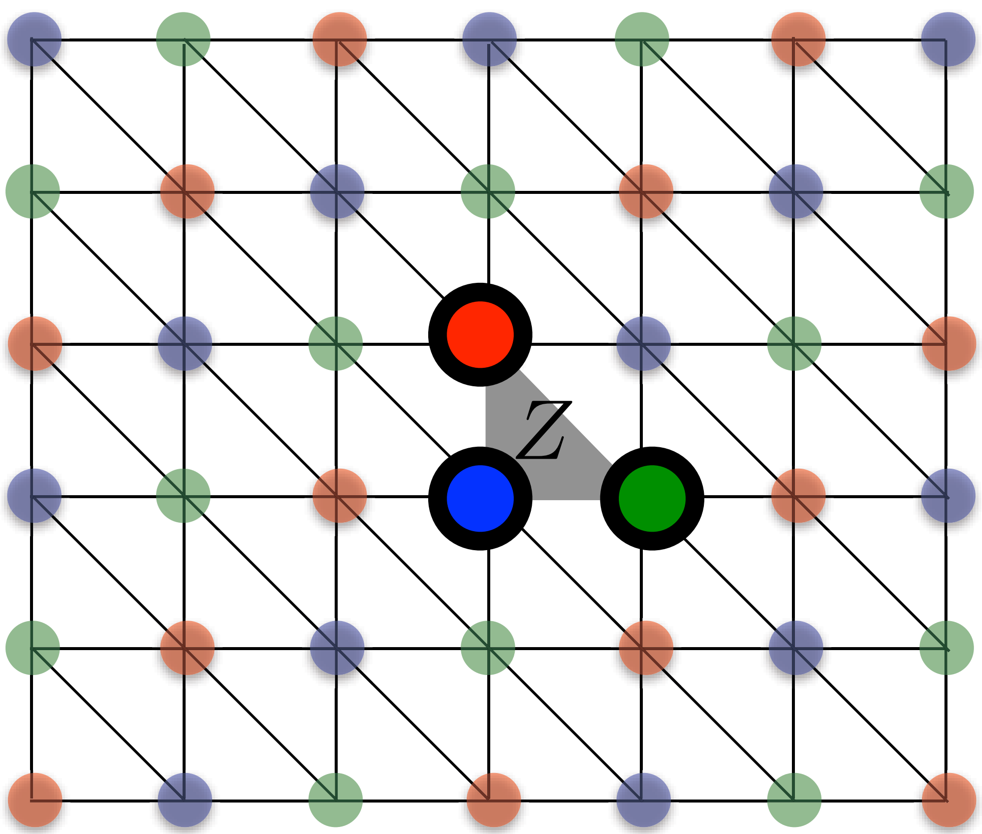
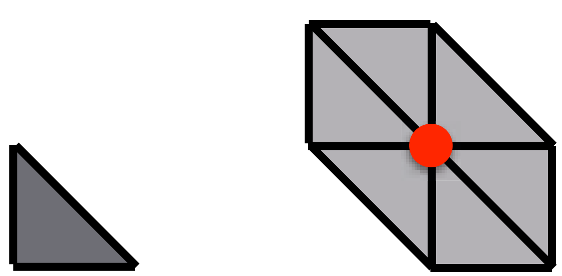
TOPOLOGICAL CODE: 2D COLOR CODE

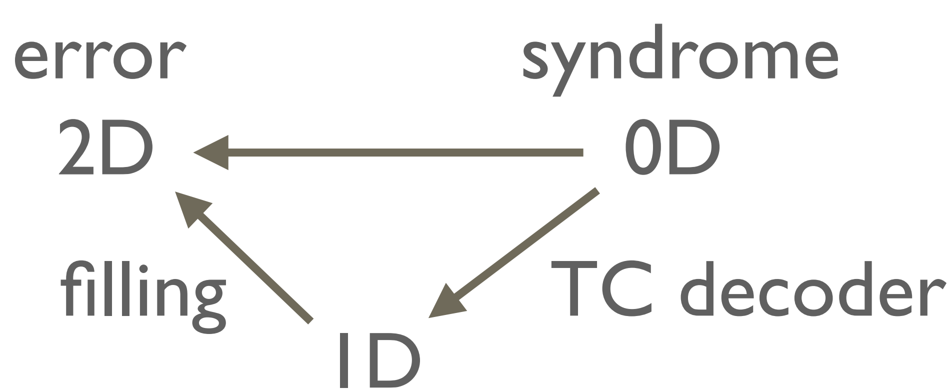


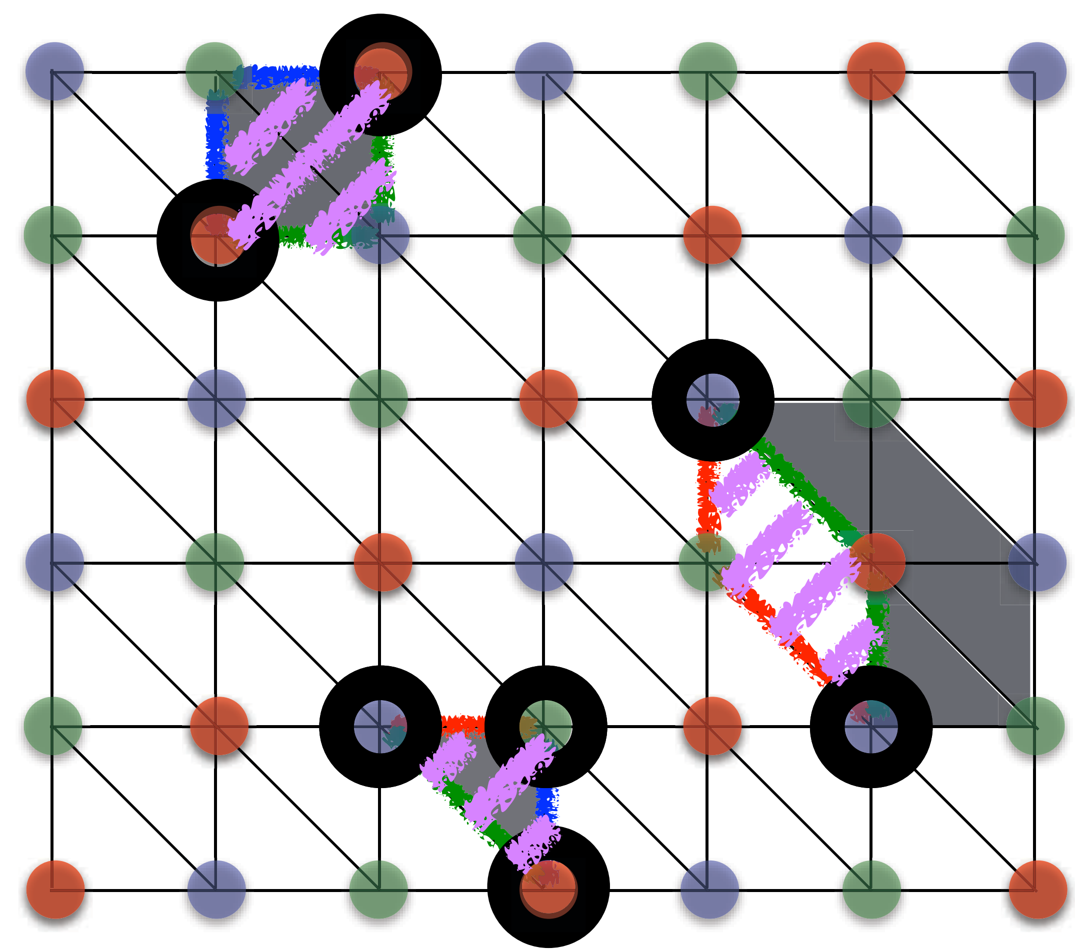


- **(Dual) lattice:** made of triangles and vertices are 3-colorable.
- **2D color code** (*Bombin*):
  - qubits = triangles,
  - stabilizers = X- & Z-vertices.
- Logical Clifford gates are transversal, code switching and gauge fixing, …
- Decoding seems to be more challenging: excitations created in triples!

HOW TO DECODE COLOR CODES?

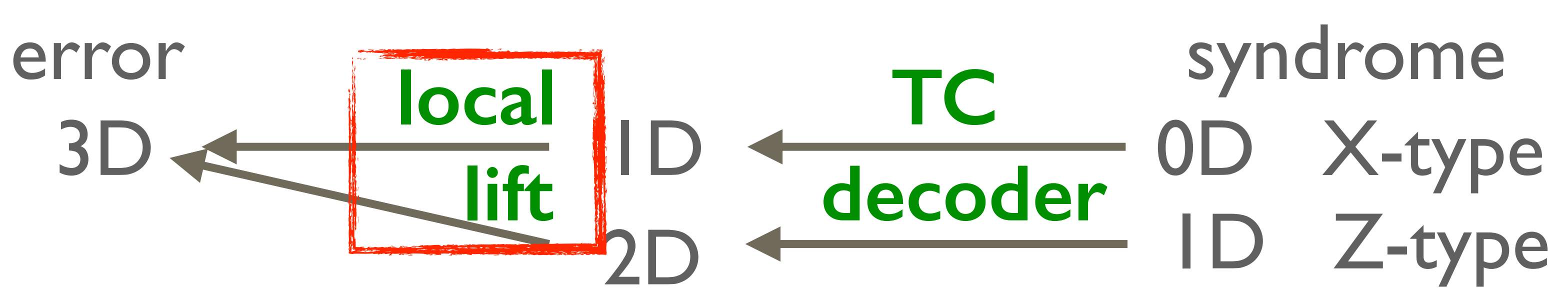
- **Idea:** color and toric codes are related (*Kubica et al.'15*) — can we use existing toric code decoders?
- Noise changes — correlated errors!
- 2D projection decoder (*Delfosse'14*)
  - TC decoder on three sublattices,
  - global filling.

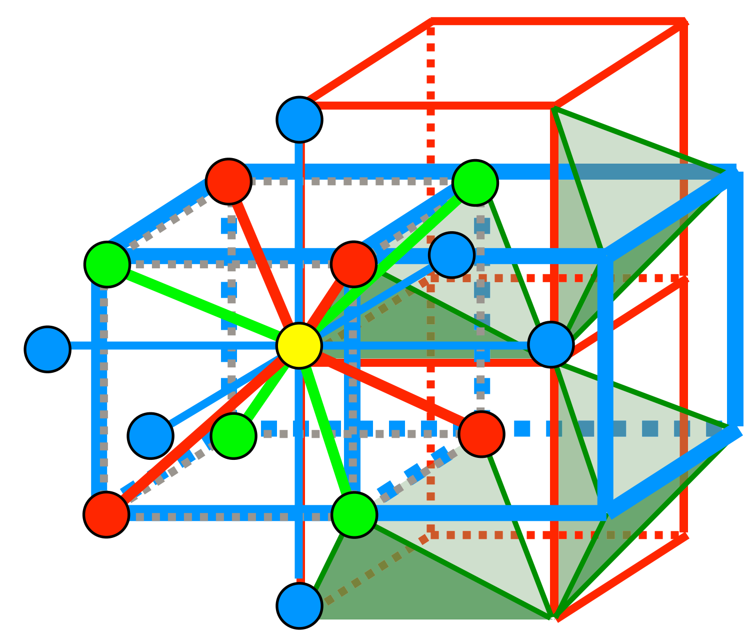




LOCAL COLOR CODE DECODERS IN D≥2 DIM

- Beyond 2D not really explored! Similar ideas work.
- <u>**Our result:**</u> decoder w/ **local** reduction and lifting in **d ≥ 2 dim**.
- **3D color code** (bcc lattice):
  - qubits = tetrahedra,
  - stabilizers = X-vertices and Z-edges.





- Any toric code decoder can be used! Fully local for loop-like excitations.
- Toric code thresholds allows to lower bound color code threshold!

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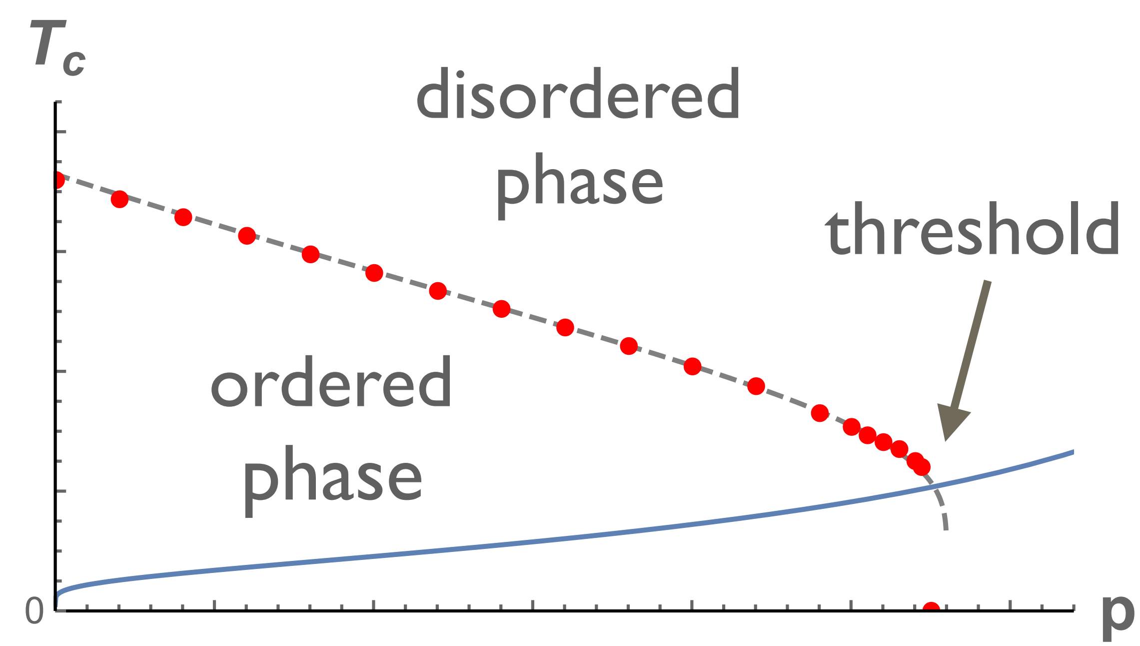
THRESHOLDS FROM STATISTICAL MECHANICS

- Analytic bounds on threshold (very) low and far from actual values.
- Values of thresholds relevant for: overhead estimates, comparing codes and decoders, experiment, …
- *Dennis et al.'02:* connection between toric code decoding and a classical spin model (random-bond Ising).

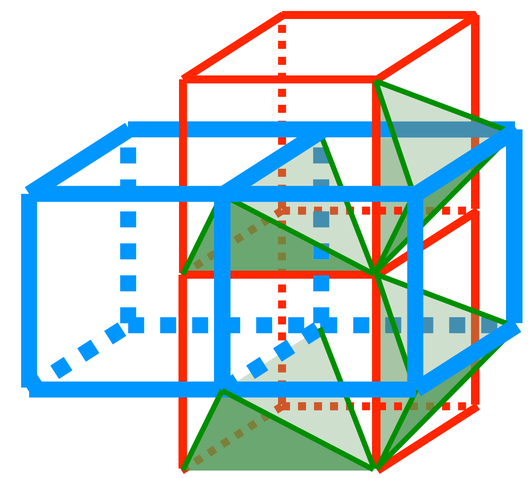


- Ordered phase = successful correction.
- **Our results:** <u>new spin models relevant for 3D color code & their phase diagrams, thresholds of 3D color code.</u>

arXiv: **1708.07131**

RANDOM COUPLING ISING MODEL AND 3D COLOR CODE

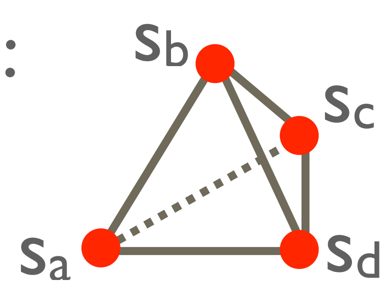
- 3D bcc lattice:
  - qubits = tetrahedra,
  - stabilizers = X-vertices (A) and Z-edges (B).



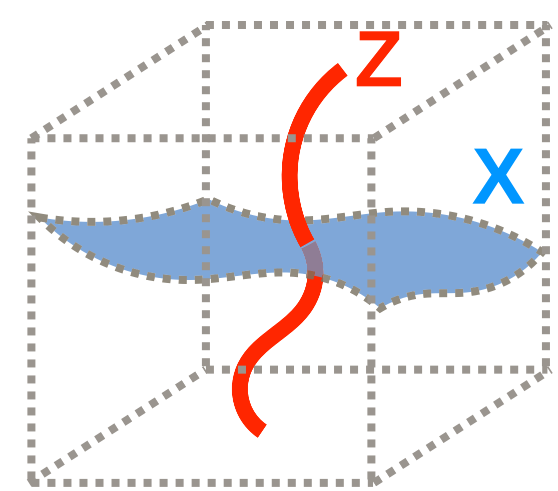
- Z/X-errors lead to 0D point-/ 1D loop-like excitations. Logical Z/X operators are 1D string-/ 2D sheet-like.

- 4-body w/ spins on vertices (A):

$$H = -\sum \varkappa_{abcd} s_a s_b s_c s_d$$

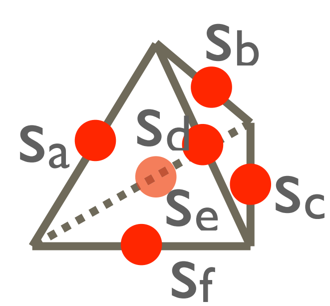


$$\varkappa_{abcd} = \pm 1$$



- 6-body w/ spins on edges (B):

$$H = -\sum \varkappa_{abcdef} s_a s_b s_c s_d s_e s_f$$

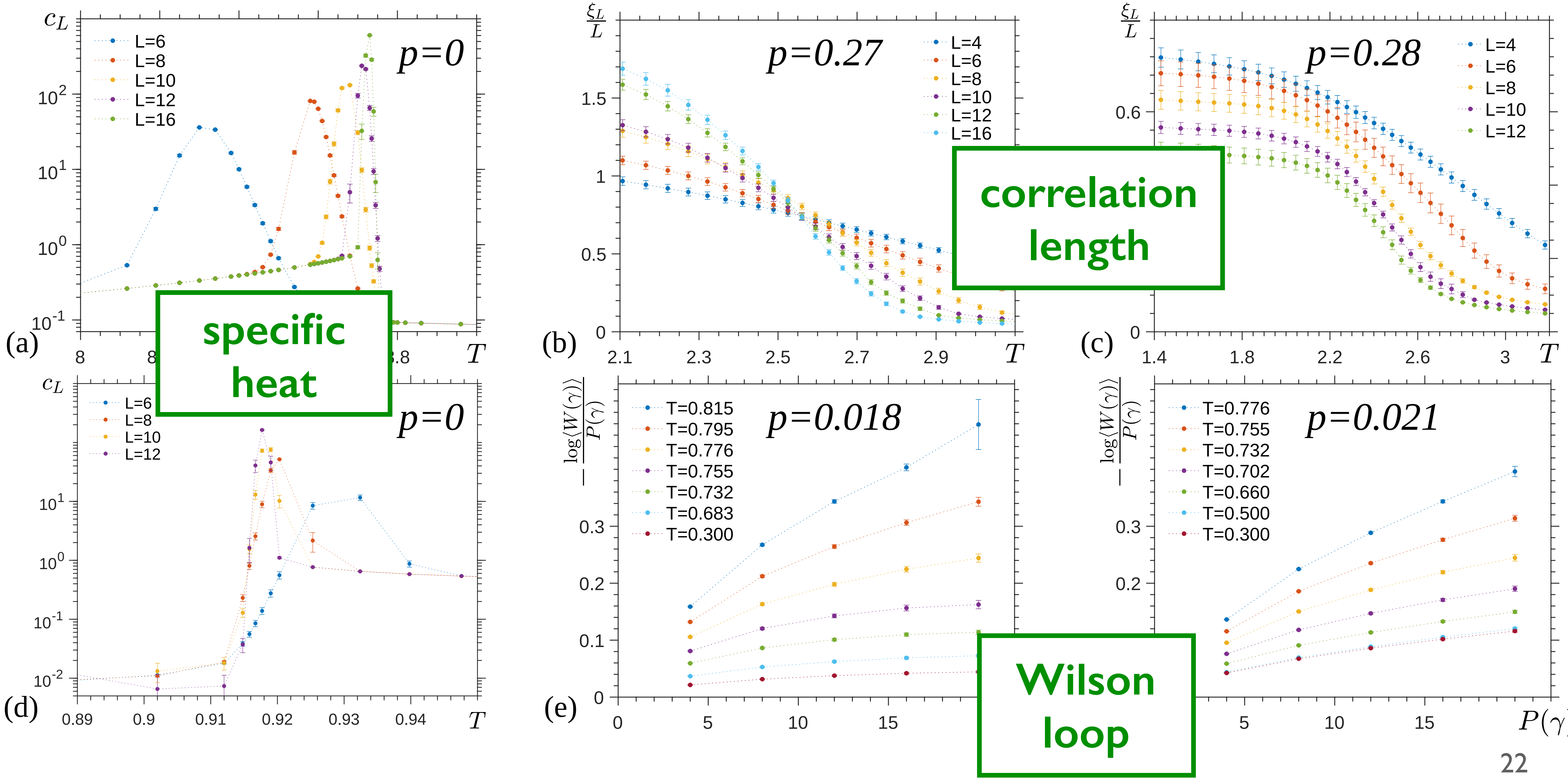


$$\varkappa_{abcdef} = \pm 1$$

- For p=0 models are dual (low- and high-T expansions match).

arXiv: **1708.07131**

NUMERICS



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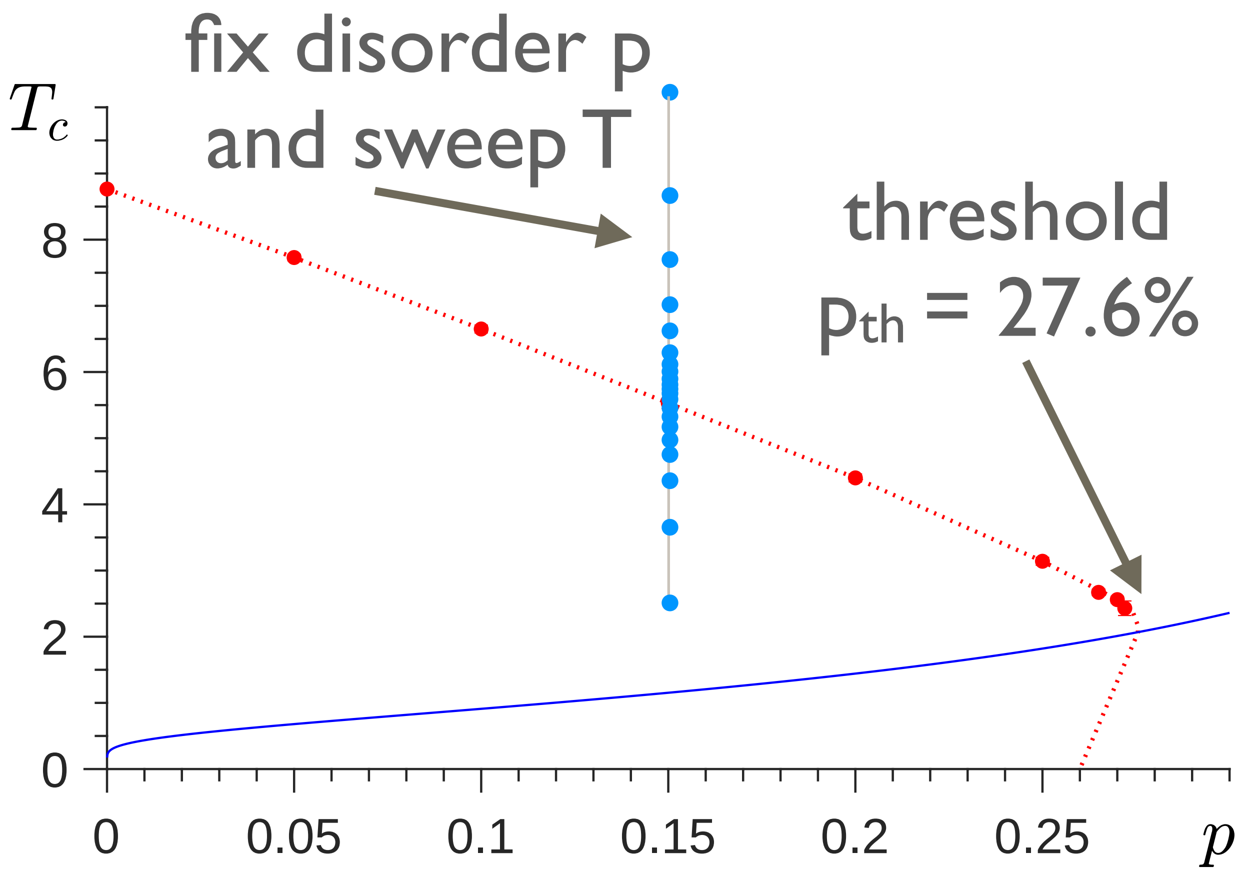
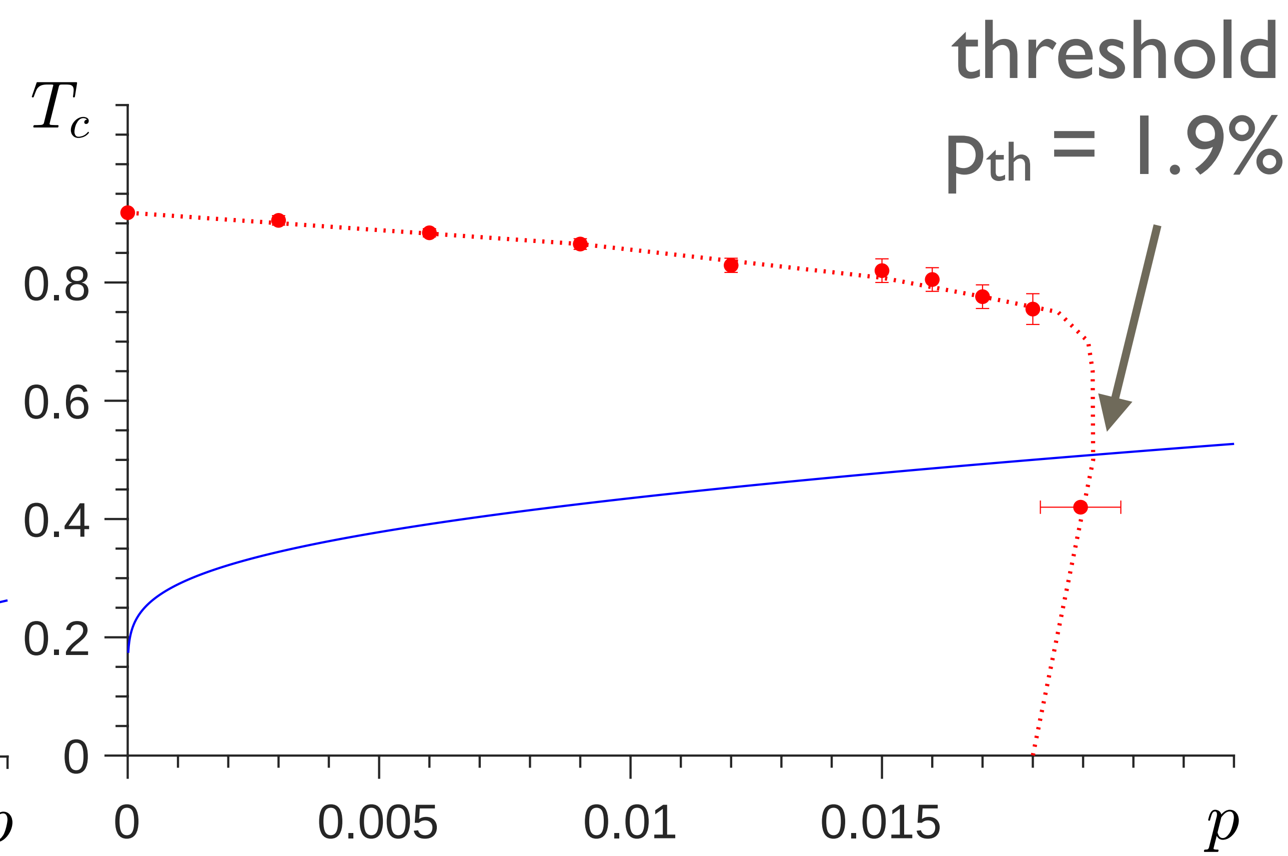
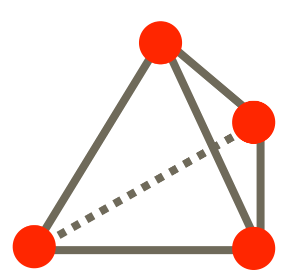
3D COLOR CODE THRESHOLDS FROM PHASE DIAGRAMS

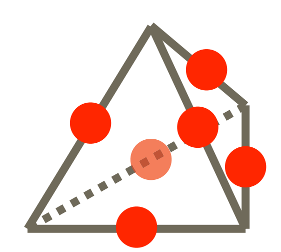




- 4-body RCIM
- threshold for 2D sheet-like logical X



- 6-body RCIM
- threshold for 1D string-like logical Z



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DISCUSSION

- **Our results:**
  - local decoders of toric and color codes w/ provable thresholds,
  - noisy measurements — 3D TC sustainable threshold $p_{TC}^{(2)} \approx 2\%$.
  - 3D color code optimal thresholds from stat-mech:
    $p^{(1)} \approx 1.9\%$ and $p^{(2)} \approx 27.6\%$.
- **3D gauge color code:** threshold $p^{(1)}$ for 1D string-like (*Brown et al.'16*).

THANK YOU FOR YOUR ATTENTION!