The Disjointness of Stabilizer Codes and Transversal Gates

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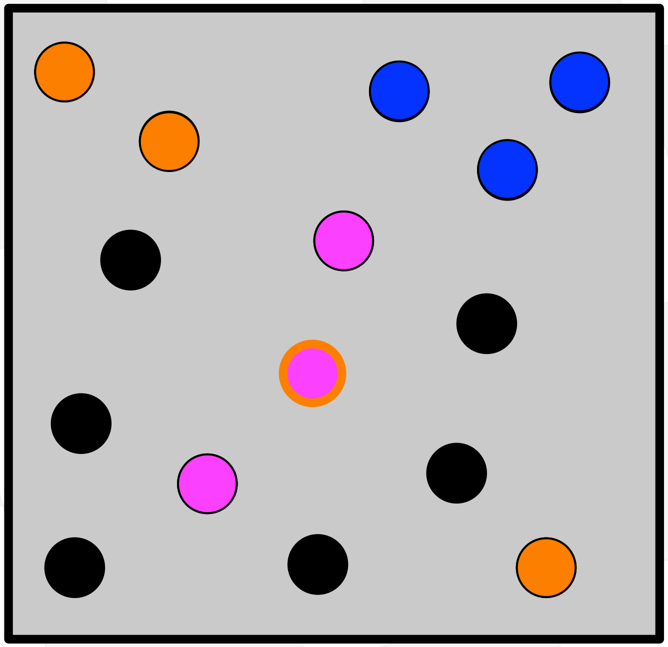
Stabilizer codes



$[[n,k,d]]$

n physical, k logical, distance d

$$S = \{s_1, s_2, s_3, s_4, s_5, \ldots\}$$

$$s_i \in \mathcal{P}_i, \; s_i s_j \in S, \; -I \notin S.$$

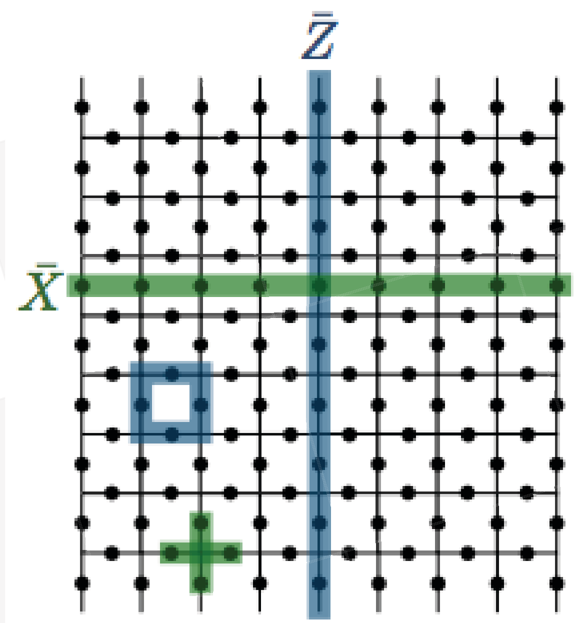
$$|S| = 2^{n-k}$$

$$N = \{\bar{X}_1, \bar{Y}_1, \bar{Z}_1, \bar{X}_2, \bar{Y}_2, \bar{Z}_2, \ldots\}$$

$$[\bar{A}_i, \bar{B}_j] = [A_i, B_j], \; [\bar{A}_i, s_j] = 0.$$

Group commutator: $[A,B] = ABA^\dagger B^\dagger$

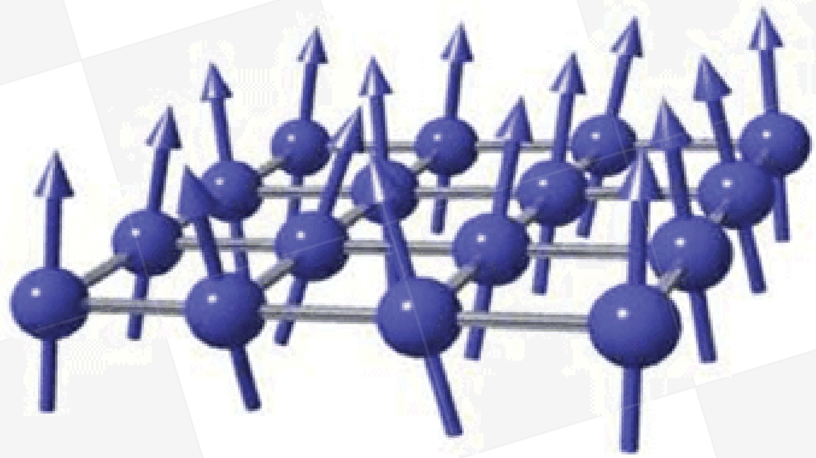
Single code vs. code family

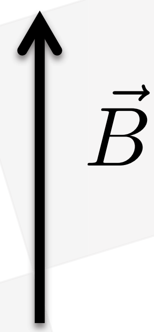


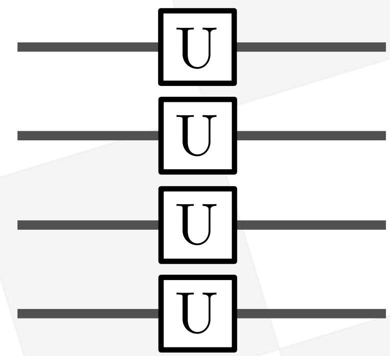
[Bravyi, Kitaev '98]

What is transversal?

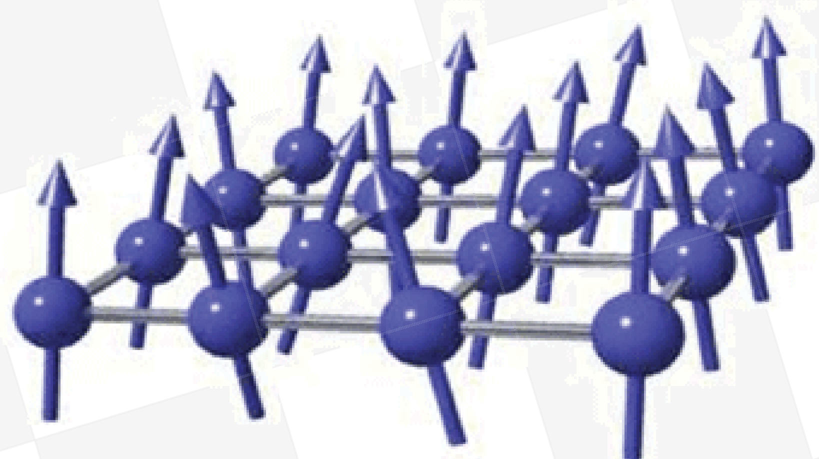
A uniform magnetic field:







A changing reference frame:

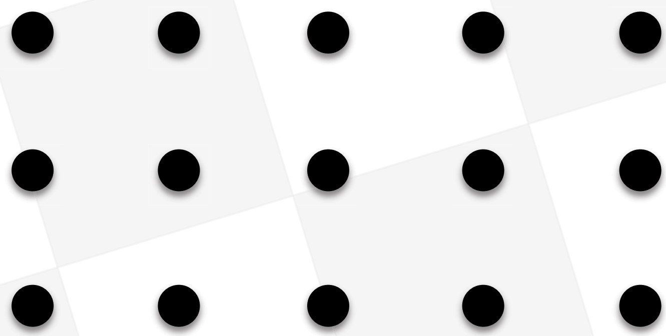


Hayden, Nezami, Popescu, Salton '17

What is transversal?

A

B



C

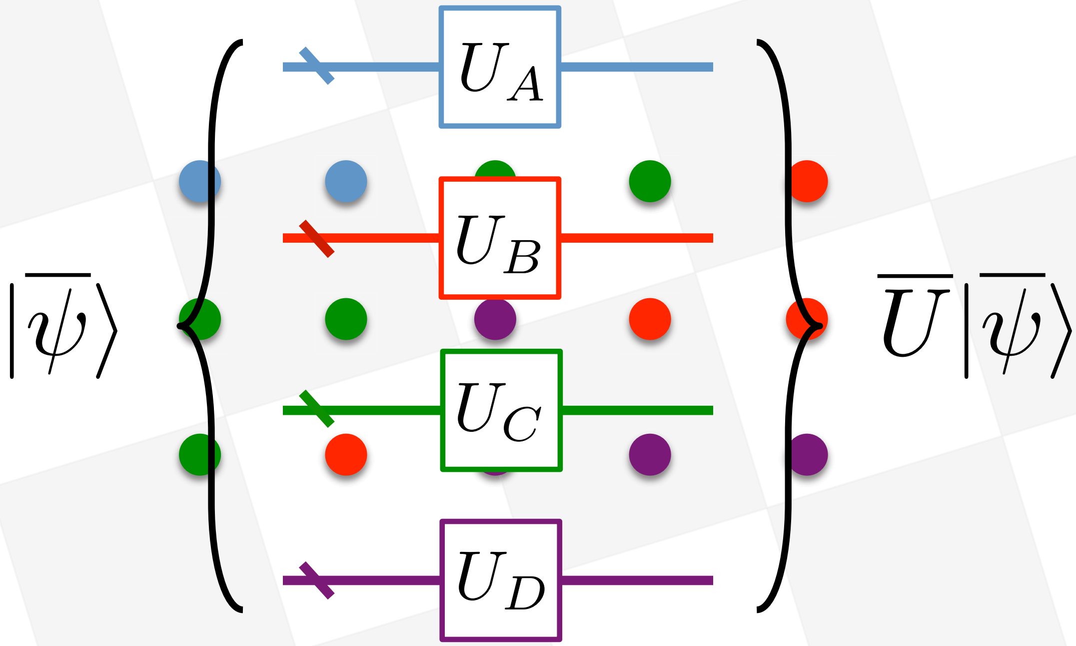
D

What is transversal?

$$Q_A \cup Q_B \cup Q_C \cup Q_D = \{1, 2, \ldots, n\}$$

A

B



C

D

Even $d-1$ untrustworthy parties cannot destroy the encoded qubits.

Can we establish limits on the possible transversal gates on a stabilizer code?

transversal → constant depth

code → code family

Prior work

✓ Zeng, Cross, Chuang 2008 – the group of transversal gates on any stabilizer code is not universal

✓ Conjecture: transversal gates on stabilizer codes are in the Clifford hierarchy (proved with new quantity: the *disjointness*)

New!

Eastin, Knill 2009 – the group of transversal gates on any quantum code is not universal

Bravyi, König 2013 – constant-depth local circuits on topological stabilizer codes in D-dimensions are in the D^{th} level of the Clifford hierarchy

Pastawski, Yoshida 2015 – the same but for topological subsystem codes with a threshold

The Clifford hierarchy

Gottesman, Chuang 1999 – the l^{th} level can be implemented using appropriate initial states and the $(l-1)^{\text{th}}$ level.

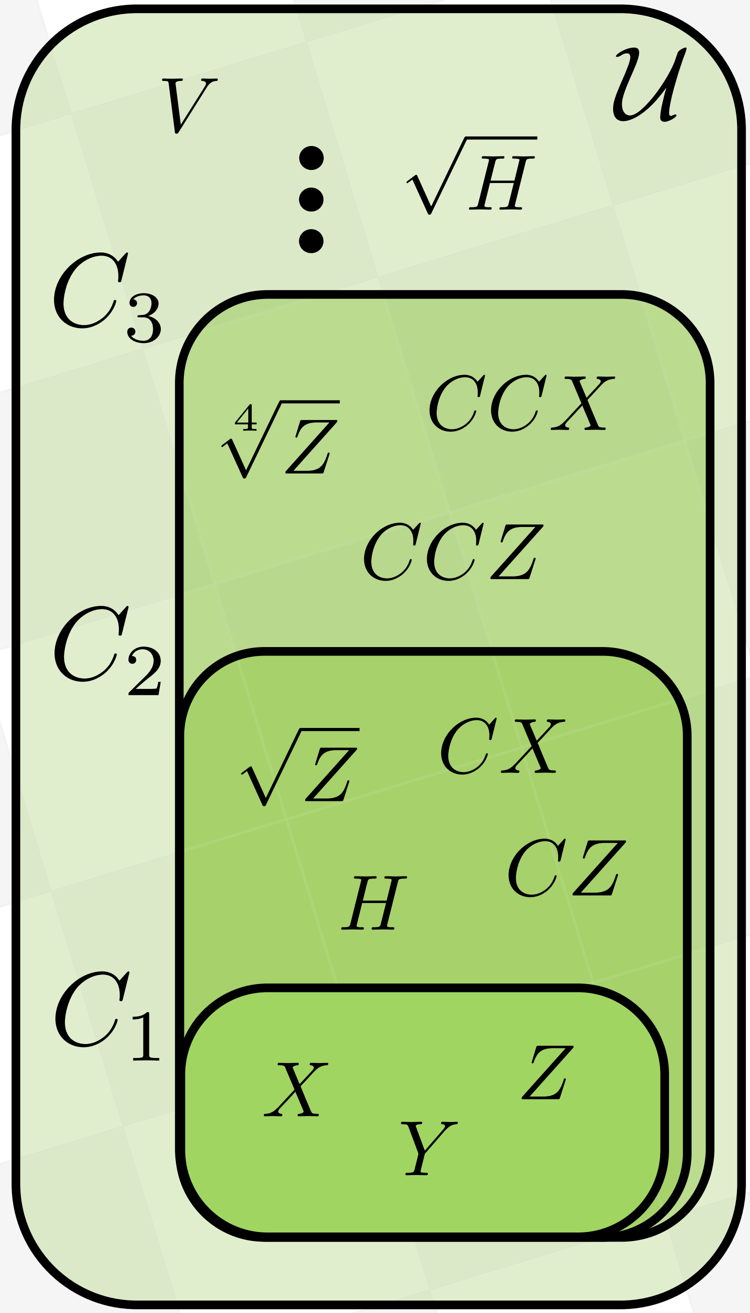
Group commutator: $[U, V] = UVU^\dagger V^\dagger$

$C_1 = \bigotimes_{i=1}^{n} \{I, X, Y, Z\}$

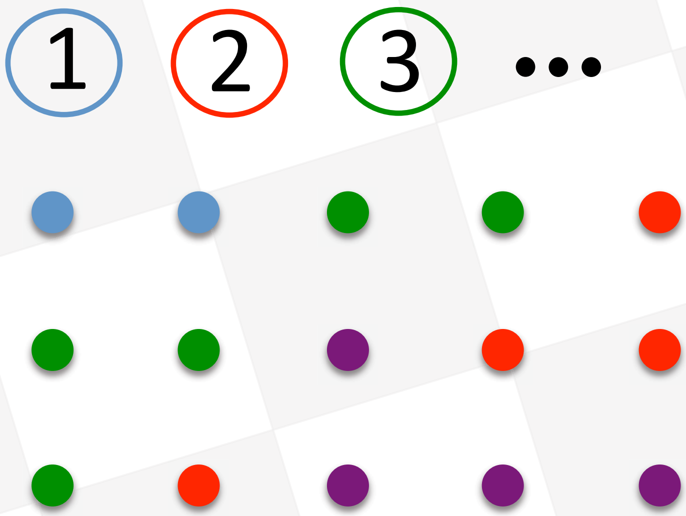
$U \in C_m$ if $[U, p] \in C_{m-1}$ for all $p \in C_1$

Descend by nesting commutators:

$$
\begin{aligned}
&U \in C_M \\
&[U, p_1] \in C_{M-1}, \quad \forall p_1 \in \mathcal{P} \\
&[[U, p_1], p_2] \in C_{M-2}, \quad \forall p_1, p_2 \in \mathcal{P} \\
&\quad \vdots \\
&[\ldots[[U, p_1], p_2] \ldots p_M] = \pm I, \quad \forall p_1, \ldots p_M \in \mathcal{P}
\end{aligned}
$$



Partition & support



Partition: $\{Q_i\}$

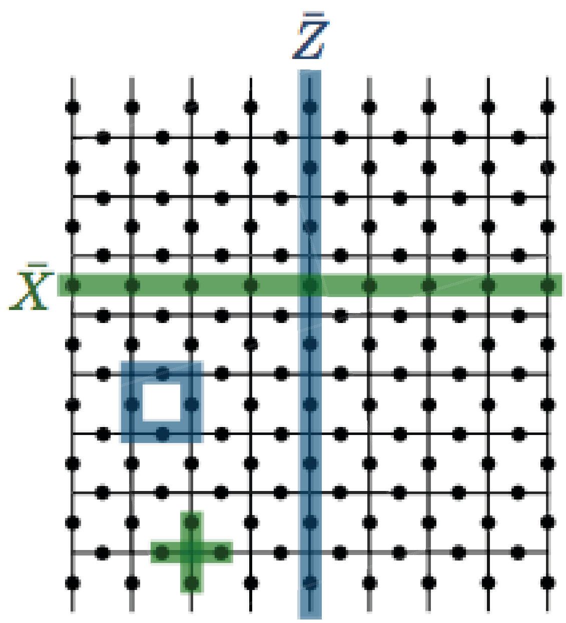
$$Q_1 \cup Q_2 \cup \cdots \cup Q_N = \{1, 2, \ldots, n\}$$

E.g. the single-qubit partition $Q_i = \{i\}$

Support of operator U:

$$\text{supp}(U) = \{i : U \text{ acts on qubits in } Q_i\}$$

E.g.



The transversal commutator fact:

If A, B transversal (w.r.t. the same partition $\{Q_i\}$), then

$$\text{supp}(ABA^\dagger B^\dagger) \subseteq \text{supp}(A) \cap \text{supp}(B).$$

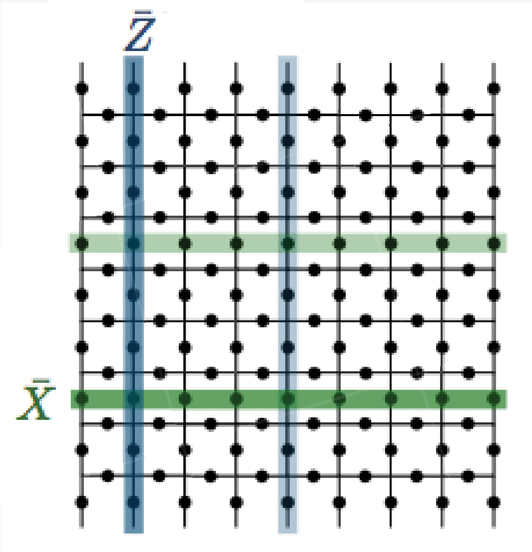
e.g. $A = \bar{X}$, $B = \bar{Z}$ in surface code

Distance & disjointness

$\bar{X}$ can be implemented in many different ways: $\bar{X} \sim \bar{X}s,\ s \in S$

Likewise, $\bar{Y}$ and $\bar{Z}$. Let $\mathcal{X} = \bar{X}S$, and $\mathcal{Y}$, $\mathcal{Z}$ similarly.

Let $\mathcal{L} = \{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\} = \bar{\mathcal{P}}S$.



Define:

"weight of smallest logical Pauli"
$d_\downarrow = \min_{G \in \mathcal{L}} \min_{p \in G} |\mathrm{supp}(p)| = d$

"weight of largest logical Pauli"
$d_\uparrow = \max_{G \in \mathcal{L}} \min_{p \in G} |\mathrm{supp}(p)|$

"fastest rate at which G can be applied" = c-disjointness of G

$$\Delta_c(G) = \frac{1}{c} \max\{|A| : A \subseteq G, \text{ at most } c \text{ elements in } A \text{ have support on any } Q_i\} \quad \text{for } G \in \mathcal{L}$$

"the rate for the slowest G" = disjointness of the code
$\Delta = \min_{G \in \mathcal{L}} \max_{c>0} \Delta_c(G)$

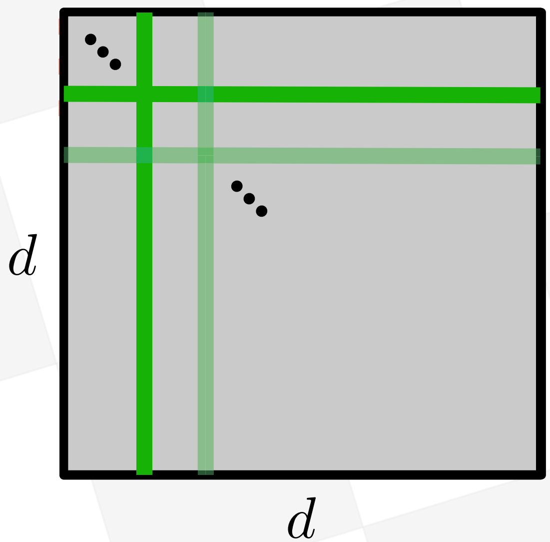
Calculating disjointness

$$\Delta_c(G) = \frac{1}{c}\max\{|A| : A \subseteq G, \text{ at most } c \text{ elements in } A \text{ have support on any } Q_i\}$$

$$\Delta = \min_{G\in\mathcal{L}} \max_{c>0} \Delta_c(G)$$

General upper bounds: $\Delta \leq \min(d_\downarrow, N/d_\uparrow)$

Code specific lower bounds: (single-qubit partition)



Surface code

$\Delta_1(\mathcal{X}) = d$ $\quad d_\downarrow = d$

$\Delta_1(\mathcal{Z}) = d$ $\quad d_\uparrow = 2d - 1$

$\Delta_2(\mathcal{Y}) \geq d/2$

$\Rightarrow \Delta \geq d/2$

More disjointness facts

1) "If the code works, then there is a way to speed up Pauli application"

$$d_{\downarrow} > 1 \text{ iff } \Delta > 1.$$

2) Given a set of regions $H \subseteq \{Q_i\}$ and $G \in \mathcal{L}$,

"If we can apply G at high rate, then we don't need many qubits from H to apply G once."

$\forall c, \exists g \in G, \text{ s.t. } \lvert H \cap \mathrm{supp}(g) \rvert \leq \lvert H \rvert / \Delta_c(G)$	**scrubbing lemma**

compare with **cleaning lemma** [Bravyi-Terhal 09, Yoshida-Chuang 10]

Given $H \subseteq \{Q_i\}$ ($\lvert H \rvert \geq d_{\downarrow} - 1$) and $G \in \mathcal{L}$,

$$\exists g \in G \text{ s.t. } \lvert H \cap \mathrm{supp}(g) \rvert \leq \lvert H \rvert - (d_{\downarrow} - 1)$$

A bound on transversal gates

If $d_\uparrow < d_\downarrow \Delta^{M-1}$, then all transversal gates are in C_M.

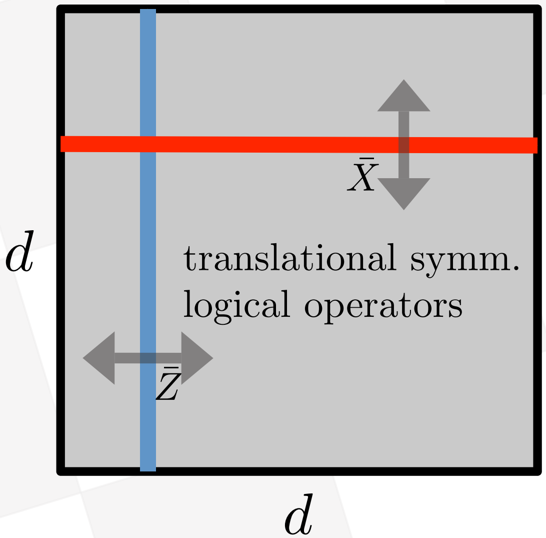
Recall $d_\downarrow > 1$ iff $\Delta > 1$, so all transversal gates are in C_{M_0} with $M_0 = \lfloor \log_\Delta (d_\uparrow / d_\downarrow) + 2 \rfloor$.

proving Zeng et al.'s conjecture

Corrolaries:

- As each C_M is finite, this also implies transversal non-universality as a corollary.
- Also, asymmetry $d_\uparrow > d_\downarrow$ is necessary for non-Clifford gates.

Disjointness examples



Surface code

$\Delta_1(\mathcal{X}) = d$ $\quad d_\downarrow = d$

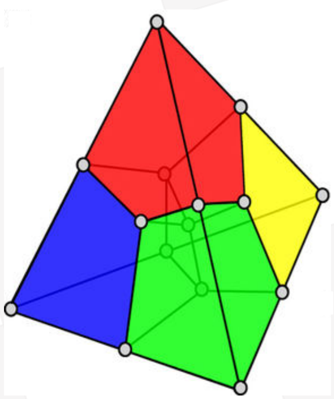
$\Delta_1(\mathcal{Z}) = d$ $\quad d_\uparrow = 2d - 1$

$\Delta_2(\mathcal{Y}) \geq d/2$ $\quad \Rightarrow d_\uparrow < d_\downarrow \Delta^{2-1}$

$\Rightarrow \Delta \geq d/2$ $\quad$ So transversal gates are in C_2

(reproduces Bravyi-König)

105-qubit code: [Jochym-O'Connor, Laflamme '14]



[Campbell-Terhal-Vuillot '17]

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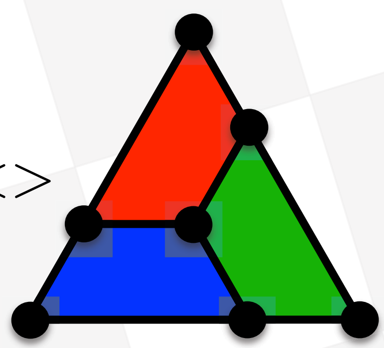

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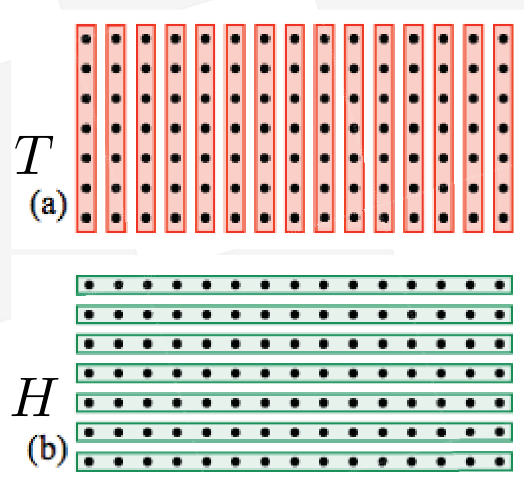


$d_\downarrow = 3$

$d_\uparrow = 7$

$\Delta = 15/7$

$d_\uparrow < d_\downarrow \Delta^{3-1}$

$U \in C_3$

$d_\downarrow = 3$

$d_\uparrow = 3$

$\Delta = 7/3$

$d_\uparrow < d_\downarrow \Delta^{2-1}$

$U \in C_2$

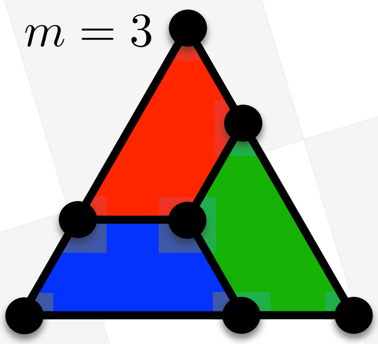
Optimality

If $d_\uparrow < d_\downarrow \Delta^{M-1}$, then all transversal gates are in C_M.

Geometric behavior matches known code families

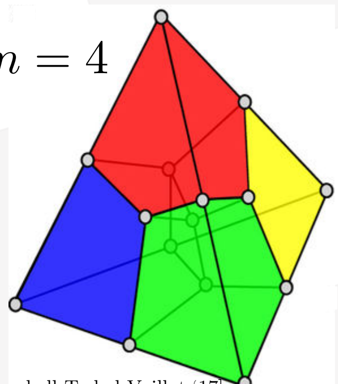
E.g. Reed-Muller family ($d = 3$ color codes)

$m = 3$



transversal $Z^{1/2}$

$m = 4$



[Campbell-Terhal-Vuillot '17]

transversal $Z^{1/4}$

...

$$n = 2^m - 1$$
$$d_\downarrow = 3$$
$$d_\uparrow = 2^{m-1} - 1$$
$$\Delta = n/d_\uparrow$$
$$\Rightarrow U \in C_{m-1}$$

Proof sketch

If $d_\uparrow < d_\downarrow \Delta^{M-1}$, then all transversal gates are in C_M.

- Let $K_0 = U$ be a transversal gate.
- Choose any sequence $G_1, G_2, G_3, \cdots \in \mathcal{L}$
- Find $g_j \in G_j$ so that $K_j = [K_{j-1}, g_j]$ has smaller support than K_{j-1}.
- Since K_{j-1} and g_j are both transversal,

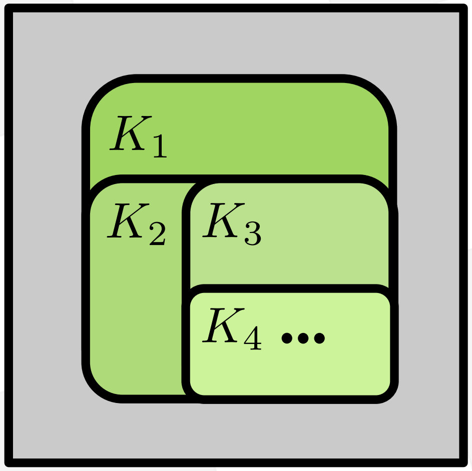
$$|\mathrm{supp}([K_{j-1}, g_j])| \leq |\mathrm{supp}(K_{j-1}) \cap \mathrm{supp}(g_j)|.$$

scrubbing lemma

$$\forall c, \exists g \in G, \text{ s.t. } |\mathrm{supp}(K_{j-1}) \cap \mathrm{supp}(g)| \leq |\mathrm{supp}(K_{j-1})|/\Delta_c(G)$$
$$\leq |\mathrm{supp}(K_{j-1})|/\Delta.$$

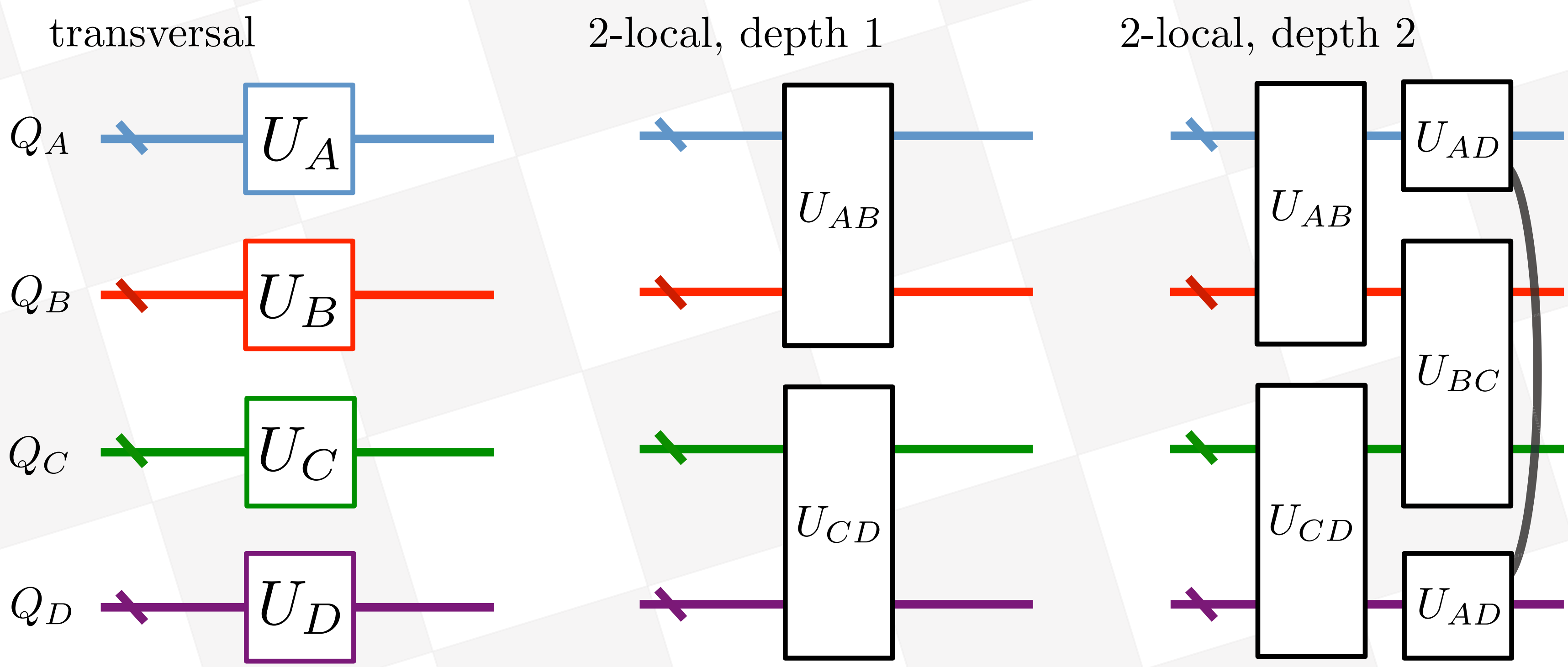
So,

$$|\mathrm{supp}(K_1)| \leq d_\uparrow,$$
$$|\mathrm{supp}(K_2)| \leq d_\uparrow/\Delta,$$
$$|\mathrm{supp}(K_3)| \leq d_\uparrow/\Delta^2,$$
$$\vdots$$
$$|\mathrm{supp}(K_M)| \leq d_\uparrow/\Delta^{M-1} < d, \quad \Rightarrow U \in C_M.$$



Constant depth circuits

- A circuit is q-local with depth 1 with respect to partition $\{Q_i\}$ if it is transversal with respect to "coarse-grained" partition $\{R_i\}$ where each R_i is the union of at most q Q_i. (local $\neq$ geometrically local)
- A q-local, depth h circuit is a sequence of h q-local, depth 1 circuits.



Constant depth circuits

Generalize the **scrubbing lemma**:

$$|\mathrm{supp}(K_j)| \leq q^{h_{j-1}} |\mathrm{supp}(K_{j-1})|/\Delta$$

when K_{j-1} is q-local, depth h_{j-1}.

which leads to a generalization of our bounding theorem:

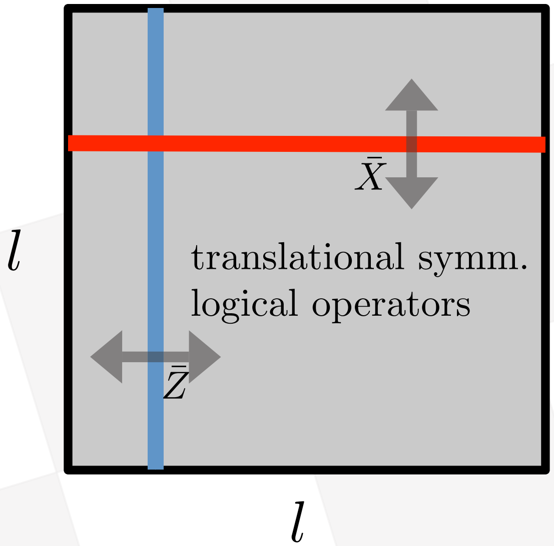
If $d_\uparrow \prod_{j=0}^{M-1} q^{h_j} < d_\downarrow \Delta^{M-1}$ then q-local, depth h_0 gates are in C_M.

In a code family $[[n(l), k(l), d(l)]]$, $d_\uparrow, d_\downarrow, \Delta$ depend on l.

If $\lim_{l \to \infty} \frac{d_\uparrow}{d_\downarrow \Delta^{M-1}} = 0$ then constant-local, constant-depth gates are in C_M.

Constant-depth circuits – surface code

What is the power of constant-depth, non-geometrically-local circuits on the surface code?



$$d_{\downarrow} = l$$

$$d_{\uparrow} = 2l - 1$$

$$\Delta = \Theta(l)$$

$$\lim_{l \to \infty} \frac{d_{\uparrow}}{d_{\downarrow}\Delta} = 0$$

$$\Rightarrow U \in C_2.$$

Generalizes Bravyi-König's conclusion to non-geometrically-local circuits (for this code)

More with disjointness...

– All these theorems work for qudits as well as qubits

– "permutation-transversal" operators PU for permutation P (of the regions Q_i) and unitary U are covered by a similar bound

$$\text{If} \quad 2d_{\uparrow} < d_{\downarrow}\Delta^{M-1} \quad \text{then } PU \in C_M.$$

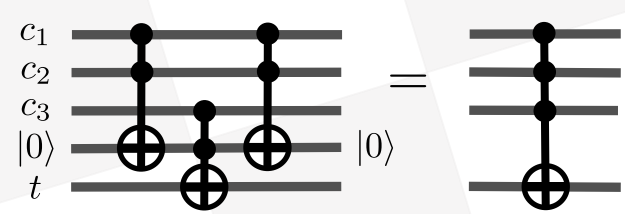
– transversal morphisms from code A to code B are covered

$$\text{If} \quad d_{\uparrow}^{(A)} < d_{\downarrow}^{(B)}\Delta^{(B)M-1} \text{ then } U \in C_M.$$

– Bounding transversal gates between r codeblocks can be done in terms of the parameters $d_{\downarrow}, d_{\uparrow}, \Delta$ of one codeblock.

– Transversal Toffoli is impossible on stabilizer codes.

(alternative proof and special case of [Newman, Shi '17])



Open questions

– We know that the disjointness bound is not always tight. Can it be strengthened?

– What is the value of coarse-graining a partition?

– Do properties of topologically local codes generically simplify the calculation of disjointness?

– Use these no-gos to design codes!

Thank you!

Scrubbing vs. cleaning and optimality

Replace scrubbing with cleaning in the proof.

cleaning lemma

$$|\text{supp}(K_{j-1}) \cap \text{supp}(g_j)| \leq |\text{supp}(K_{j-1})| - (d_\downarrow - 1)$$

Now K_j decrease in size arithmetically:

$$\begin{aligned}
&|\text{supp}(K_1)| \leq d_\uparrow \\
&|\text{supp}(K_2)| \leq d_\uparrow - (d_\downarrow - 1) \\
&|\text{supp}(K_3)| \leq d_\uparrow - 2(d_\downarrow - 1) \\
&\quad\vdots \\
&|\text{supp}(K_M)| \leq d_\uparrow - (M-1)(d_\downarrow - 1)
\end{aligned}$$

And we get (unpublished [Beverland, Preskill '14])

If $d_\uparrow < d_\downarrow + (M-1)(d_\downarrow - 1)$, then all transversal gates in C_M.

Scrubbing vs. cleaning and optimality

Theorem (cleaning):

If $d_\uparrow < d_\downarrow + (M-1)(d_\downarrow - 1)$, then all transversal gates in C_M.

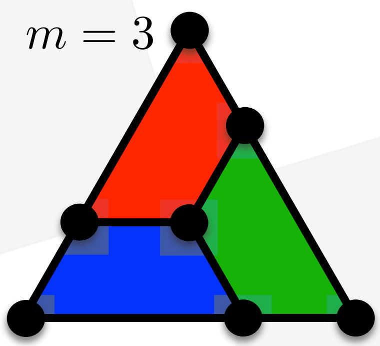
Theorem (scrubbing):

If $d_\uparrow < d_\downarrow \Delta^{M-1}$, then all transversal gates are in C_M.

Exponential behavior of the latter matches known code families.

E.g. Reed-Muller family ($d = 3$ color codes)

$m = 3$



transversal $Z^{1/2}$

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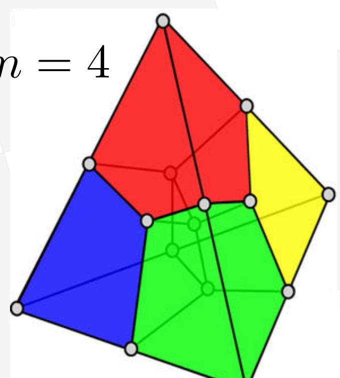

[Campbell-Terhal-Vuillot '17]

transversal $Z^{1/4}$

…

$n = 2^m - 1$

$d_\downarrow = 3$

$d_\uparrow = 2^{m-1} - 1$

$\Delta = n/d_\uparrow$

$\Rightarrow U \in C_{m-1}$