# All Pure Bipartite Entangled States can be Self-Tested

Andrea Coladangelo, Koon Tong Goh and Valerio Scarani

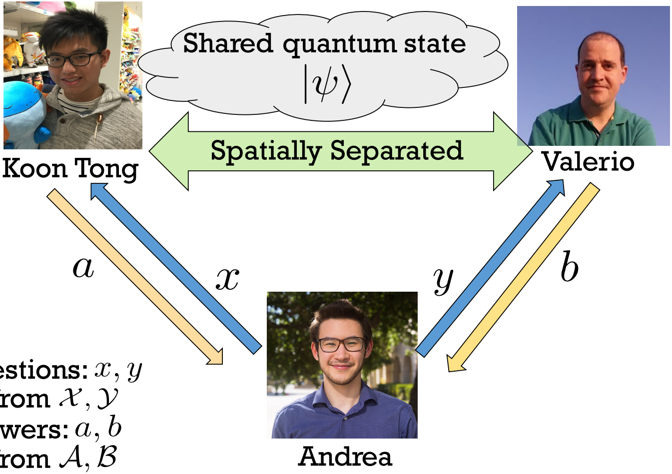
Caltech & CQT

QIP 2018, January 16

# Outline

- Background
- Main result
- Self-testing correlations
- Open questions

# The Setup:




Questions: $x, y$
from $\mathcal{X}, \mathcal{Y}$
Answers: $a, b$
from $\mathcal{A}, \mathcal{B}$

# Correlations:

- A given strategy by Alice and Bob determines a collection of conditional probability distributions of answers given questions

$$\{p(a, b|x, y)\}_{x,y \in \mathcal{X} \times \mathcal{Y}} \tag{1}$$

# Correlations:

- A given strategy by Alice and Bob determines a collection of conditional probability distributions of answers given questions

$$\{p(a, b|x, y)\}_{x,y \in \mathcal{X} \times \mathcal{Y}} \tag{1}$$

- We refer to each such collection as a *correlation*. And we call it *quantum* if it's obtained via some quantum strategy.

# Correlations:

- A given strategy by Alice and Bob determines a collection of conditional probability distributions of answers given questions

$$\{p(a, b|x, y)\}_{x,y \in \mathcal{X} \times \mathcal{Y}} \tag{1}$$

- We refer to each such collection as a *correlation*. And we call it *quantum* if it's obtained via some quantum strategy.
- A quantum strategy is specified by a joint state $|\psi\rangle$, and projective measurements $\{\Pi_x^a\}_{x \in \mathcal{X}}$ and $\{\Pi_y^b\}_{y \in \mathcal{Y}}$ for Alice and Bob respectively, such that

$$p(a, b|x, y) = \langle\psi| \Pi_x^a \Pi_y^b |\psi\rangle \tag{2}$$

# Correlations:

- A given strategy by Alice and Bob determines a collection of conditional probability distributions of answers given questions

$$\{p(a,b|x,y)\}_{x,y\in\mathcal{X}\times\mathcal{Y}} \tag{1}$$

- We refer to each such collection as a *correlation*. And we call it *quantum* if it's obtained via some quantum strategy.
- A quantum strategy is specified by a joint state $|\psi\rangle$, and projective measurements $\{\Pi_x^a\}_{x\in\mathcal{X}}$ and $\{\Pi_y^b\}_{y\in\mathcal{Y}}$ for Alice and Bob respectively, such that

$$p(a,b|x,y) = \langle\psi|\,\Pi_x^a\Pi_y^b\,|\psi\rangle \tag{2}$$

- Let $\mathcal{C}_q^{m,n,r,s}$ be the set of quantum correlations, where $\mathcal{X},\mathcal{Y},\mathcal{A},\mathcal{B}$ have sizes $m,n,r,s$ respectively.

# Device-independent self-testing

- The aim is to certify the behaviour of a quantum system, based only on the observed correlations.

# Device-independent self-testing

- The aim is to certify the behaviour of a quantum system, based only on the observed correlations.
- Nothing is assumed other than a no-signalling constraint on Alice and Bob, and that they obey quantum mechanics.

- The aim is to certify the behaviour of a quantum system, based only on the observed correlations.
- Nothing is assumed other than a no-signalling constraint on Alice and Bob, and that they obey quantum mechanics.

**Definition: (self-testing, informal)**

We say that a correlation *self-tests* a state $|\psi\rangle_{target}$ if it can be uniquely achieved when Alice and Bob make local measurements on $|\psi\rangle_{target}$, up to local isometries.

# Example: CHSH

Questions $x, y \in \{0,1\}$, answers $a, b \in \{-1,1\}$.
Maximal violation of CHSH:

$$\langle\psi| A_0(B_0 + B_1) + A_1(B_0 - B_1) |\psi\rangle = 2\sqrt{2}$$

# Example: CHSH

Questions $x, y \in \{0, 1\}$, answers $a, b \in \{-1, 1\}$.
Maximal violation of CHSH:

$$\langle\psi| A_0(B_0 + B_1) + A_1(B_0 - B_1) |\psi\rangle = 2\sqrt{2}$$

The correlation in the quantum correlations set $\mathcal{C}_q^{2,2,2,2}$ that achieves this maximal violation self-tests the maximally entangled pair of qubits[1].

[1] S. Popescu, D. Rohrlich (1992)

# Some applications of self-testing in cryptography

[2] C. A. Miller, Y. Shi (2014)
[3] U. Vazirani, T. Vidick (2014)

# Some applications of self-testing in cryptography

- Randomness expansion and key distribution [2] [3]

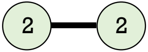
[2] C. A. Miller, Y. Shi (2014)
[3] U. Vazirani, T. Vidick (2014)
[4] B. W. Reichardt, F. Unger, U. Vazirani (2012)
[5] M. McKague (2013)

# Some applications of self-testing in cryptography

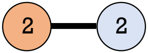
- Randomness expansion and key distribution [2] [3]
- Delegated computation [4] [5]

[2] C. A. Miller, Y. Shi (2014)
[3] U. Vazirani, T. Vidick (2014)
[4] B. W. Reichardt, F. Unger, U. Vazirani (2012)
[5] M. McKague (2013)

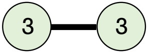
# So far, which (bipartite) states can we self-test?



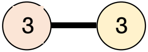

Maximally Entangled Qubits (Singlet)
[Mayers-Yao 2004]



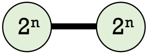

All pure Partially Entangled Qubits
[Bamps-Pironio 2015]



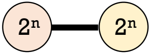

Maximally Entangled Qutrits (numerical)
[Salavrakos et al. 2016]




A certain Pair of Partially Entangled Qutrits
[Yang et al. 2014]




n Singlets in parallel
[McKague 2015]




A small class of Partially Entangled Qudits, for $d = 2^n$
[C. 2016]

It seems like a lot of states can be self-tested…

… but can we self-test *all* bipartite entangled states?

# Our Result

**Theorem:**

Let $|\psi\rangle_{target} = \sum_{i=0}^{d-1} c_i |ii\rangle$. There exists a correlation in $\mathcal{C}_q^{3,4,d,d}$ that self-tests $|\psi\rangle_{target}$. Moreover, the local measurements that achieve it are also unique up to local isometries.

# Our Result

**Theorem:**

Let $|\psi\rangle_{target} = \sum_{i=0}^{d-1} c_i |ii\rangle$. There exists a correlation in $\mathcal{C}_q^{3,4,d,d}$ that self-tests $|\psi\rangle_{target}$. Moreover, the local measurements that achieve it are also unique up to local isometries.

(Mixed states can't be self-tested.)

**Tilted CHSH inequality:**[6]

$$\langle\psi|\, \alpha A_0 + A_0(B_0 + B_1) + A_1(B_0 - B_1)\, |\psi\rangle \leq 2 + \alpha \qquad (3)$$

[6] A. Acín, S. Massar, S. Pironio (2012)

**Tilted CHSH inequality:**[6]

$$\langle\psi| \; \boxed{\alpha A_0} + A_0(B_0 + B_1) + A_1(B_0 - B_1) \, |\psi\rangle \leq 2 + \boxed{\alpha} \tag{3}$$

[6] A. Acín, S. Massar, S. Pironio (2012)

**Tilted CHSH inequality:**[6]

$$\langle\psi|\, \alpha A_0 + A_0(B_0 + B_1) + A_1(B_0 - B_1)\, |\psi\rangle \leq 2 + \alpha \qquad (3)$$

- Maximal quantum violation is $I_\alpha = \sqrt{8 + 2\alpha^2}$

[6] A. Acín, S. Massar, S. Pironio (2012)

**Tilted CHSH inequality:**[6]

$$\langle\psi|\, \alpha A_0 + A_0(B_0 + B_1) + A_1(B_0 - B_1)\, |\psi\rangle \leq 2 + \alpha \qquad (3)$$

- Maximal quantum violation is $I_\alpha = \sqrt{8 + 2\alpha^2}$
- Maximal quantum violation self-tests the tilted EPR pair

$$|\psi_\theta\rangle = \cos\theta\, |00\rangle + \sin\theta\, |11\rangle \qquad (4)$$

for $\sin\theta = \sqrt{\frac{4-\alpha^2}{4+\alpha^2}}$.

[6] A. Acín, S. Massar, S. Pironio (2012)

# Tilted CHSH

- The self-testing result is also robust: [7]

$$\langle\psi|\, \alpha A_0 + A_0(B_0 + B_1) + A_1(B_0 - B_1)\, |\psi\rangle \geq I_\alpha - \epsilon$$

$\Rightarrow|\psi\rangle$ is $O(\sqrt{\epsilon})$-close to $|\psi_\theta\rangle$, up to a local isometry.

[7] C.Bamps and S.Pironio (2015)

# Tilted CHSH

- The self-testing result is also robust: [7]

$$\langle\psi|\,\alpha A_0 + A_0(B_0 + B_1) + A_1(B_0 - B_1)\,|\psi\rangle \geq I_\alpha - \epsilon$$

$\Rightarrow|\psi\rangle$ is $O(\sqrt{\epsilon})$-close to $|\psi_\theta\rangle$, up to a local isometry.

- Maximal violation also self-tests the measurements (with the same robustness):

$$\begin{aligned} A_0 &= Z, & B_0 &= \cos\mu Z + \sin\mu X, \\ A_1 &= X, & B_1 &= \cos\mu Z - \sin\mu X \end{aligned}$$

[7] C.Bamps and S.Pironio (2015)

- Represent correlations via *correlation tables*. We specify a correlation by specifying, for each possible question $(x, y) \in \mathcal{X} \times \mathcal{Y}$, the table $T_{x,y}$ with entries $T_{x,y}(a, b) := p(a, b|x, y)$.

- Represent correlations via *correlation tables*. We specify a correlation by specifying, for each possible question $(x, y) \in \mathcal{X} \times \mathcal{Y}$, the table $T_{x,y}$ with entries $T_{x,y}(a, b) := p(a, b|x, y)$.
- For example, the ideal tilted CHSH correlation for angle $\theta$ is specified by four tables, one for each $(x, y) \in \{0, 1\}^2$, of the form

| | |
|---|---|
| $p_\theta(1, 1\|x, y)$ | $p_\theta(1, -1\|x, y)$ |
| $p_\theta(-1, 1\|x, y)$ | $p_\theta(-1, -1\|x, y)$ |

# Some Intuition behind the Self-Testing Correlations

Recall that we are trying to self-test $|\psi\rangle = \sum_{i=0}^{d-1} c_i |ii\rangle$

# Some Intuition behind the Self-Testing Correlations

Recall that we are trying to self-test $|\psi\rangle = \sum_{i=0}^{d-1} c_i |ii\rangle$

- The structure of our correlations is inspired by [Yang and Navascués (2013)]:
  Question sets $\mathcal{X} = \{0, 1, 2\}, \mathcal{Y} = \{0, 1, 2, 3\}$,
  Answer sets $\mathcal{A} = \mathcal{B} = \{0, 1, .., d-1\}$.

# Some Intuition behind the Self-Testing Correlations

Recall that we are trying to self-test $|\psi\rangle = \sum_{i=0}^{d-1} c_i |ii\rangle$

- The structure of our correlations is inspired by [Yang and Navascués (2013)]:
  Question sets $\mathcal{X} = \{0, 1, 2\}, \mathcal{Y} = \{0, 1, 2, 3\}$,
  Answer sets $\mathcal{A} = \mathcal{B} = \{0, 1, .., d-1\}$.
- **Intuitively, we want to make use of the tilted CHSH self-test to certify $|\psi\rangle_{\textbf{target}}$ "portion by portion".**

Recall that we are trying to self-test $|\psi\rangle = \sum_{i=0}^{d-1} c_i |ii\rangle$

- The structure of our correlations is inspired by [Yang and Navascués (2013)]:
  Question sets $\mathcal{X} = \{0, 1, 2\}, \mathcal{Y} = \{0, 1, 2, 3\}$,
  Answer sets $\mathcal{A} = \mathcal{B} = \{0, 1, .., d-1\}$.
- **Intuitively, we want to make use of the tilted CHSH self-test to certify $|\psi\rangle_{\textbf{target}}$ "portion by portion".**
  We start with the case $d = 4$. So
  $|\psi\rangle_{target} = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$.

Recall that we are trying to self-test $|\psi\rangle = \sum_{i=0}^{d-1} c_i |ii\rangle$

- The structure of our correlations is inspired by [Yang and Navascués (2013)]:
  Question sets $\mathcal{X} = \{0, 1, 2\}, \mathcal{Y} = \{0, 1, 2, 3\}$,
  Answer sets $\mathcal{A} = \mathcal{B} = \{0, 1, .., d-1\}$.
- **Intuitively, we want to make use of the tilted CHSH self-test to certify $|\psi\rangle_{\textbf{target}}$ "portion by portion".**
  We start with the case $d = 4$. So
  $|\psi\rangle_{target} = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$.

For $x, y \in \{0, 1\}$ we use Alice and Bob's $\{0, 1\}$ answers to certify the portion $c_0 |00\rangle + c_1 |11\rangle$, and their $\{2, 3\}$ answers to certify $c_2 |22\rangle + c_3 |33\rangle$.

# The ideal measurements

$\mathcal{X} = \{0, 1, 2\}, \mathcal{Y} = \{0, 1, 2, 3\}$ and $\mathcal{A} = \mathcal{B} = \{0, 1, .., 3\}$

# The ideal measurements

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$

**Ideal state:** $|\Psi\rangle = c_0\,|00\rangle + c_1\,|11\rangle + c_2\,|22\rangle + c_3\,|33\rangle$

# The ideal measurements

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$

**Ideal state:** $|\Psi\rangle = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$

Let $A_0^{(\theta)}, A_1^{(\theta)}$, $B_0^{(\theta)}, B_1^{(\theta)}$ be the ideal tilted CHSH qubit measurements for $\theta = \arctan \frac{c_1}{c_0}$.

# The ideal measurements

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$

**Ideal state:** $|\Psi\rangle = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$

Let $A_0^{(\theta)}, A_1^{(\theta)}$, $B_0^{(\theta)}, B_1^{(\theta)}$ be the ideal tilted CHSH qubit measurements for $\theta = \arctan \frac{c_1}{c_0}$.

# The ideal measurements

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$

**Ideal state:** $|\Psi\rangle = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$

Let $A_0^{(\theta)}, A_1^{(\theta)}$, $B_0^{(\theta)}, B_1^{(\theta)}$ be the ideal tilted CHSH qubit measurements for $\theta = \arctan \frac{c_1}{c_0}$.
And $A_0^{(\phi)}, A_1^{(\phi)}$, $B_0^{(\phi)}, B_1^{(\phi)}$ for $\phi = \arctan \frac{c_3}{c_2}$.

# The ideal measurements

$\mathcal{X} = \{0, 1, 2\}, \mathcal{Y} = \{0, 1, 2, 3\}$ and $\mathcal{A} = \mathcal{B} = \{0, 1, .., 3\}$

**Ideal state:** $|\Psi\rangle = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$

Let $A_0^{(\theta)}, A_1^{(\theta)}, B_0^{(\theta)}, B_1^{(\theta)}$ be the ideal tilted CHSH qubit measurements for $\theta = \arctan \frac{c_1}{c_0}$.
And $A_0^{(\phi)}, A_1^{(\phi)}, B_0^{(\phi)}, B_1^{(\phi)}$ for $\phi = \arctan \frac{c_3}{c_2}$.

# The ideal measurements

$\mathcal{X} = \{0, 1, 2\}, \mathcal{Y} = \{0, 1, 2, 3\}$ and $\mathcal{A} = \mathcal{B} = \{0, 1, .., 3\}$

**Ideal state:** $|\Psi\rangle = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$

Let $A_0^{(\theta)}, A_1^{(\theta)}$, $B_0^{(\theta)}, B_1^{(\theta)}$ be the ideal tilted CHSH qubit measurements for $\theta = \arctan \frac{c_1}{c_0}$.
And $A_0^{(\phi)}, A_1^{(\phi)}$, $B_0^{(\phi)}, B_1^{(\phi)}$ for $\phi = \arctan \frac{c_3}{c_2}$.

**Ideal measurements** $\tilde{\mathbf{A}}_\mathbf{0}, \tilde{\mathbf{A}}_\mathbf{1}, \tilde{\mathbf{A}}_\mathbf{2}, \tilde{\mathbf{B}}_\mathbf{0}, \tilde{\mathbf{B}}_\mathbf{1}, \tilde{\mathbf{B}}_\mathbf{2}, \tilde{\mathbf{B}}_\mathbf{3}$:

# The ideal measurements

$\mathcal{X} = \{0, 1, 2\}, \mathcal{Y} = \{0, 1, 2, 3\}$ and $\mathcal{A} = \mathcal{B} = \{0, 1, .., 3\}$

**Ideal state:** $|\Psi\rangle = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$

Let $A_0^{(\theta)}, A_1^{(\theta)}, B_0^{(\theta)}, B_1^{(\theta)}$ be the ideal tilted CHSH qubit measurements for $\theta = \arctan \frac{c_1}{c_0}$.
And $A_0^{(\phi)}, A_1^{(\phi)}, B_0^{(\phi)}, B_1^{(\phi)}$ for $\phi = \arctan \frac{c_3}{c_2}$.

**Ideal measurements** $\tilde{\mathbf{A}}_\mathbf{0}, \tilde{\mathbf{A}}_\mathbf{1}, \tilde{\mathbf{A}}_\mathbf{2}, \tilde{\mathbf{B}}_\mathbf{0}, \tilde{\mathbf{B}}_\mathbf{1}, \tilde{\mathbf{B}}_\mathbf{2}, \tilde{\mathbf{B}}_\mathbf{3}$:

$$\tilde{A}_0 := \begin{bmatrix} A_0^{(\theta)} & 0 \\ 0 & A_0^{(\phi)} \end{bmatrix} \qquad \tilde{B}_0 := \begin{bmatrix} B_0^{(\theta)} & 0 \\ 0 & B_0^{(\phi)} \end{bmatrix}$$

$$\tilde{A}_1 := \begin{bmatrix} A_1^{(\theta)} & 0 \\ 0 & A_1^{(\phi)} \end{bmatrix}, \qquad \tilde{B}_1 := \begin{bmatrix} B_1^{(\theta)} & 0 \\ 0 & B_1^{(\phi)} \end{bmatrix}$$

# The ideal measurements

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$

**Ideal state:** $|\Psi\rangle = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$

Let $A_0^{(\theta)}, A_1^{(\theta)}$, $B_0^{(\theta)}, B_1^{(\theta)}$ be the ideal tilted CHSH qubit measurements for $\theta = \arctan \frac{c_1}{c_0}$.
And $A_0^{(\phi)}, A_1^{(\phi)}$, $B_0^{(\phi)}, B_1^{(\phi)}$ for $\phi = \arctan \frac{c_3}{c_2}$.

**Ideal measurements $\tilde{\mathbf{A}}_0, \tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_2, \tilde{\mathbf{B}}_0, \tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2, \tilde{\mathbf{B}}_3$:**

$$\tilde{A}_0 := \begin{bmatrix} A_0^{(\theta)} & 0 \\ 0 & A_0^{(\phi)} \end{bmatrix} \sim \text{comput. basis meas.}, \qquad \tilde{B}_0 := \begin{bmatrix} B_0^{(\theta)} & 0 \\ 0 & B_0^{(\phi)} \end{bmatrix}$$

$$\tilde{A}_1 := \begin{bmatrix} A_1^{(\theta)} & 0 \\ 0 & A_1^{(\phi)} \end{bmatrix}, \qquad \tilde{B}_1 := \begin{bmatrix} B_1^{(\theta)} & 0 \\ 0 & B_1^{(\phi)} \end{bmatrix}$$

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$.
$|\Psi\rangle = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$.
$|\Psi\rangle = c_0\,|00\rangle + c_1\,|11\rangle + c_2\,|22\rangle + c_3\,|33\rangle$

- The ideal measurements from the last slide imply that, for each $(x,y) \in \{0,1\}^2$, $T_{x,y}$ takes the form

<table>
<tr><th>$a \backslash b$</th><th>0</th><th>1</th><th>2</th><th>3</th></tr>
<tr><td>0</td><td colspan="2" rowspan="2">$C_{x,y}$</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>2</td><td>0</td><td>0</td><td colspan="2" rowspan="2">$C'_{x,y}$</td></tr>
<tr><td>3</td><td>0</td><td>0</td></tr>
</table>

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$.
$|\Psi\rangle = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$

- The ideal measurements from the last slide imply that, for each $(x,y) \in \{0,1\}^2$, $T_{x,y}$ takes the form

<table>
<tr><th>$a \backslash b$</th><th>0</th><th>1</th><th>2</th><th>3</th></tr>
<tr><td>0</td><td colspan="2" rowspan="2">$C_{x,y}$</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>2</td><td>0</td><td>0</td><td colspan="2" rowspan="2">$C'_{x,y}$</td></tr>
<tr><td>3</td><td>0</td><td>0</td></tr>
</table>

  where $C_{x,y}$ contains ideal tilted CHSH correlations on question $(x,y)$ for angle $\theta = \arctan \frac{c_1}{c_0}$, weighted by $c_0^2 + c_1^2$

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$.
$|\Psi\rangle = c_0 |00\rangle + c_1 |11\rangle + c_2 |22\rangle + c_3 |33\rangle$

- The ideal measurements from the last slide imply that, for each $(x,y) \in \{0,1\}^2$, $T_{x,y}$ takes the form

<table>
<tr><th>$a \backslash b$</th><th>0</th><th>1</th><th>2</th><th>3</th></tr>
<tr><td>0</td><td colspan="2" rowspan="2">$C_{x,y}$</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>2</td><td>0</td><td>0</td><td colspan="2" rowspan="2">$C'_{x,y}$</td></tr>
<tr><td>3</td><td>0</td><td>0</td></tr>
</table>

where $C_{x,y}$ contains ideal tilted CHSH correlations on question $(x,y)$ for angle $\theta = \arctan \frac{c_1}{c_0}$, weighted by $c_0^2 + c_1^2$

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$.
$|\Psi\rangle = c_0\,|00\rangle + c_1\,|11\rangle + c_2\,|22\rangle + c_3\,|33\rangle$

- The ideal measurements from the last slide imply that, for each $(x,y) \in \{0,1\}^2$, $T_{x,y}$ takes the form

<table>
<tr><th>$a\backslash b$</th><th>0</th><th>1</th><th>2</th><th>3</th></tr>
<tr><td>0</td><td colspan="2" rowspan="2">$C_{x,y}$</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>2</td><td>0</td><td>0</td><td colspan="2" rowspan="2">$C'_{x,y}$</td></tr>
<tr><td>3</td><td>0</td><td>0</td></tr>
</table>

where $C_{x,y}$ contains ideal tilted CHSH correlations on question $(x,y)$ for angle $\theta = \arctan\frac{c_1}{c_0}$, weighted by $c_0^2 + c_1^2$ and $C'_{x,y}$ for angle $\theta' = \arctan\frac{c_3}{c_2}$, weighted by $c_2^2 + c_3^2$

$\mathcal{X} = \{0,1,2\}, \mathcal{Y} = \{0,1,2,3\}$ and $\mathcal{A} = \mathcal{B} = \{0,1,..,3\}$.

$|\Psi\rangle = c_0\,|00\rangle + c_1\,|11\rangle + c_2\,|22\rangle + c_3\,|33\rangle$

- The ideal measurements from the last slide imply that, for each $(x,y) \in \{0,1\}^2$, $T_{x,y}$ takes the form

<table>
<tr><th>$a\backslash b$</th><th>0</th><th>1</th><th>2</th><th>3</th></tr>
<tr><td>0</td><td colspan="2" rowspan="2">$C_{x,y}$</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>2</td><td>0</td><td>0</td><td colspan="2" rowspan="2">$C'_{x,y}$</td></tr>
<tr><td>3</td><td>0</td><td>0</td></tr>
</table>

where $C_{x,y}$ contains ideal tilted CHSH correlations on question $(x,y)$ for angle $\theta = \arctan\frac{c_1}{c_0}$, weighted by $c_0^2 + c_1^2$ and $C'_{x,y}$ for angle $\theta' = \arctan\frac{c_3}{c_2}$, weighted by $c_2^2 + c_3^2$

As you can expect, then the correlation table for questions $x, y \in \{0, 1\}$ for the case of general $d$ (even) is:

<table>
<tr><th>$a \backslash b$</th><th>0</th><th>1</th><th>2</th><th>3</th><th>$\cdots$</th><th>$d-2$</th><th>$d-1$</th></tr>
<tr><td>0</td><td colspan="2" rowspan="2">$C_{x,y}^{(0)}$</td><td>0</td><td>0</td><td>$\cdots$</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>$\cdots$</td><td>0</td><td>0</td></tr>
<tr><td>2</td><td>0</td><td>0</td><td colspan="2" rowspan="2">$C_{x,y}^{(1)}$</td><td>$\cdots$</td><td>0</td><td>0</td></tr>
<tr><td>3</td><td>0</td><td>0</td><td>$\cdots$</td><td>0</td><td>0</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\vdots$</td><td>$\ddots$</td><td>$\vdots$</td><td>$\vdots$</td></tr>
<tr><td>$d-2$</td><td>0</td><td>0</td><td>0</td><td>0</td><td>$\cdots$</td><td colspan="2" rowspan="2">$C_{x,y}^{(\frac{d}{2}-1)}$</td></tr>
<tr><td>$d-1$</td><td>0</td><td>0</td><td>0</td><td>0</td><td>$\cdots$</td></tr>
</table>

where, the $m$th block contains ideal tilted CHSH correlations for angle $\theta_m = \arctan \frac{c_{2m+1}}{c_{2m}}$, weighted by $c_{2m}^2 + c_{2m+1}^2$.

Are the correlations from the previous slide for questions $x, y \in \{0, 1\}$ uniquely achieved by measuring $|\psi\rangle_{target} = \sum_{i=0}^{d-1} c_i |ii\rangle$, up to a local isometry?

Are the correlations from the previous slide for questions $x, y \in \{0, 1\}$ uniquely achieved by measuring $|\psi\rangle_{target} = \sum_{i=0}^{d-1} c_i |ii\rangle$, up to a local isometry?

**NO!** (For one thing $\mathcal{X} = \{0, 1, 2\}$ and $\mathcal{Y} = \{0, 1, 2, 3\}$, so you should expect us to use the other questions as well)

Are the correlations from the previous slide for questions $x, y \in \{0, 1\}$ uniquely achieved by measuring $|\psi\rangle_{target} = \sum_{i=0}^{d-1} c_i |ii\rangle$, up to a local isometry?

**NO!** (For one thing $\mathcal{X} = \{0, 1, 2\}$ and $\mathcal{Y} = \{0, 1, 2, 3\}$, so you should expect us to use the other questions as well)

Here is a simple counterexample for the case $d = 4$: Consider the mixed state represented by the mixture

$$\Big\{ \big( c_0^2 + c_1^2, \cos\theta\, |00\rangle + \sin\theta\, |11\rangle \big), \big( c_2^2 + c_3^2, \cos\theta'\, |22\rangle + \sin\theta'\, |33\rangle \big) \Big\},$$

where $\theta = \arctan \frac{c_1}{c_0}$ and $\theta' = \arctan \frac{c_3}{c_2}$.

The point is that it is NOT enough to certify just the “even-odd” pairs $(0,1)$, $(2,3)$,..,$(d-2, d-1)$.

The point is that it is NOT enough to certify just the "even-odd" pairs $(0, 1)$, $(2, 3)$,..,$(d-2, d-1)$.

We need to enforce the same structure also on the "odd-even" pairs $(d-1, 0)$,$(1, 2)$,..,$(d-3, d-2)$.

We also need to use questions $\mathbf{x} \in \{\mathbf{0}, \mathbf{2}\}$ **and** $\mathbf{y} \in \{\mathbf{2}, \mathbf{3}\}$. The ideal observables are $\tilde{A}_0$ (already defined earlier),
and $\tilde{A}_2$, $\tilde{B}_2$, $\tilde{B}_3$ same as before except shifted down by one basis element.

We also need to use questions $\mathbf{x} \in \{\mathbf{0}, \mathbf{2}\}$ **and** $\mathbf{y} \in \{\mathbf{2}, \mathbf{3}\}$. The ideal observables are $\tilde{A}_0$ (already defined earlier),
and $\tilde{A}_2$, $\tilde{B}_2$, $\tilde{B}_3$ same as before except shifted down by one basis element.

For $x, y \in \{0, 1\}$,
$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

We also need to use questions $\mathbf{x} \in \{\mathbf{0}, \mathbf{2}\}$ **and** $\mathbf{y} \in \{\mathbf{2}, \mathbf{3}\}$. The ideal observables are $\tilde{A}_0$ (already defined earlier),
and $\tilde{A}_2$, $\tilde{B}_2$, $\tilde{B}_3$ same as before except shifted down by one basis element.

For $x, y \in \{0, 1\}$,
$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

For $\mathbf{x} \in \{\mathbf{0}, \mathbf{2}\}$ **and** $\mathbf{y} \in \{\mathbf{2}, \mathbf{3}\}$,
$$\begin{bmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{bmatrix}$$

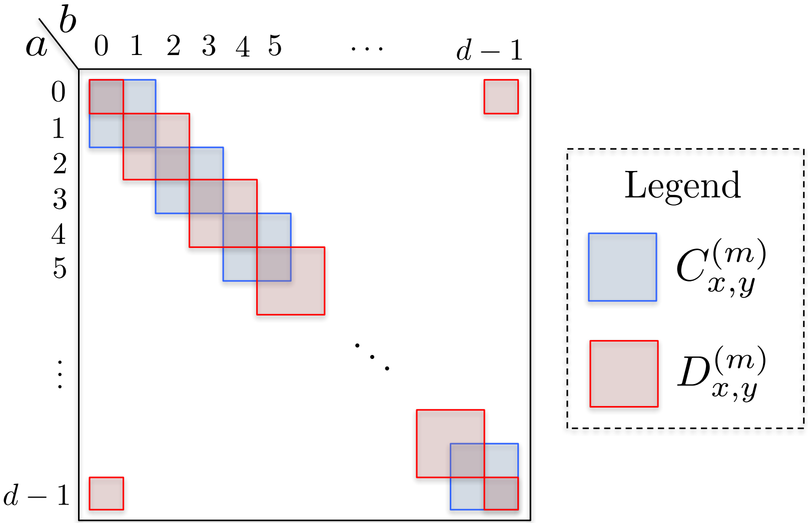
For the case of general $d$ even this means that for $\mathbf{x} \in \{\mathbf{0}, \mathbf{2}\}$ **and** $\mathbf{y} \in \{\mathbf{2}, \mathbf{3}\}$, the correlation tables have the form:

For the case of general $d$ even this means that for $\mathbf{x} \in \{\mathbf{0}, \mathbf{2}\}$ **and** $\mathbf{y} \in \{\mathbf{2}, \mathbf{3}\}$, the correlation tables have the form:

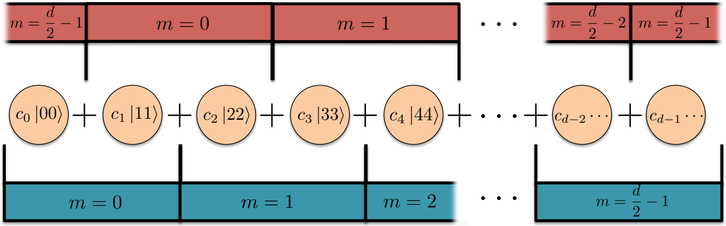
| $a \backslash b$ | $d-1$ | $0$ | $1$ | $2$ | $\cdots$ | $d-3$ | $d-2$ |
|---|---|---|---|---|---|---|---|
| $d-1$ | $D_{x,y}^{(0)}$ | | $0$ | $0$ | $\cdots$ | $0$ | $0$ |
| $0$ | | | $0$ | $0$ | $\cdots$ | $0$ | $0$ |
| $1$ | $0$ | $0$ | $D_{x,y}^{(1)}$ | | $\cdots$ | $0$ | $0$ |
| $2$ | $0$ | $0$ | | | $\cdots$ | $0$ | $0$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| $d-3$ | $0$ | $0$ | $0$ | $0$ | $\cdots$ | $D_{x,y}^{(\frac{d}{2}-1)}$ | |
| $d-2$ | $0$ | $0$ | $0$ | $0$ | $\cdots$ | | |

where, the $m$th block contains ideal tilted CHSH correlations for angle $\theta'_m = \arctan \frac{c_{2m+2}}{c_{2m+1}}$, weighted by $c_{2m+1}^2 + c_{2m+2}^2$.

More suggestively:



Or:




Questions $(x, y) \in \{0, 1\}^2$ serve to certify the even-odd pairs.
Questions $(x, y) \in \{0, 2\} \times \{2, 3\}$ the odd-even pairs.

Finally, the self-test is also robust [C. , Stark '17]:

**Theorem:**

Let $|\psi\rangle_{target} = \sum_{i=0}^{d-1} c_i |ii\rangle$. If Alice and Bob produce, on a joint state $|\psi\rangle$, a correlation that is $\epsilon$-close to the self-testing correlation described earlier, then their joint state is $O(d^3 \epsilon^{\frac{1}{4}})$-close to $|\psi\rangle_{target}$, up to a local isometry.

# Open questions

- Can we formulate these self-tests in terms of maximal violation of some Bell inequality?

# Open questions

- Can we formulate these self-tests in terms of maximal violation of some Bell inequality?
- Can we come up with non-local games that self-test a partially entangled state? What properties do they need to have?

# Open questions

- Can we formulate these self-tests in terms of maximal violation of some Bell inequality?
- Can we come up with non-local games that self-test a partially entangled state? What properties do they need to have?
- What about multipartite states? Less explored.

# Open questions

- Can we formulate these self-tests in terms of maximal violation of some Bell inequality?
- Can we come up with non-local games that self-test a partially entangled state? What properties do they need to have?
- What about multipartite states? Less explored.
    - Not guaranteed to have Schmidt decomposition

# Open questions

- Can we formulate these self-tests in terms of maximal violation of some Bell inequality?
- Can we come up with non-local games that self-test a partially entangled state? What properties do they need to have?
- What about multipartite states? Less explored.
  - Not guaranteed to have Schmidt decomposition
  - If they do they can be self-tested [Šupić, C. , Augusiak, Acín '17]

# Open questions

- Can we formulate these self-tests in terms of maximal violation of some Bell inequality?
- Can we come up with non-local games that self-test a partially entangled state? What properties do they need to have?
- What about multipartite states? Less explored.
    - Not guaranteed to have Schmidt decomposition
    - If they do they can be self-tested [Šupić, C. , Augusiak, Acín '17]
    - Some states cannot be self-tested due to complex conjugation ambiguity

# Open questions

- Can we formulate these self-tests in terms of maximal violation of some Bell inequality?
- Can we come up with non-local games that self-test a partially entangled state? What properties do they need to have?
- What about multipartite states? Less explored.
  - Not guaranteed to have Schmidt decomposition
  - If they do they can be self-tested [Šupić, C. , Augusiak, Acín '17]
  - Some states cannot be self-tested due to complex conjugation ambiguity
  - Can we self-test all multipartites states modulo complex conjugation?

# A conjecture for a Bell inequality

**GOAL**: For each pure bipartite entangled state, find a Bell inequality whose maximal violation is achieved uniquely by measurements on that state.

# A conjecture for a Bell inequality

**GOAL**: For each pure bipartite entangled state, find a Bell inequality whose maximal violation is achieved uniquely by measurements on that state.
i.e. an inequality self-testing that state.

# A candidate family

Some notation.

$$[CHSH]_p = \sum_{a,b,x,y \in \{0,1\}} (-1)^{a \oplus b - xy} p(a,b|x,y)$$

# A candidate family

Some notation.

$$[CHSH]_p = \sum_{a,b,x,y \in \{0,1\}} (-1)^{a \oplus b - xy} p(a,b|x,y)$$

Let $p \in \mathcal{C}_q^{3,4,d,d}$,

$$[CHSH^{(m)}]_p = \sum_{a,b \in \{2m,2m+1\}, x,y \in \{0,1\}} (-1)^{a \oplus b - xy} p(a,b|x,y)$$

# A candidate family

Some notation.

$$[CHSH]_p = \sum_{a,b,x,y\in\{0,1\}} (-1)^{a\oplus b - xy} p(a,b|x,y)$$

Let $p \in \mathcal{C}_q^{3,4,d,d}$,

$$[CHSH^{(m)}]_p := \sum_{a,b\in\{2m,2m+1\},x,y\in\{0,1\}} (-1)^{a\oplus b - xy} p(a,b|x,y)$$

# A candidate family

Some notation.

$$[CHSH]_p = \sum_{a,b,x,y\in\{0,1\}} (-1)^{a\oplus b - xy} p(a,b|x,y)$$

Let $p \in \mathcal{C}_q^{3,4,d,d}$,

$$[CHSH^{(m)}]_p = \sum_{a,b\in\{2m,2m+1\},x,y\in\{0,1\}} (-1)^{a\oplus b - xy} p(a,b|x,y)$$

$$\begin{aligned}[tCHSH^{(m)}(\alpha)]_p =& \alpha\left(p(a=2m|x=0) - p(a=2m+1|x=0)\right) \\ &+ [CHSH^{(m)}]_p\end{aligned}$$

# A candidate family

Define $[CHSH^{(m)\prime}]_p$, $[tCHSH^{(m)\prime}(\alpha)]_p$ in the same way but over answers $\{2m+1, 2m+2\}$ and questions $x \in \{0, 2\}, y \in \{2, 3\}$.

# A candidate family

Define $[CHSH^{(m)\prime}]_p$, $[tCHSH^{(m)\prime}(\alpha)]_p$ in the same way but over answers $\{2m+1, 2m+2\}$ and questions $x \in \{0,2\}, y \in \{2,3\}$.

**Candidate Bell Operator** - maximally entangled case, $|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$:

$$[\mathcal{B}]_p = \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)}]_p + \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)\prime}]_p$$

.

$$[\mathcal{B}]_p = \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)}]_p + \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)\prime}]_p$$

$$[\mathcal{B}]_p = \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)}]_p + \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)\prime}]_p$$

**Hope**: Any correlation that maximally violates the above must have the same block-diagonal structure as the self-testing correlations.

$$[\mathcal{B}]_p = \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)}]_p + \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)\prime}]_p$$

**Hope**: Any correlation that maximally violates the above must have the same block-diagonal structure as the self-testing correlations.

Then, there are weights $w_m$, $w'_m$, with $\sum_m w_m = \sum_m w'_m = 1$ such that

$$\begin{aligned}[\mathcal{B}]_p &\leq \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} w_m \cdot 2\sqrt{2} + \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} w'_m \cdot 2\sqrt{2} \\ &\leq 2\end{aligned}$$

Unfortunately, the fact that we are hoping for is still a conjecture.

Unfortunately, the fact that we are hoping for is still a conjecture.

We can try to add penalty terms to enforce the desired block-diagonal structure.

$$[\mathcal{B}]_p = \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)}]_p + \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)\prime}]_p - [CROSS]_p - [CROSS']_p$$

Unfortunately, the fact that we are hoping for is still a conjecture.

We can try to add penalty terms to enforce the desired block-diagonal structure.

$$[\mathcal{B}]_p = \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)}]_p + \frac{1}{2\sqrt{2}} \sum_{m=0}^{\frac{d}{2}-1} [CHSH^{(m)\prime}]_p - [CROSS]_p - [CROSS']_p$$

**This works for the maximally entangled case!**

**Candidate Bell Operator** - tilted case, $|\Psi\rangle = \sum_{i=0}^{d-1} c_i \, |ii\rangle$:

$$[\mathcal{B}]_p = \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{I_{\alpha_m}} [tCHSH^{(m)}(\alpha_m)]_p + \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{I_{\alpha'_m}} [tCHSH^{(m)\prime}(\alpha'_m)]_p$$

.

**Candidate Bell Operator** - tilted case, $|\Psi\rangle = \sum_{i=0}^{d-1} c_i |ii\rangle$:

$$[\mathcal{B}]_p = \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{I_{\alpha_m}} [tCHSH^{(m)}(\alpha_m)]_p + \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{I_{\alpha'_m}} [tCHSH^{(m)\prime}(\alpha'_m)]_p$$

.

where $I_\alpha = \sqrt{8 + 2\alpha^2}$, and $\alpha_m$, $\alpha'_m$ are the appropriate angles.

**Candidate Bell Operator** - tilted case, $|\Psi\rangle = \sum_{i=0}^{d-1} c_i \, |ii\rangle$:

$$[\mathcal{B}]_p = \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{I_{\alpha_m}} [tCHSH^{(m)}(\alpha_m)]_p + \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{I_{\alpha'_m}} [tCHSH^{(m)\prime}(\alpha'_m)]_p$$
$$- C \cdot \big([CROSS]_p - [CROSS']_p\big)$$

where $I_\alpha = \sqrt{8 + 2\alpha^2}$, and $\alpha_m$, $\alpha'_m$ are the appropriate angles.

**Candidate Bell Operator** - tilted case, $|\Psi\rangle = \sum_{i=0}^{d-1} c_i |ii\rangle$:

$$[\mathcal{B}]_p = \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{I_{\alpha_m}} [tCHSH^{(m)}(\alpha_m)]_p + \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{I_{\alpha'_m}} [tCHSH^{(m)\prime}(\alpha'_m)]_p - C \cdot \left([CROSS]_p - [CROSS']_p\right)$$

where $I_\alpha = \sqrt{8 + 2\alpha^2}$, and $\alpha_m$, $\alpha'_m$ are the appropriate angles.

**Still a conjecture for the tilted case!**

**Candidate Bell Operator** - tilted case, $|\Psi\rangle = \sum_{i=0}^{d-1} c_i |ii\rangle$:

$$[\mathcal{B}]_p = \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{I_{\alpha_m}}[tCHSH^{(m)}(\alpha_m)]_p + \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{I_{\alpha'_m}}[tCHSH^{(m)\prime}(\alpha'_m)]_p$$
$$- C \cdot \left([CROSS]_p - [CROSS']_p\right)$$

where $I_\alpha = \sqrt{8+2\alpha^2}$, and $\alpha_m$, $\alpha'_m$ are the appropriate angles.

**Still a conjecture for the tilted case!**

THANK YOU!

(Find me at coffee break if you want to chat more!)