Non-closure of the set of quantum correlations via graphs

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(Joint work with Vern I. Paulsen and Ken Dykema)

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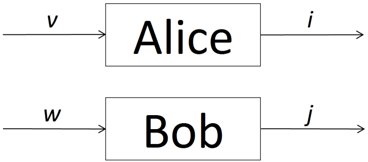

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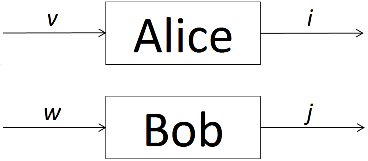

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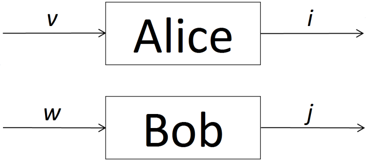

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Tsirelson [Ts1993] considered different mathematical models to describe these correlations and studied relationships among them.

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$$p(0,0|v,v) = t, \qquad p(0,1|v,v) = p(1,0|v,v) = 0, \qquad p(1,1|v,v) = 1-t,$$
$$p(0,0|v,w) = \frac{t(5t-1)}{4}, \qquad p(1,1|v,w) = \frac{(1-t)(4-5t)}{4},$$
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Then $(p(i,j|v,w)) \in \overline{C_q(5,2)} \setminus C_q(5,2)$

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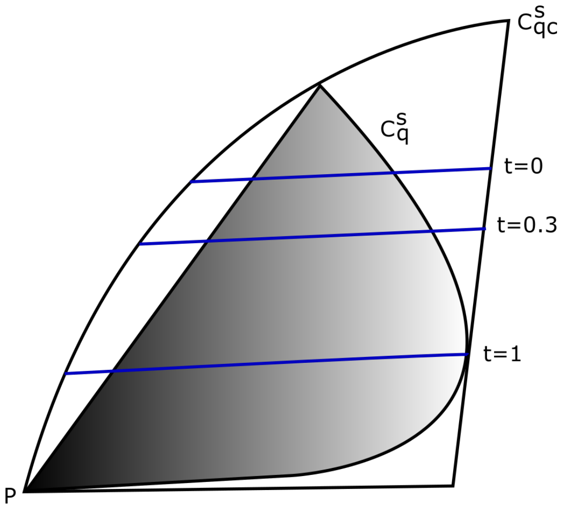
Theorem (PSSTW)

A correlation $(p(i,j|v,w))$ belongs to $C_{qc}^s(n,k)$ (resp. $C_q^s(n,k)$) if and only if there exists a unital C^*-algebra $\mathcal{A}$ (resp. finite dimensional C^*-algebra) with a tracial state τ and projections $\{e_{v,i} : 1 \le v \le n, 1 \le i \le k\}$ such that $\sum_i e_{v,i} = 1$ and

$$p(i,j|v,w) = \tau(e_{v,i} e_{w,j})$$

Idea



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$$\Gamma_r(t) = \{(p(i,j|v,w)) \in C_r^s(n,2) : p_A(0|v) = p_B(0|w) = t \ \forall v, w\},$$

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where $p_A(0|v) = \sum_j p(0,j|v,w)$ and $p_B(0|w) = \sum_i p(i,0|v,w)$. Given a graph $G = (V,E)$, define $f_r : [0,1] \to \mathbb{R}$ by

$$f_r(t) = \inf \left\{ \sum_{(v,w)\in E} p(0,0|v,w) : (p(i,j|v,w)) \in \Gamma_r(t) \right\}.$$

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By the above characterisation,

$$\begin{aligned} f_q(t) = \inf\{ \sum_{(v,w)\in E} \tau(e_v e_w) &: e_v \in \mathcal{A} \text{ projections}, \tau(e_v) = t \ \forall v \in V, \\ &\mathcal{A} \text{ finite-dimensional C*-algebra with tracial state } \tau\} \\ f_{qc}(t) = \inf\{ \sum_{(v,w)\in E} \tau(e_v e_w) &: e_v \in \mathcal{A} \text{ projections}, \tau(e_v) = t \ \forall v \in V, \\ &(\mathcal{A}, \tau) \text{ unital C*-algebra with tracial state } \tau\}. \end{aligned}$$

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Theorem (Vertex and edge transitive graphs)

Let $G = (V, E)$ be a vertex and edge transitive graph on n vertices and let $t \in [0, 1]$ be irrational. If $f_q(t)$ attains the infimum then there exists a nondegenerate interval $[r, s]$ having rational endpoints such that $t \in [r, s]$ and the restriction of f_q to $[r, s]$ is linear.

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Corollary ($G = K_5$)

If $C_q(5, 2)$ is closed, then for each irrational $t \in [0, 1]$ there exists a nondegenerate interval $[r, s]$ with $t \in [r, s]$ such that $f_q|_{[r,s]}$ is linear.

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- $\Sigma_3 = \left\{0, 1, \frac{3}{2}, 2, 3\right\}$

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Easy to compute:

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Theorem (Kruglyak, Rabanovich, Samoĭlenko)

Let $n \geq 5$ and let $\alpha \in (\beta_n, n - \beta_n)$. Then there exist finite dimensional projections $P_1, \cdots, P_n$ such that $\sum_{j=1}^{n} P_j = \alpha I$ if and only if $\alpha \in \mathbb{Q}$.

$G = K_5$

Theorem (Dykema, Paulsen, P.)

$C_q^s(5,2)$ *is not closed.*

Proof

1. Show that
$$f_q(t) \geq 5t(5t-1) \text{ if } t \in \left[\frac{1}{5}, \frac{4}{5}\right].$$
2. Show that $f_q(t) = 5t(5t-1)$ for all rational $t \in \frac{1}{5}[\beta_5, 5-\beta_5] \subset \left[\frac{1}{5}, \frac{4}{5}\right]$. Enough to show that $f_q(t) \leq 5t(5t-1)$ on that interval.

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   Let $5t \in [\beta_5, 5-\beta_5]$ be rational.

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Let $5t \in [\beta_5, 5-\beta_5]$ be rational. By the previous theorem, there exist five projections $P_1, \ldots, P_5 \in \mathbb{M}_k$ for some k such that $P_1 + \ldots + P_5 = 5tI_k$.

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Let $5t \in [\beta_5, 5-\beta_5]$ be rational. By the previous theorem, there exist five projections $P_1, \ldots, P_5 \in \mathbb{M}_k$ for some k such that $P_1 + \ldots + P_5 = 5tI_k$. Homogenise to get projections $\widetilde{P}_1, \ldots, \widetilde{P}_5 \in \mathbb{M}_{5k}$ such that

$$\mathrm{tr}_{5k}(\widetilde{P}_i) = \frac{1}{5k}\sum_{j=1}^{5}\mathrm{Tr}(P_j) = \frac{1}{5k}\mathrm{Tr}\left(\sum_{j=1}^{5}P_j\right) = \frac{1}{5k}(5tk) = t.$$

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Hence, we have five projections $\widetilde{P}_1, \ldots, \widetilde{P}_5 \in \mathbb{M}_{5k}$ such that $\mathrm{tr}_{5k}(\widetilde{P}_i) = t$, and $\sum_{j=1}^{5} \widetilde{P}_j = 5tI_{5k}$. Squaring, $\sum_{i\neq j} \widetilde{P}_i\widetilde{P}_j = 5t(5t-1)I_{5k}$. Taking normalised trace,
$$\sum_{i\neq j} \mathrm{tr}_{5k}(\widetilde{P}_i\widetilde{P}_j) = 5t(5t-1).$$

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3. Contradicts "piecewise" linearity.

Thank You.