

Recovering a local Hamiltonian from a single eigenstate

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With Xiaoliang Qi

I give you the eigenstate of a
Hamiltonian.

Can you find the Hamiltonian?

Motivation

- Hamiltonian tomography – experimental applications?
- Insights about many-body physics?

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The plan

- Introduce the **k -correlation matrix** as method for Hamiltonian recovery
- Consider sensitivity to noise, thermodynamic limit, continuum limit...
- Find interesting physics along the way?

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restricted type of

Can you find the Hamiltonian?

~~No.~~

Well, depends...

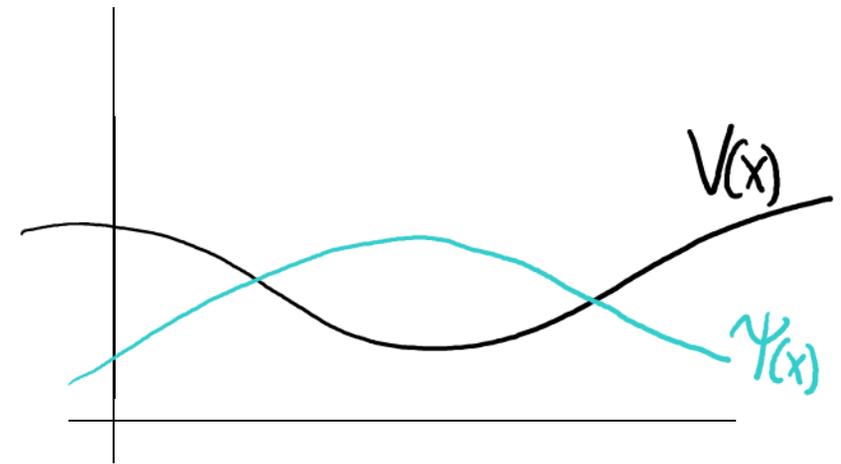
Toy example: Particle on a line

If you *know* the Hamiltonian is of the form: $H = p^2 + V(x)$

and I tell you an eigenstate $\psi(x)$

then you can find H by solving for $V(x)$,

$$V(x) = \frac{\psi''(x)}{\psi(x)} + \text{const.}$$



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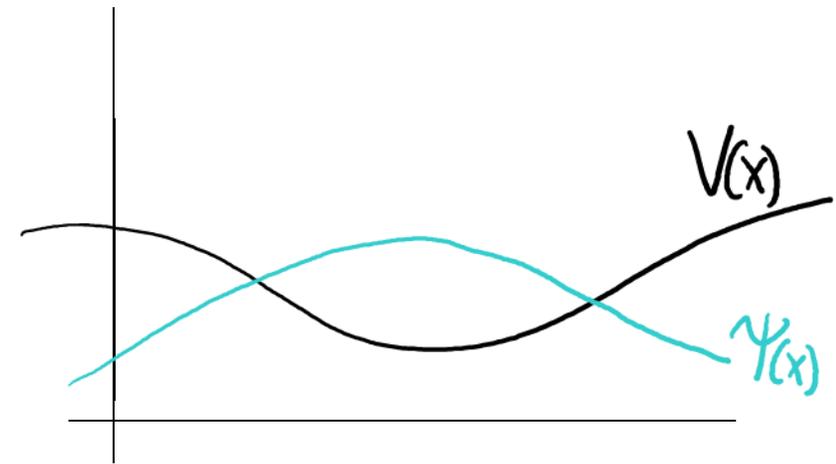
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and I tell you an eigenstate $\psi(x)$

Complete description of the eigenstate,
not just a single quantum copy!

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$$V(x) = \frac{\psi''(x)}{\psi(x)} + \text{const.}$$



I give you the eigenstate of a
Hamiltonian.

¹
local, many-body

Can you find the Hamiltonian?

~~No.~~

Let's see...

Local Hamiltonians

Finite-dimensional local Hilbert spaces, with finite-range local interactions, on finite (for now) lattices.

$$\mathcal{H} = \bigotimes_x \mathcal{H}_x$$

For instance, 1D spin chains with nearest-neighbor couplings:

$$H = \sum_{i=1}^n \sigma_i^z \sigma_{i+1}^z + \sigma_i^x + \sigma_i^z$$

Do we expect we can uniquely recover the Hamiltonian from the eigenstate?

Dimension counting

Most general nearest-neighbor Hamiltonian on chain of qubits:

$$H = \sum_{i=1}^n H_i = \sum_{i=1}^n \sum_{a=0}^3 \sum_{b=1}^3 c_{ab}^i \sigma_a^i \sigma_b^{i+1}$$

with $12n$ real parameters c_{ab}^i .

Linear in volume...

Most general state:

$$\dim(\mathcal{H}) = 2^n$$

parameters.

Exponential in volume...

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So maybe the full eigenstate has enough information?

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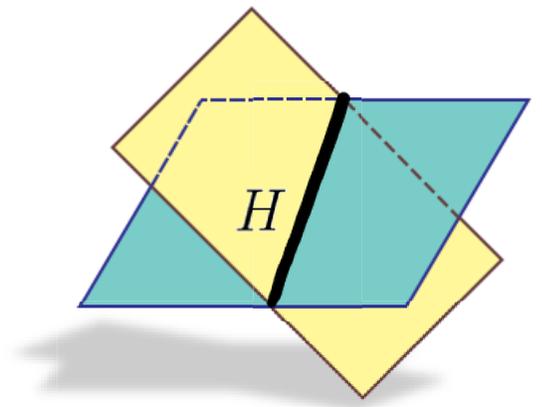
Related work: Harrison + Grover, 2015

- Focuses on the physics of the “eigenstate thermalization hypothesis” (ETH)
- Does not apply to ground states or other low-lying states (which I’ll focus on here)
- Somewhat different than present work, but fascinating physics! Ask me about it.

Most naïve method of recovery

If you actually knew the full eigenstate \mathcal{V} (component by component...), you could try to calculate the intersection of two linear subspaces in the space of operators:

- (1) Local Hamiltonians
- (2) Operators with \mathcal{V} as eigenstate



The intersection yields all candidate Hamiltonians.

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It's just calculating the intersection of linear subspaces... but in an exponentially large space!

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And needing to know the full quantum state in Hilbert space is very limiting.

(1)

But maybe using the full state is overkill?

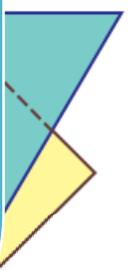
(2)

Fact: An eigenstate of a local Hamiltonian is fully determined by its two-body reduced density matrices.

The intersection yields all candidate Hamiltonians.

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Recovering the Hamiltonian from two-point correlations

- Simple but powerful
- Leads to efficient reconstructions, given the two-point correlations of all local variables (*w.r.t.* the state)
- Leads to interesting physics

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Follow the math with me for just a few lines!

Recovering the Hamiltonian from two-point correlations

Consider space of all local Hamiltonians on your system, e.g. nearest-neighbor spin chains:

$$LocHam = \text{span}(\{\sigma_i^a \sigma_{i+1}^b\}) \subset \mathcal{B}(\mathcal{H})$$

We're looking for operators $O \in LocHam$ that have our given state \mathcal{V} as an eigenstate.

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First simple observation:

$$\mathcal{V} \text{ is an eigenstate of } O \iff \langle O^2 \rangle_{\mathcal{V}} - \langle O \rangle_{\mathcal{V}}^2 = 0$$

Recovering the Hamiltonian from two-point correlations

Define the correlation matrix $M_{ij}^{(v)}$ for a state \mathcal{V} :

$$M_{ij}^{(v)} = \frac{1}{2} \langle L_i L_j + L_j L_i \rangle_v - \langle L_i \rangle \langle L_j \rangle_v$$

for an orthonormal basis $\{L_i\}$ of $LocHam$,

e.g. $\{L_i\} = \{\sigma_j^a \sigma_{j+1}^b\}$

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When considering k -nearest-neighbor Hamiltonians, then $M_{ij}^{(v)}$ is the symmetrized matrix of connected correlations of range- k local observables.

Recovering the Hamiltonian from two-point correlations

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Consider the correlation matrix as a bilinear form on space of range- k local operators, sending operators to their fluctuation w.r.t. the state.

Express operator O in basis $\{L_i\}$ as $O = c_i L_i$

and the eigenstate condition becomes:
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So vectors in the kernel of $M_{ij}^{(v)}$ have components c_i corresponding to local Hamiltonians $H = c_i L_i$ that have \mathcal{U} as an eigenstate.

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Find kernel of correlation matrix. (Diagonalize a matrix whose dimension grows like lattice size.)

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Each vector in the kernel is a set of coefficients for the terms in a local Hamiltonian. Each such Hamiltonian has the given state as an eigenstate.

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If kernel is one-dimensional, you've identified the unique Hamiltonian up to scaling!

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Does it actually work?

Let's try it numerically: Generate a 12-qubit, nearest-neighbor Hamiltonian with random couplings.

$$H = \sum_{i=1}^n H_i = \sum_{i=1}^n \sum_{a=0}^3 \sum_{b=1}^3 c_{ab}^i \sigma_a^i \sigma_b^{i+1}$$

Given an eigenstate, can we recover the Hamiltonian?

Yes, within error due to machine precision.

Okay, but does it work in general?

Knowing that recovery is possible for some particular Hamiltonian, you can **prove** that it works for almost all Hamiltonians on that geometry and system size. So, e.g.:

Now we know it works for all 12-qubit nearest-neighbor Hamiltonians, except on a measure zero set.

Examples of Hamiltonians in the measure zero set:

Systems that are non-interacting or have locally conserved charges.

But I meant, for systems that aren't tiny?

No rigorous proof yet. We can speculate and talk about proof strategies soon.

But first:

- Robustness of reconstruction to error
- What happens when you only know the state on a sub-region
- The physics of the correlation spectrum

Correlation spectrum: Spectrum of $M_{ij}^{(v)}$

Define the k -correlation spectrum of a state v as the spectrum of the correlation matrix $M_{ij}^{(v)}$ built using correlations of range- k local observables.

If the k -correlation spectrum has exactly one zero, v is the eigenstate of a k -local Hamiltonian, which can be uniquely recovered.

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We can define the k -correlation spectrum for *any* state!

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It tells you about more than just Hamiltonian recovery:

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Gives a local-unitary-invariant measure of correlations.

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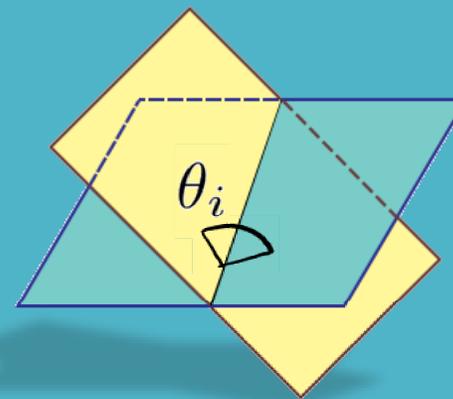
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The k -correlation spectrum is *also* the list of *principal angles* between two planes in operator space:

(a) plane of k -local operators, vs.

(b) plane of operators with zero fluctuation in the state



Sensitivity of reconstruction to error

Imagine we have the wrong correlation matrix – maybe we measured the correlations wrong, or we didn't start with an exact eigenstate.

How sensitive is the reconstruction?

$$\frac{1}{2} \sin 2\theta \leq \frac{\|\Delta M^{(v)}\|}{\lambda_2}$$

θ – angle between the true and reconstructed Hamiltonian

$\Delta M^{(v)}$ – error between true and given correlation matrix

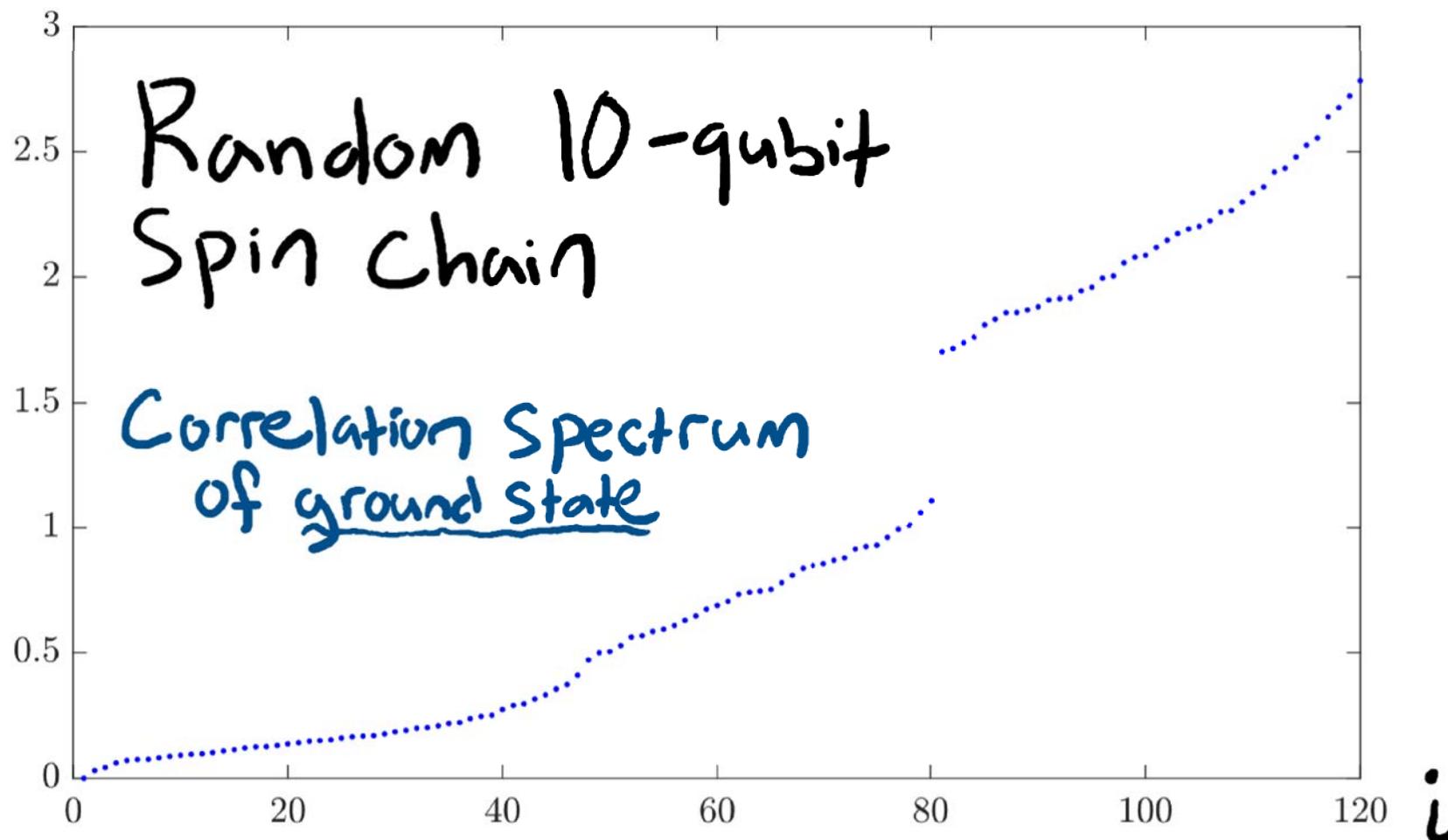
λ_2 – second-smallest eigenvalue of correlation spectrum

Taking a look at correlation spectra

Eigenvale λ_i

$$\lambda_1 \leq \dots \leq \lambda_{120}$$

$$\begin{aligned} \dim(\text{Loc Ham}) \\ &= 12 \cdot n \\ &= 120 \end{aligned}$$

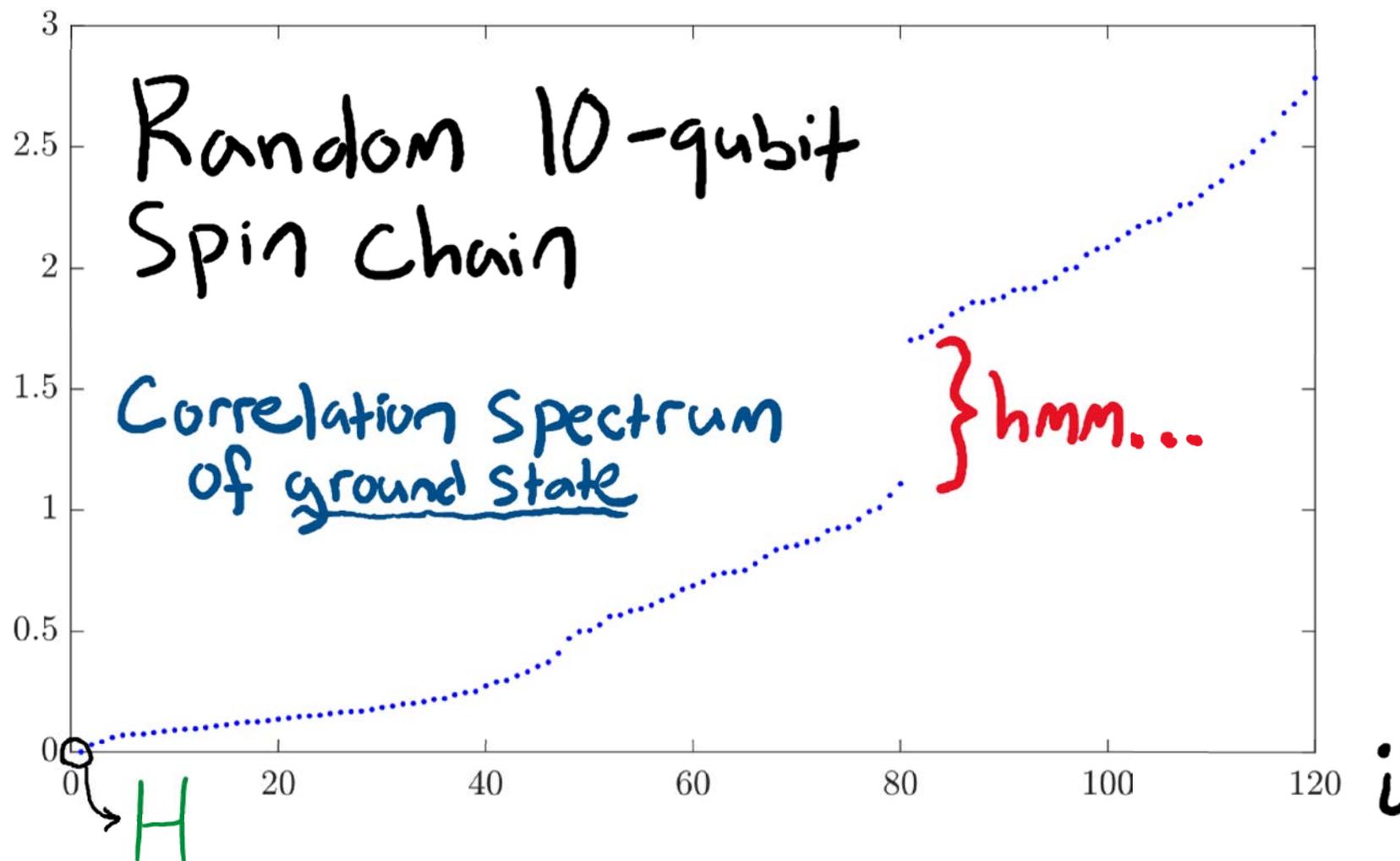


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Correlation spectra for translation-invariant states

The eigenvalues of the correlation matrix have corresponding eigenoperators.

For the correlation matrix of a translation-invariant state, the eigenoperators are translation invariant up to a phase, i.e. they're of the form:

$$O = \sum_{x=1}^n O_x e^{i\frac{q}{n}2\pi x}$$

for some “momentum number” q , where O_x is some fixed k -site interaction term centered at lattice site x .

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→ Associate a momentum q to every correlation eigenvalue.

Correlation spectra for translation

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For gapped ground states, we have exponential decay of correlations.

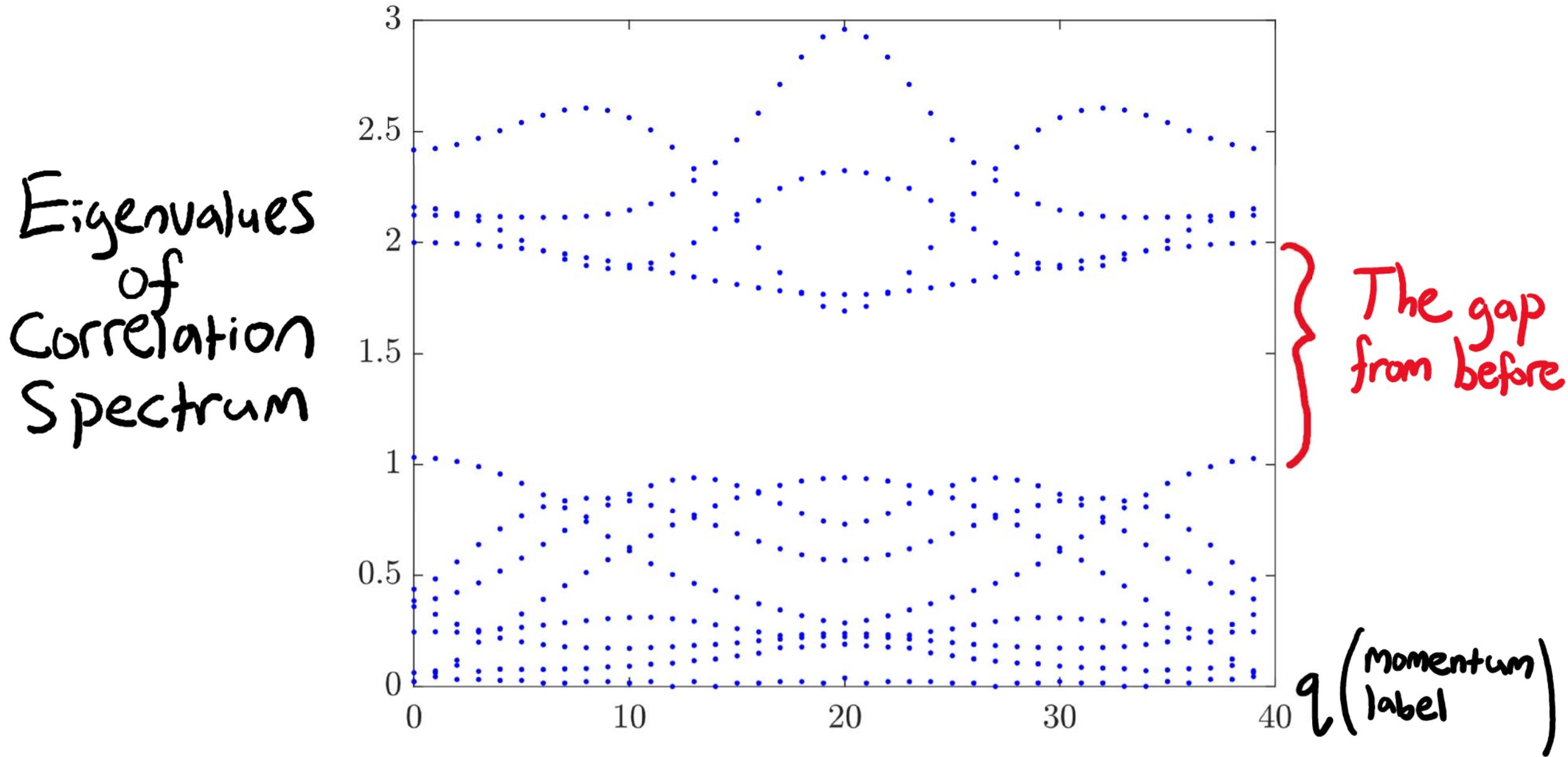
Leads to smooth, banded correlation spectrum in the thermodynamic limit.

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Ground state of a random, translation-invariant, 40-qubit chain



Ground state of a random, translation-invariant, 40-qubit chain

The gap in the correlation spectrum only appears for the ground state. Associated with area law, exponential decay of correlations?

What can we learn from band structure of state?

Note the smallest band, with connection to energy diffusion.

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0 10 20 30 40 (label momentum)

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In the thermodynamic limit, does reconstruction remain robust?

- **Yes (speculation)** for translation-invariant gapped ground states in the thermodynamic limit. And you can approximately reproduce the correlation functions with access to only a finite sub-region (easy fact).
- **I don't know** for lots of other interesting cases.

Taking stock

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- Found interesting structure in bands of the correlation spectrum. What else can you learn from the spectrum of eigenoperators?
- Experimental application? Measure the ground state correlations of a qudit system as a method of determining or verifying the Hamiltonian?

More material

A nod at the continuum

We can ask about recovering the Hamiltonian from the ground state for lattice-regularized QFTs, and then what happens as we shrink the lattice spacing (while tuning the couplings using the renormalization group).

For Lorentz-invariant QFTs, there's a trick for getting the Hamiltonian from the ground state, using the modular Hamiltonian. This method should work approximately for the lattice-regularized case.

More generally, **any** finite-energy-density state (doesn't need to be an eigenstate!) will have "same UV structure" — so any such state might determine the Hamiltonian? You just need to probe the state at the UV (lattice cutoff) scale.

Advertisement: Semi-related work

We've been assuming we know the proper tensor factorization of the Hilbert space into local degrees of freedom, i.e.

$$\mathcal{H} = \bigotimes_x \mathcal{H}_x$$

What if we didn't know that tensor factorization already – what if we just knew the trajectory of some non-equilibrium state in the Hilbert space?

Could we recover the right local degrees of freedom – finding a sort of preferred local basis in Hilbert space – knowing only the the dynamics of the state?

Paper: *“Locality from the Spectrum”* (Cotler, Penington, DR)

Advertisement: Things I'd love to know

While we're on the topic of generic local lattice Hamiltonians...

Are they usually **gapped**? [Quiz question! Answer: No.]

What about translation-invariant systems? [I'd guess gapped, but others say no!]

And... do they have **zero-momentum ground states**?

And... do they have **quasi-particle behavior**?

Previous work: Harrison+Grover, 2015

If we assume the Hamiltonian satisfies the “eigenstate thermalization hypothesis” (ETH),

then for an *excited eigenstate at nonzero energy-density*, the reduced density matrix on any region satisfies:

$$\begin{aligned}\rho_A &\approx \frac{\text{Tr}_{\bar{A}}(e^{-\beta H})}{\text{Tr}(e^{-\beta H})} \\ &\approx \frac{e^{-\beta H_A}}{\text{Tr}(e^{-\beta H_A})}\end{aligned}$$

Then we can find the Hamiltonian on the region:

$$H_A \approx \log(\rho_A) + \text{const.}$$

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Leads to a fascinating study of thermalization
(H+G, 2008).

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$\rho_A \approx$

Limited as method of Hamiltonian recovery:

- Doesn't work for ground states or other low-lying states (which I'll focus on most)
- Requires knowing full quantum state on a sub-region
- Requires ETH

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Wait: the ground-state has no UV information!

Well, if it's the exact ground state, it has some information about the UV when you probe it at short distances.

Wait: Do approximate ground state projectors ruin everything?

Given the ground state of a range- k Hamiltonian, there will generically be many quasi-local Hamiltonians with the same ground state. But such Hamiltonians will be non-generic among quasi-local Hamiltonians of similar interaction length.

Heuristic intuition for when recovery is possible

It can get confusing thinking about quasi-local Hamiltonians, continuum limits, and QFTs.

Heuristic intuition:

You can recover the Hamiltonian from the ground state when the state's correlation length is longer than the length scale over which DOF are directly coupled.