Generic Local Hamiltonians are Gapless

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QIP 2018, Delft (Jan. 2018)

Interactions in quantum mechanics are modeled by "Hermitian" matrices and operators:

$$H = H^\dagger$$

Therefore the eigenvalues are real and can be ordered:

$$E_0 \leq E_1 \leq E_2 \leq \cdots \quad .$$

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What are the generic properties of H?

Definition

Generic means typical behavior. Mathematically, Generic means almost surely, or with probability one.

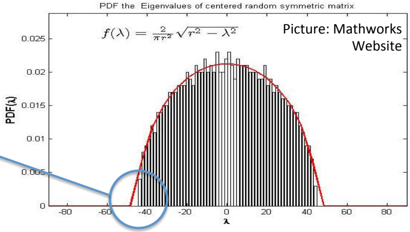
Generic instances are modeled by random matrices.

1. Generic properties of H?

1 Generic properties of *H*?

- Eigenvalues follow a **Wigner-Dyson** Semi-circle law
- The edge Statistics follow the **Tracy-Widom** Laws
- The eigenvalue repulsion rigorously proved in the bulk
- Their (non-commuting algebra) is described by **Free Probability Theory**

Random Matrix Theory



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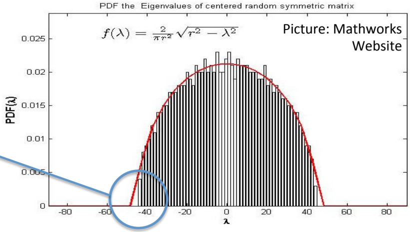

2 What subset is physical?

① Generic properties of *H*?

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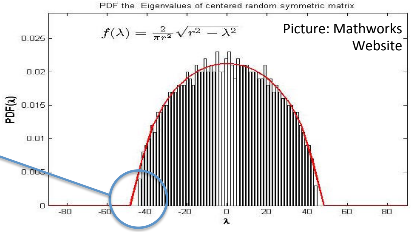


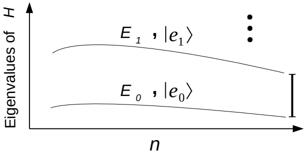
② What subset is physical?

Very Non-Generic !

$$H = \sum_{j=1}^{2n-1} H_{j,j+1}$$

$\cdots$ $H_{j,j+1}$ (j — $j+1$) $\cdots$

Locality !



$(E_1 - E_0) = \Delta$: Gap

Ground state Gap !

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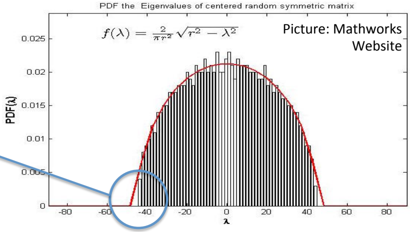

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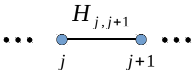

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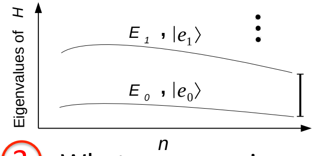

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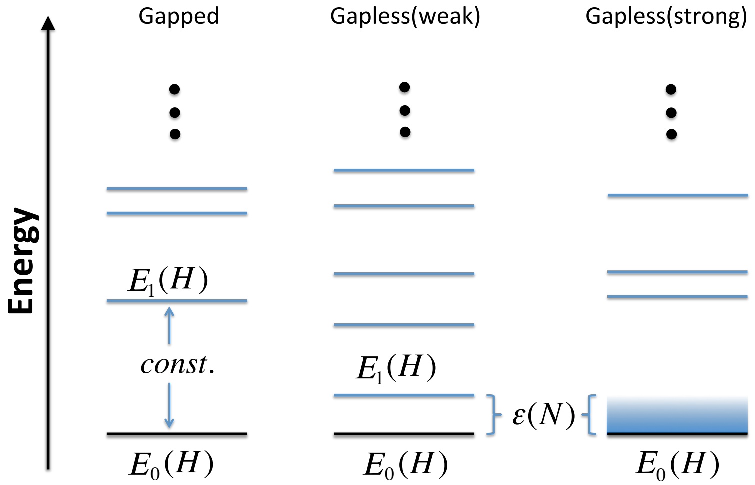
③ What are generic properties of <u>physical</u> Hamiltonians?

Since you might be interested ...

Quantity	Result	Reference
Density of States, generic spin chains	Well captured by ideas in Random matrix Theory	RM-, A. Edelman, (PRL 2011)
Density of States, Anderson model	Well captured by Free Probability Theory	RM-, A. Edelman +MIT Chemists, (PRL 2012)
Frustration free-ness and G.S. degeneracy, generic spin chains	Analytically solved using matrix product representation	RM-, Farhi, Goldstone, Nagaj, Osborne, Shor, (PRA 2010)

Today: The gap of generic local Hamiltonians (any dimension)

Definition of Gapless



Importance of the Gap

- Gap and correlation functions are intimately connected
- Gap and entanglement scalings are believed to be interdependent
- Gapped systems are easier to classically simulate in one-dimension (believed to hold in higher dimensions)
- Gapless-ness is a necessary condition for quantum phase transitions : $\lim_{n\to\infty} \Delta = 0$

Previous work

Solving the gap in general is a very hard problem:

- Gap problem is undecidable. **Cubitt, Perez-Garcia, Wolf, Nature (2015)**
- Frustration free translationally invariant qubit chains ($s = 1/2$)

$$H(\psi) = \sum_{j=1}^{N-1} |\psi\rangle_{j,j+1}\langle\psi|,$$

where $|\psi\rangle$ is generic.

$H(\psi)$ is strictly translationally invariant and generically gapped.

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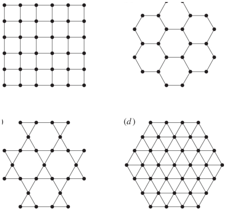
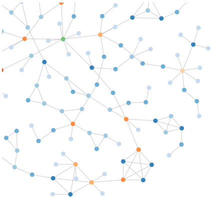
Today: Lack of an energy gap is completely a generic property in the physical submanifold.

Problem Statement

- Take the Hamiltonian

$$H = \sum_{\langle i,j \rangle} H_{ij} \quad : \qquad i,j \text{ are neighbours}$$





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Problem Statement and Assumptions

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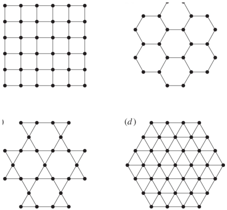
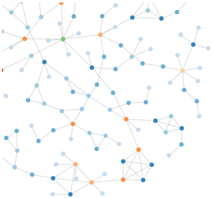




- Let each H_{ij} be independent of others and a generic matrix. H can be translationally invariant in a disordered sense.
  - E.g., GOE, GUE, GSE, Wishart [These come up in Many-Body Localization. Random exchange model. Griffiths' singularities]
  - E.g., Random projectors [Important in quantum complexity theory and quantum Satisfiability (qSAT)]

Assumption : Eigenvalues of H_{ij} *can* all get arbitrary close. Matrix close to a multiple of an identity. E.g. Gaussian ensembles:

$$G = \frac{A + A^\dagger}{2}; \qquad A \approx I$$

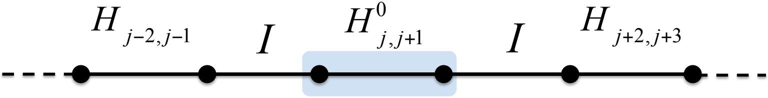
In Matlab and for qubit interactions:

```
A=randn(4)+1i*randn(4); G=0.5*(A+A');
```

Theorem

H almost surely has a continuous density of states above the ground state, if $H_{i,j}$'s are independent and each $H_{i,j}$ has a continuous joint distribution of eigenvalues that obeys Assumption.

Proof Idea: Rare local regions



$$H = \sum_k H_{k,k+1}$$

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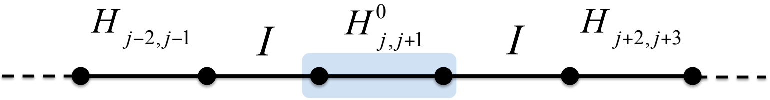

$$H = \sum_k H_{k,k+1} = H_E \otimes \mathbb{I}_{j,j+1} + \mathbb{I}_E \otimes H^0_{j,j+1}$$

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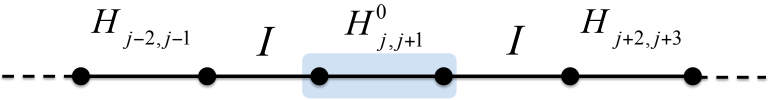

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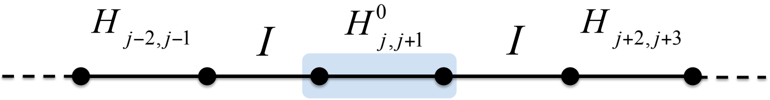

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* Apply Weyl inequalities. Perturbation theory just won't cut it!

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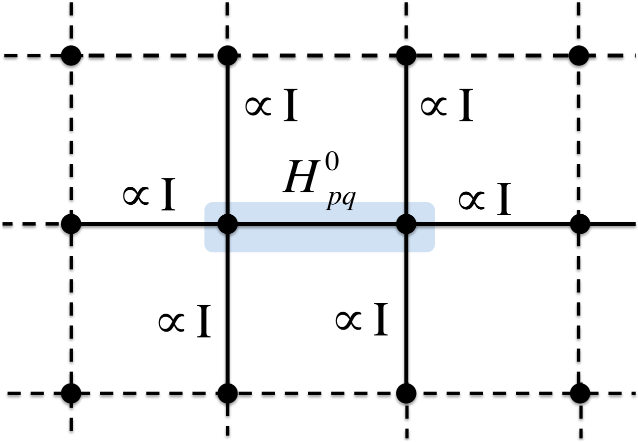

Corollary

If local eigenvalue distribution is discrete with Haar eigenvectors, then the ground state is almost surely exactly degenerate and can be represented as a product state.

Example: Gaussian unitary ensemble

Let M_n be a G(O/U/S)E matrix; the measure for general β is

$$\mu_n(\beta) = C_n(\beta)\ e^{-\frac{\beta}{4}\mathrm{tr}(M_n)^2} dM_n.$$

Let us fix a real number a, and compute the probability

$$\mathrm{P}\left[\|M_n - a\mathbb{I}_n\|_F \leq \varepsilon\right]$$

β	Entries	Matrix
1	Real	Symmetric
2	Complex	Unitary
4	Quaternion	Symplectic

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$$\mathrm{P}\left[\|M_n - a\mathbb{I}_n\|_F \leq \varepsilon\right] \quad = \quad (C_n(\beta) + o(1))\, e^{-\beta n a^2/4} \varepsilon^{n^2}$$

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$$\begin{aligned}
\mathrm{P}\left[\|M_n - a\mathbb{I}_n\|_F \le \varepsilon\right] &= \left(C_n(\beta) + o(1)\right) e^{-\beta n a^2/4}\varepsilon^{n^2} \\
\mathrm{P}\left[\exists a \in \mathbb{R} : \|M_n - a\mathbb{I}_n\|_F \le \varepsilon\right] &= \left(C_n'(\beta) + o(1)\right)\sqrt{\frac{\pi}{\beta n}}\varepsilon^{n^2}.
\end{aligned}$$

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Example: Gaussian unitary ensemble

The probability of a rare local region is (d 'spin states' and $n = d^2$ local eigenvalues)

$$p = (K_d(\beta) + o(1))\, \varepsilon^{zd^4+4}.$$

For the gap to be ε small, the expected number of terms in the Hamiltonian is (z overlapping terms)

$$N \quad \sim \quad \varepsilon^{-zd^4-4}$$

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$$\begin{aligned} N &\sim \varepsilon^{-zd^4-4} \\ \varepsilon(N) &\sim N^{-1/(zd^4+4)} \end{aligned}$$

Relaxing the assumption

Generic Projectors Are Gapless

Theorem

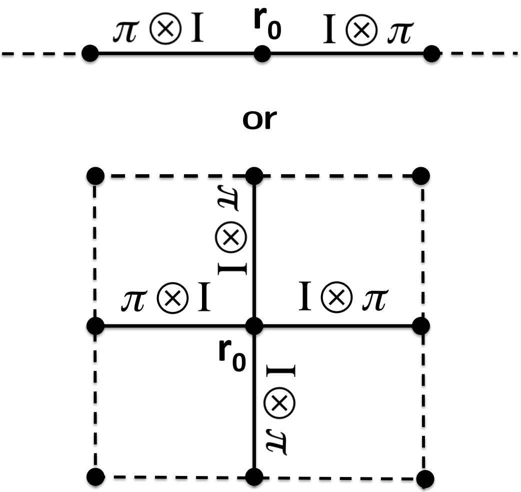
H is almost surely gapless if $H_{i,j}$ are random rank-r projectors with Haar eigenvectors and r is

1. Fixed and at most $d(d-1)$.

2. Vary randomly among the terms in the Hamiltonian.

*** * Note that we don't have the assumption anymore.**

Rare regions



Corollaries

Corollary

If local eigenvalue distribution is discrete with Haar eigenvectors, then the Hamiltonian is almost surely gapless.

Remark

- The probability of rare regions is a function of local distributions (not universal)
- The gap scaling is a function of local statistics (not universal)
- Really need Weyl inequality and perturbation theory won't do!

Open problems

- Translationally invariant Hamiltonians
- Do there exist global configurations with smaller gaps?
- Comparing our formulas against a serious numerical study of the gap scaling for G(O/U/S)E local terms

Lastly

Thank you

Proof Idea

At $\varepsilon = 0$, define

$$H_0 \equiv \mathbb{I} \otimes H^0_{p,q} + \sum_{|\langle i,j \rangle|=1} \beta_{i,j} \mathbb{I} \otimes \mathbb{I}_{i,j} + \sum_{|\langle i,j \rangle|\geq 2} \mathbb{I} \otimes H_{i,j}$$

Let λ_E be the smallest eigenvalue of $\sum_{|\langle i,j \rangle|\geq 2} \mathbb{I} \otimes H_{i,j}$.
1. $H_{pq} = H^0_{pq} + \delta H_{pq}$, where $||\delta H_{pq}|| \leq \varepsilon$
2. The summands of distant 1 terms are $H_{i,j} = \beta_{i,j}\mathbb{I}_{i,j} + \delta H_{ij}$, where $||\delta H_{i,j}|| \leq \varepsilon$.

Weyl's inequalities : the two smallest eigenvalues of H, denoted by $\lambda^{\varepsilon,k}_{min}$ with $k \in \{1,2\}$, obey

$$\lambda_E + \beta + \lambda_0 - B \leq \lambda^{\varepsilon,k}_{min} \leq \lambda_E + \beta + \lambda_0 + B, \qquad (1)$$

where $B = ||\delta H_{pq} + \sum_{|\langle i,j \rangle|=1} \delta H_{ij}|| \leq \varepsilon(z+1)$, and $\beta \equiv \sum_{|\langle i,j \rangle|=1} \beta_{i,j}$.