

Moderate deviation analysis for c-q channels (and hypothesis testing)

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QuTech, TU Delft

Two techniques

Refined small-deviation analysis:

- “*Moderate deviation analysis for classical communication over quantum channels*”, **Christopher T. Chubb**, Vincent Y.F. Tan, and Marco Tomamichel, *Communications in Mathematical Physics* (2017) 355: 1283, arXiv:1701.03114.

Refined large-deviation analysis:

- “*Moderate Deviation Analysis for Classical-Quantum Channels and Quantum Hypothesis Testing*”, **Hao-Chung Cheng** and Min-Hsiu Hsieh, *IEEE Transactions on Information Theory* (to appear), arXiv:1701.03195.
- “*Quantum Sphere-Packing Bounds with Polynomial Prefactors*”, **Hao-Chung Cheng**, Min-Hsiu Hsieh, and Marco Tomamichel, arXiv:1704.05703.

Classical communication

Suppose Alice wants to send classical information to Bob, via some channel.

There is a trade-off between three quantities:

- Number of channel uses
- Amount of information transmitted
- Error probability

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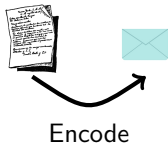


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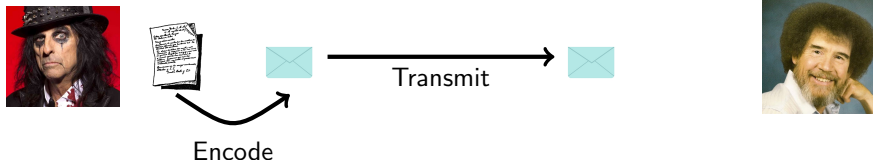


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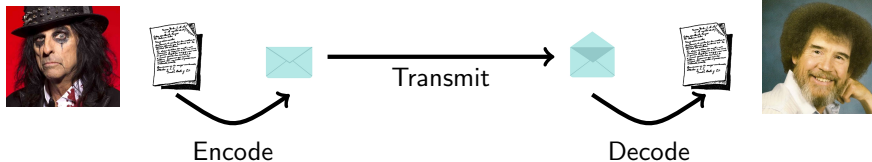


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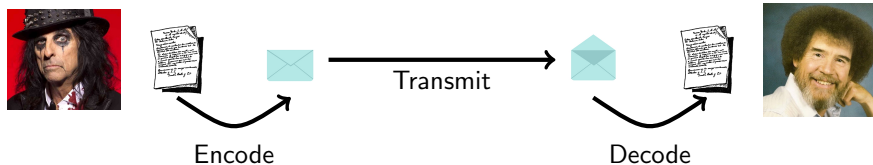


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Classical channels versus quantum channels

If we have access to a quantum channel, then quantum encoding/decoding can allow us to transmit more information with less error than classical encoding/decoding.

A simple example is a bit-flip channel

$$\mathcal{E}(\rho) = pX\rho X + (1 - p)\rho.$$

Classically: Either we send many noisy bits, or fewer encoded bits.

Quantumly: Simply transmit our bits noiselessly in the X basis $\{|+\rangle, |-\rangle\}$.

Classical communication over a quantum channel

We are going to consider coding of classical-quantum channels.

For c-q channel \mathcal{W} , a (n, R, ϵ) -code is an encoder E and decoding POVM $\{D_i\}$ such that

$$\frac{1}{2^{nR}} \sum_{m=1}^{2^{nR}} \text{Tr} \left[\mathcal{W}^{\otimes n} \left(\bigotimes_{i=1}^n E_i(m) \right) D_m \right] \geq 1 - \epsilon$$

We will be concerned with the trade-off between the block-length n , the rate R , and the error probability ϵ . We define the optimal rate/error probability as

$$R^*(\mathcal{W}; n, \epsilon) := \max \{ R \mid \exists (n, R, \epsilon)\text{-code} \},$$
$$\epsilon^*(\mathcal{W}; n, R) := \min \{ \epsilon \mid \exists (n, R, \epsilon)\text{-code} \}.$$

Asymptotics

For a constant error probability ϵ , the Strong Converse Theorem tells us the rate approaches a constant known as the capacity

$$\lim_{n \rightarrow \infty} R^*(\mathcal{W}; n, \epsilon) = C(\mathcal{W}).$$

Equivalently this means that the error probability must go to 0 to 1 either side of the capacity

$$\lim_{n \rightarrow \infty} \epsilon^*(\mathcal{W}; n, R) = \begin{cases} 0 & : R < C(\mathcal{W}) \\ 1 & : R > C(\mathcal{W}) \end{cases}$$

This tells us we can have either $R \rightarrow C$ OR $\epsilon \rightarrow 0$.

How fast are these convergences? Can we do both?

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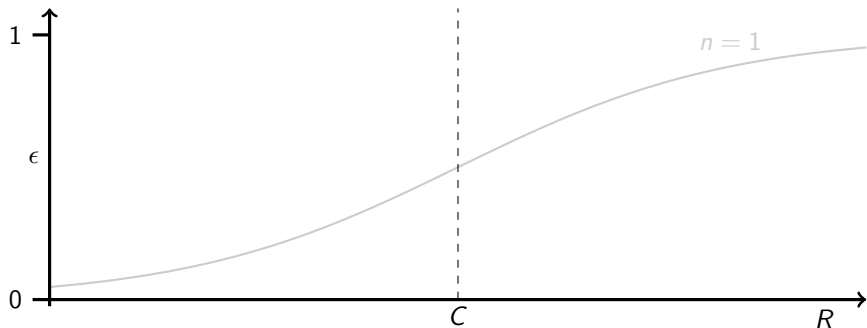
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Small and large deviations

How fast are the convergences $R \rightarrow C$ or $\epsilon \rightarrow 0$ as $n \rightarrow \infty$?



Small deviation (Tomamichel and Tan 2015)

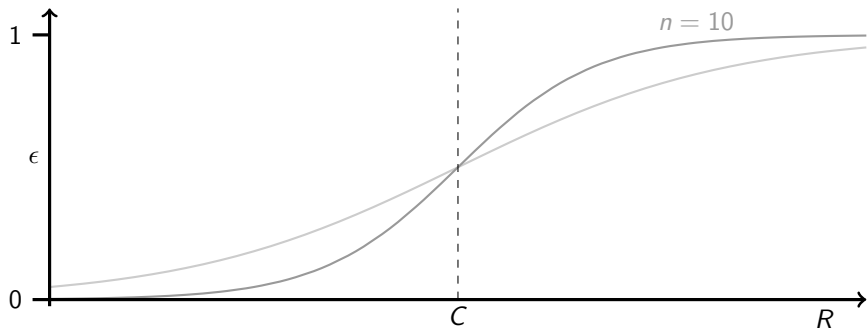
$$R^*(n, \epsilon) = C + \sqrt{\frac{V}{n}} \Phi^{-1}(\epsilon) + o\left(\frac{1}{\sqrt{n}}\right) \quad \epsilon \in (0, \frac{1}{2})$$

Large deviation (Partial progress)

$$\ln \epsilon^*(n, R) = -n \cdot E(R) + o(n) \quad R < C$$

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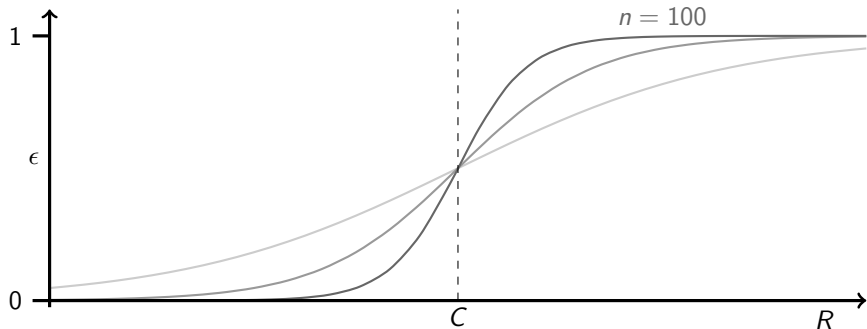
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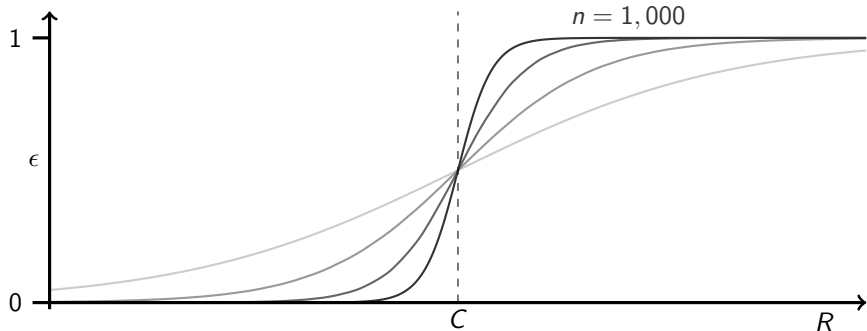
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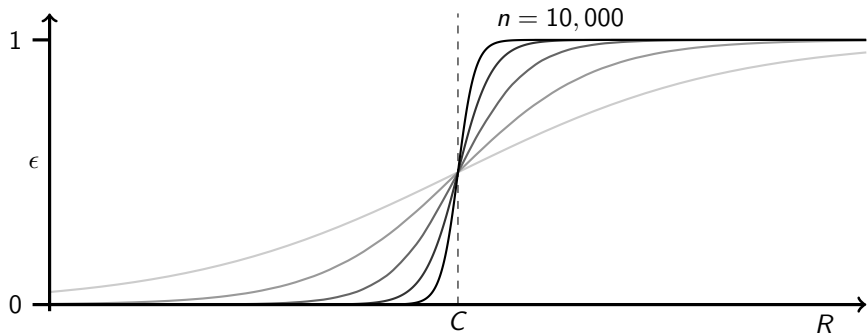
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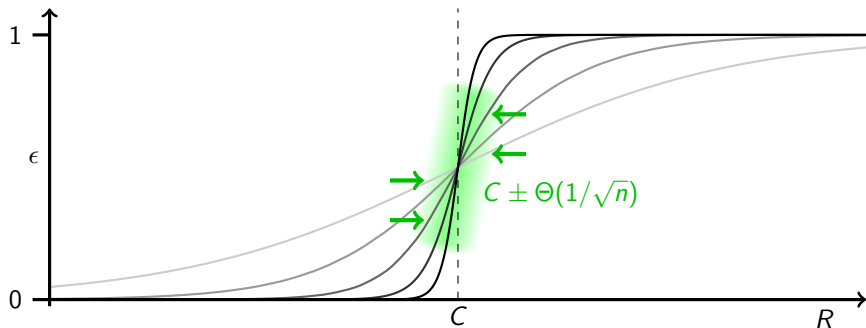
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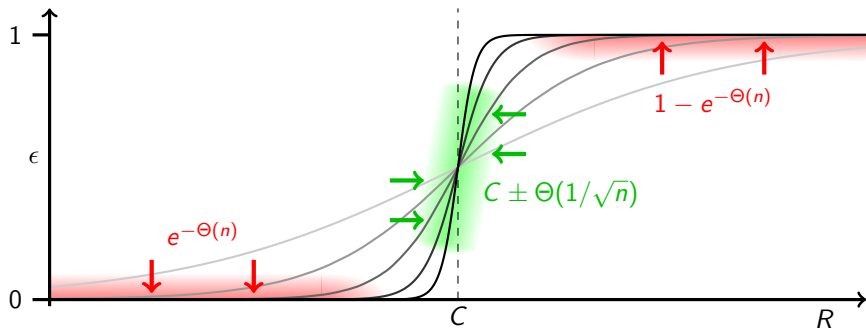
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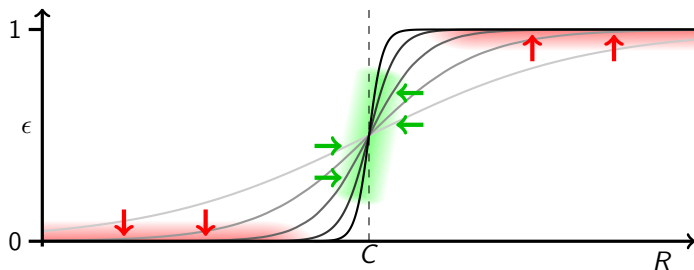
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Moderate deviations

What if we want $R \rightarrow C$ AND $\epsilon \rightarrow 0$?



Moderate deviation (This work)

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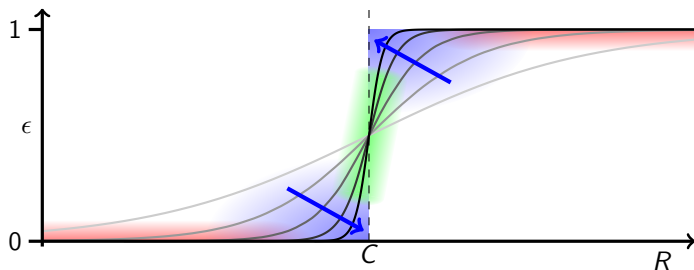
$$R^*(n, \epsilon_n) = C - \sqrt{2V}a_n + o(a_n) \quad \text{for } \epsilon_n = e^{-na_n^2},$$

or equivalently

$$\ln \epsilon^*(n, R_n) = -\frac{na_n^2}{2V} + o(na_n^2) \quad \text{for } R_n = C - a_n.$$

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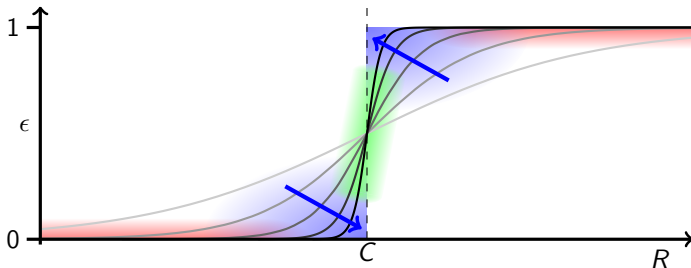
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Concentration inequalities

Take $\{X_i\}$ iid with $\mathbb{E}[X_i] = 0$ and $\text{Var}[X_i] =: V$, and $\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i$.

Asymptotic (Law of large numbers)

$$\lim_{n \rightarrow \infty} \Pr [\bar{X}_n \geq t] = \begin{cases} 1 & t < 0, \\ 0 & t > 0. \end{cases}$$

Small deviation (Berry-Esseen)

$$\Pr \left[\bar{X}_n \geq \frac{\epsilon}{\sqrt{n}} \right] = Q \left(\frac{\epsilon}{\sqrt{V}} \right) + \mathcal{O} \left(\frac{1}{\sqrt{n}} \right) \quad \epsilon \in (0, 1)$$

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Hypothesis testing

We want to test between two hypotheses, ρ and σ . For a binary POVM $\{A, I - A\}$, we define the type-I and type-II errors as

$$\alpha(A; \rho, \sigma) := \text{Tr}(I - A)\rho, \quad \beta(A; \rho, \sigma) := \text{Tr} A\sigma,$$

and the ϵ -hypothesis-testing divergence

$$D_h^\epsilon(\rho\|\sigma) := -\log \min_{0 \leq A \leq I} \{\beta(A; \rho, \sigma) \mid \alpha(A; \rho, \sigma) \leq \epsilon\}.$$

If we now consider testing between $\rho^{\otimes n}$ and $\sigma^{\otimes n}$, then the asymptotic behaviour is given by Quantum Stein's Lemma.

Asymptotics (Hiai and Petz 1991, Ogawa and Nagaoka 1999)

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Deviation results for hypothesis testing

Small deviation (Tomamichel and Hayashi 2013, Li 2014)

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Bounding the rate

For this we can use the one shot bounds

$$R^*(1, \epsilon) \geq \sup_{P_X} D_h^{\epsilon/2}(\pi_{XY} \| \pi_X \otimes \pi_Y) - \mathcal{O}(1), \quad (\text{Wang and Renner 2012})$$

$$R^*(1, \epsilon) \leq \inf_{\sigma} \sup_{\rho \in \text{Im}(\mathcal{W})} D_h^{2\epsilon}(\rho \| \sigma) + \mathcal{O}(1), \quad (\text{Tomamichel and Tan 2015})$$

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Reducing hyp. testing to concentration inequalities

To give a moderate deviation analysis of the HTD, we will use concentration bounds. First we see it is related to tail bounds of the Nussbaum-Szkoła distributions¹

$$P^{\rho,\sigma}(a, b) := r_a |\langle \phi_a | \psi_b \rangle|^2 \quad \text{and} \quad Q^{\rho,\sigma}(a, b) := s_b |\langle \phi_a | \psi_b \rangle|^2,$$

where we have eigendecomposed our states $\rho := \sum_a r_a |\phi_a\rangle\langle\phi_a|$ and $\sigma := \sum_b s_b |\psi_b\rangle\langle\psi_b|$. These reproduce the first two moments of our states

$$D(P^{\rho,\sigma} \| Q^{\rho,\sigma}) = D(\rho \| \sigma) \quad \text{and} \quad V(P^{\rho,\sigma} \| Q^{\rho,\sigma}) = V(\rho \| \sigma).$$

Specifically for iid $Z_i = \log P^{\rho,\sigma} / Q^{\rho,\sigma}$ and $(a_i, b_i) \sim P^{\rho,\sigma}$, then²

$$\frac{1}{n} D_h^{\epsilon_n}(\rho^{\otimes n} \| \sigma^{\otimes n}) \geq \sup \left\{ R \mid \Pr \left[\sum_{i=1}^n Z_i \leq \epsilon_n / 2 \right] \right\} - \mathcal{O}(\log 1 / \epsilon_n),$$

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¹Nussbaum and Szkoła 2009.

²Tomamichel and Hayashi 2013

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Specifically for iid $Z_i = \log P^{\rho,\sigma} / Q^{\rho,\sigma}$ and $(a_i, b_i) \sim P^{\rho,\sigma}$, then²

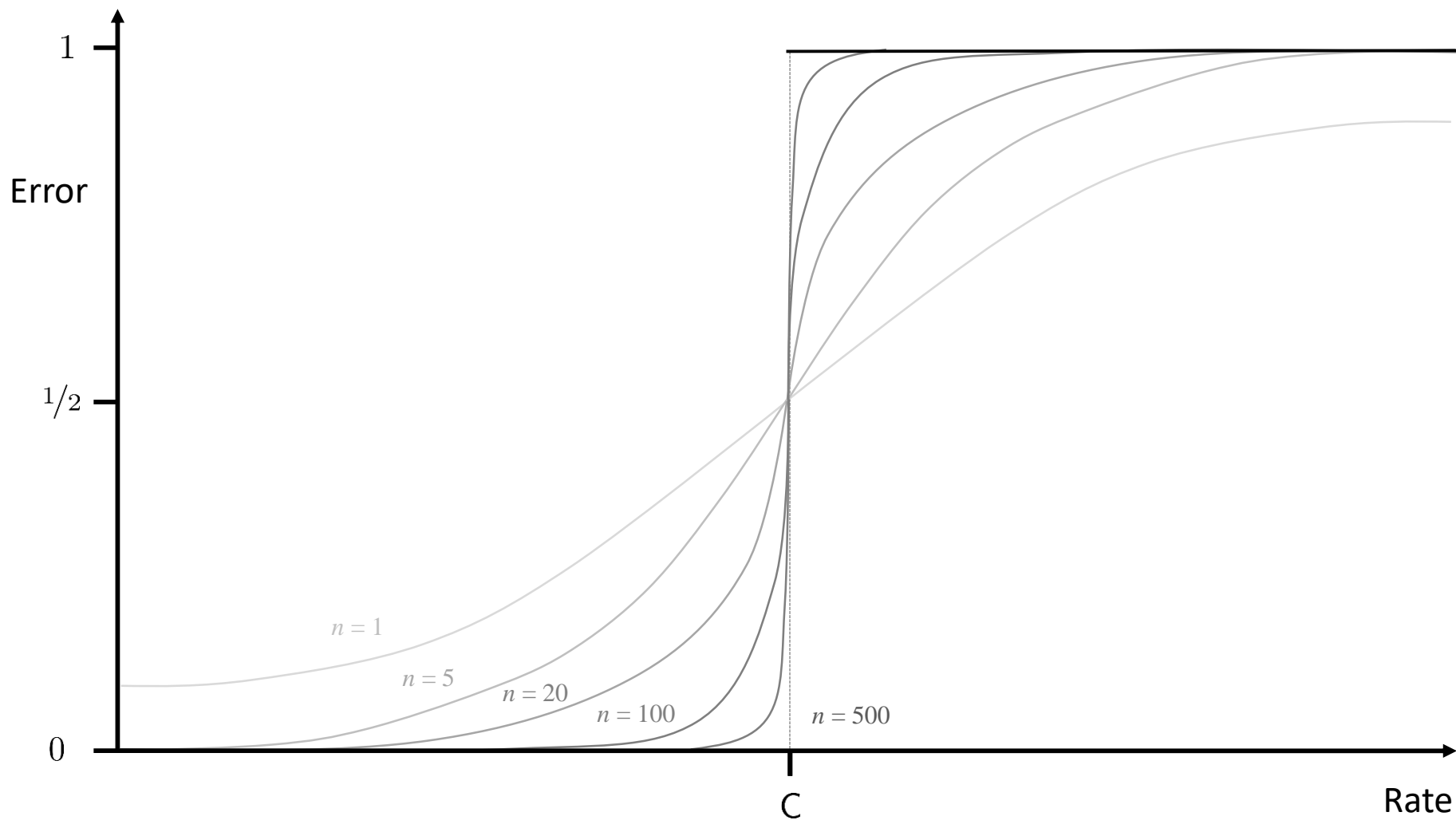
$$\frac{1}{n} D_h^{\epsilon_n}(\rho^{\otimes n} \| \sigma^{\otimes n}) \geq \sup \left\{ R \mid \Pr \left[\sum_{i=1}^n Z_i \leq \epsilon_n / 2 \right] \right\} - \mathcal{O}(\log 1 / \epsilon_n),$$

$$\frac{1}{n} D_h^{\epsilon_n}(\rho^{\otimes n} \| \sigma^{\otimes n}) \leq \sup \left\{ R \mid \Pr \left[\sum_{i=1}^n Z_i \leq 2\epsilon_n \right] \right\} + \mathcal{O}(\log 1 / \epsilon_n).$$

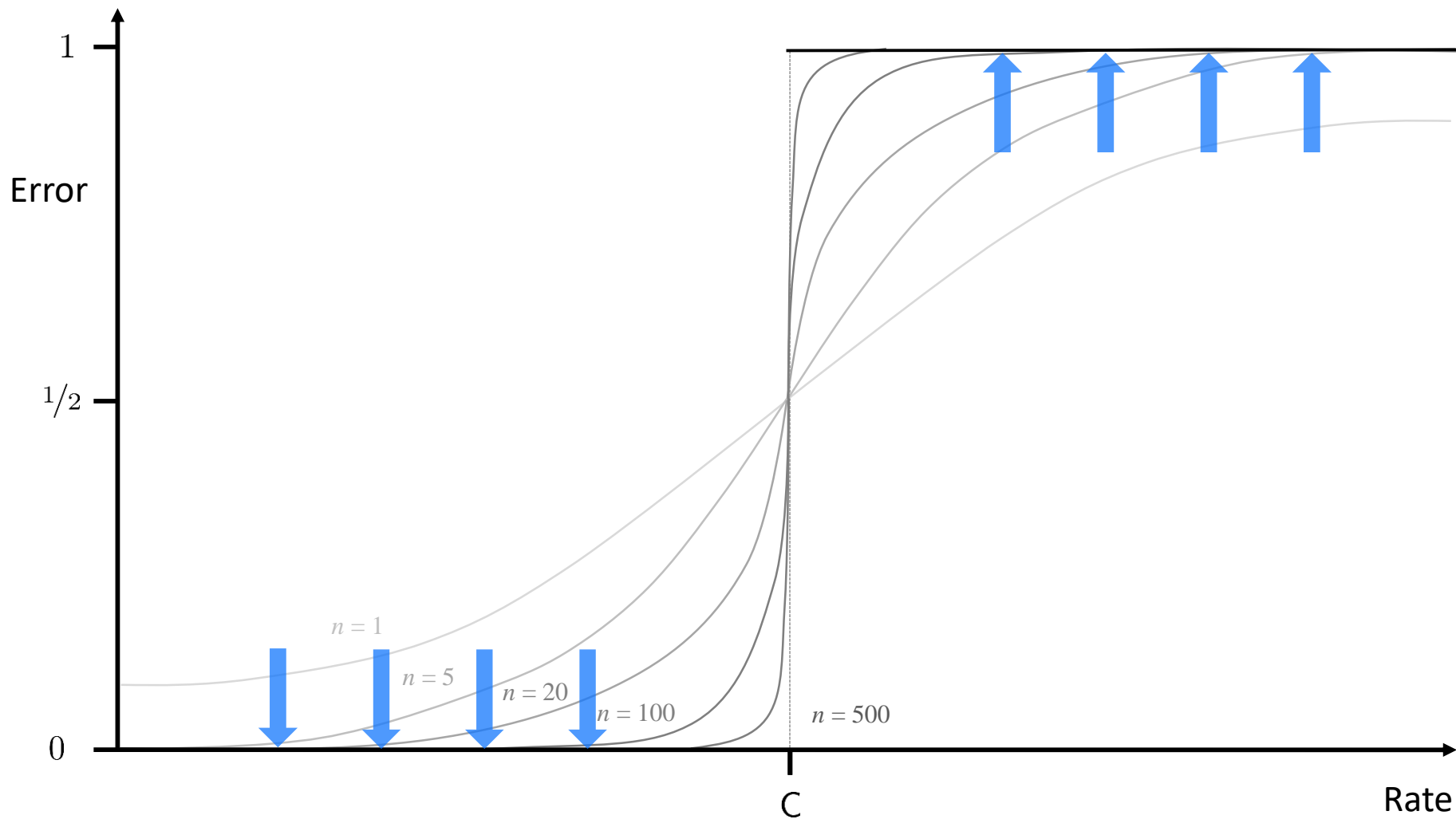
¹Nussbaum and Szkoła 2009.

²Tomamichel and Hayashi 2013

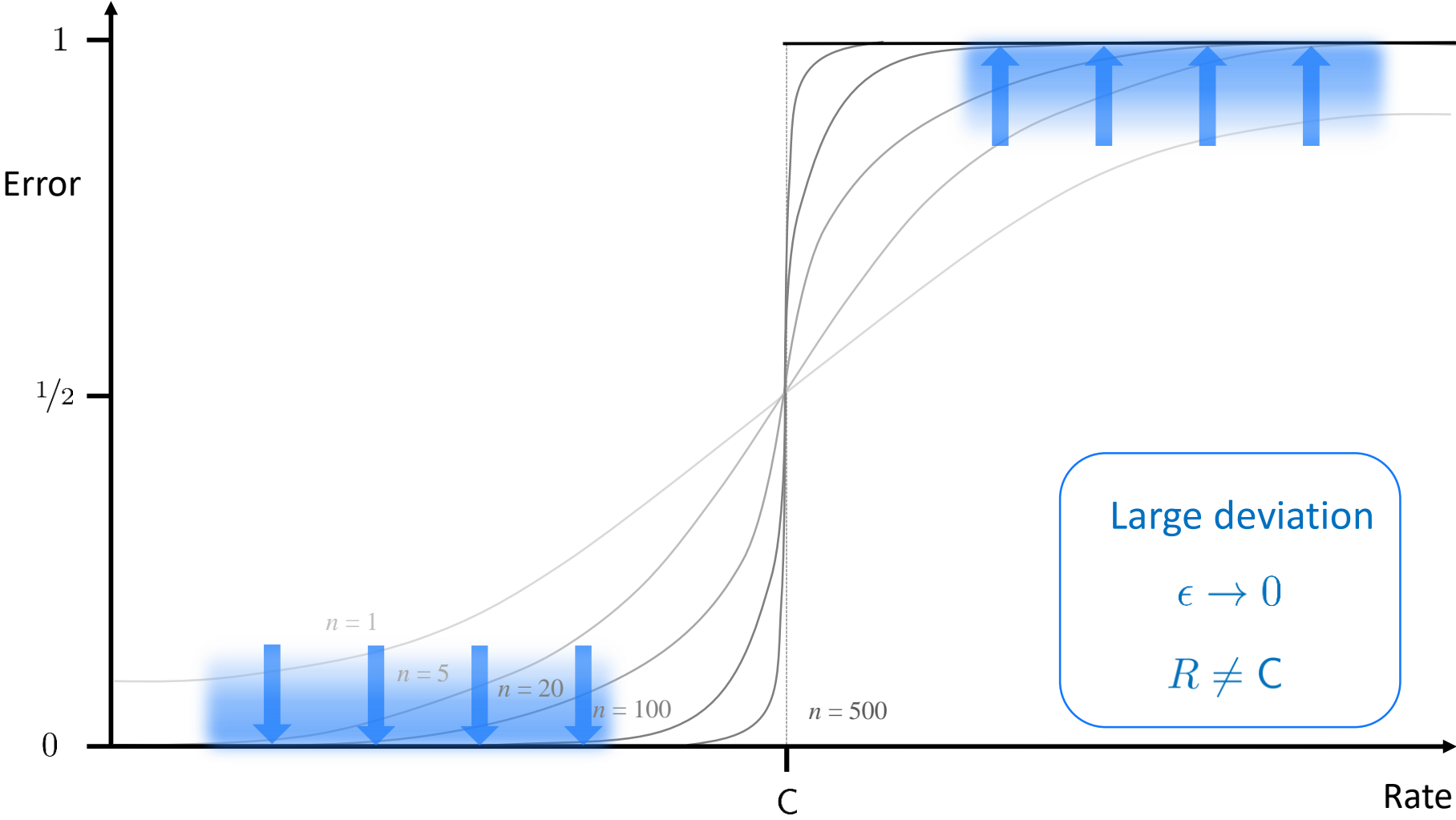
Different regimes



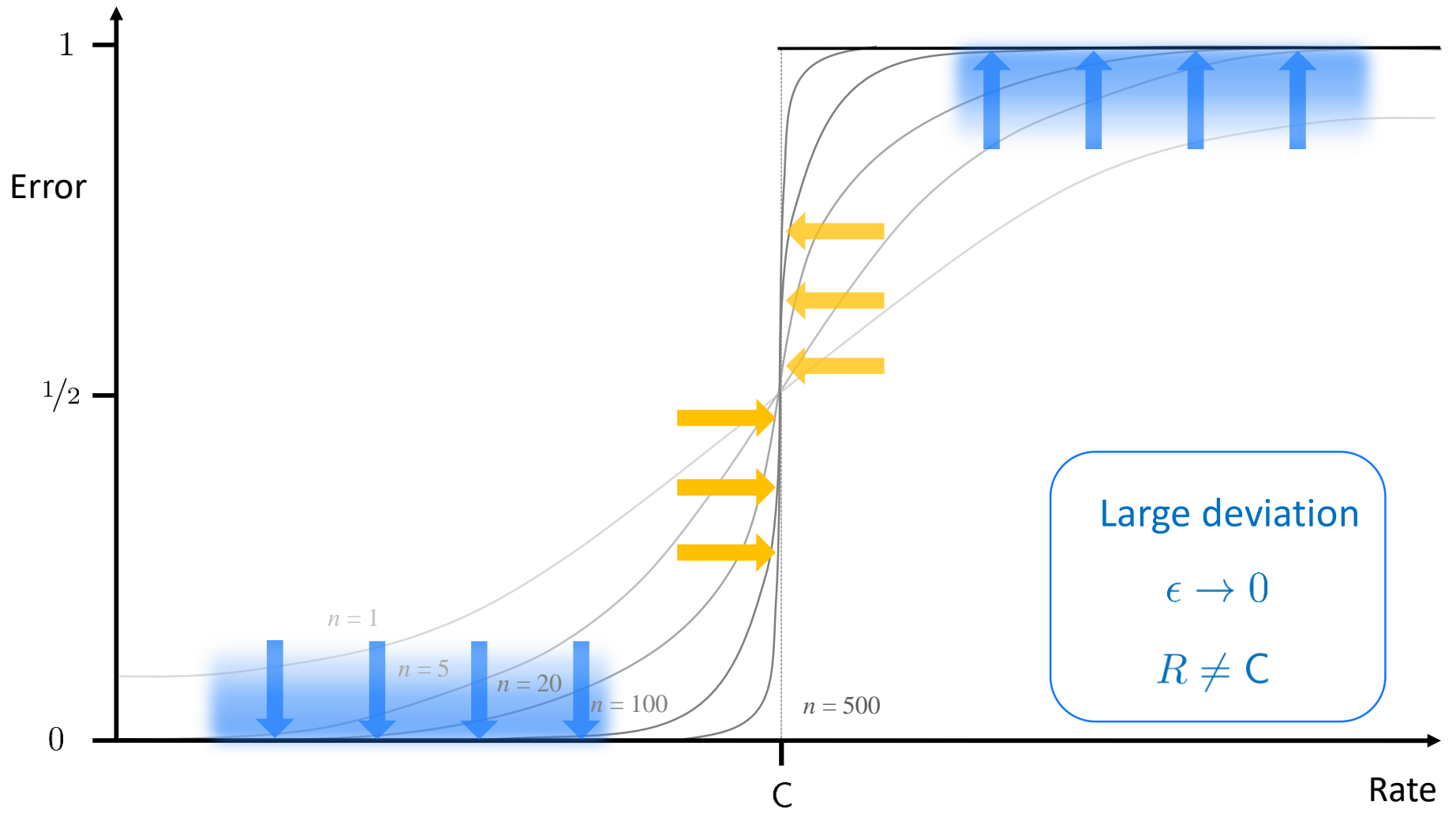
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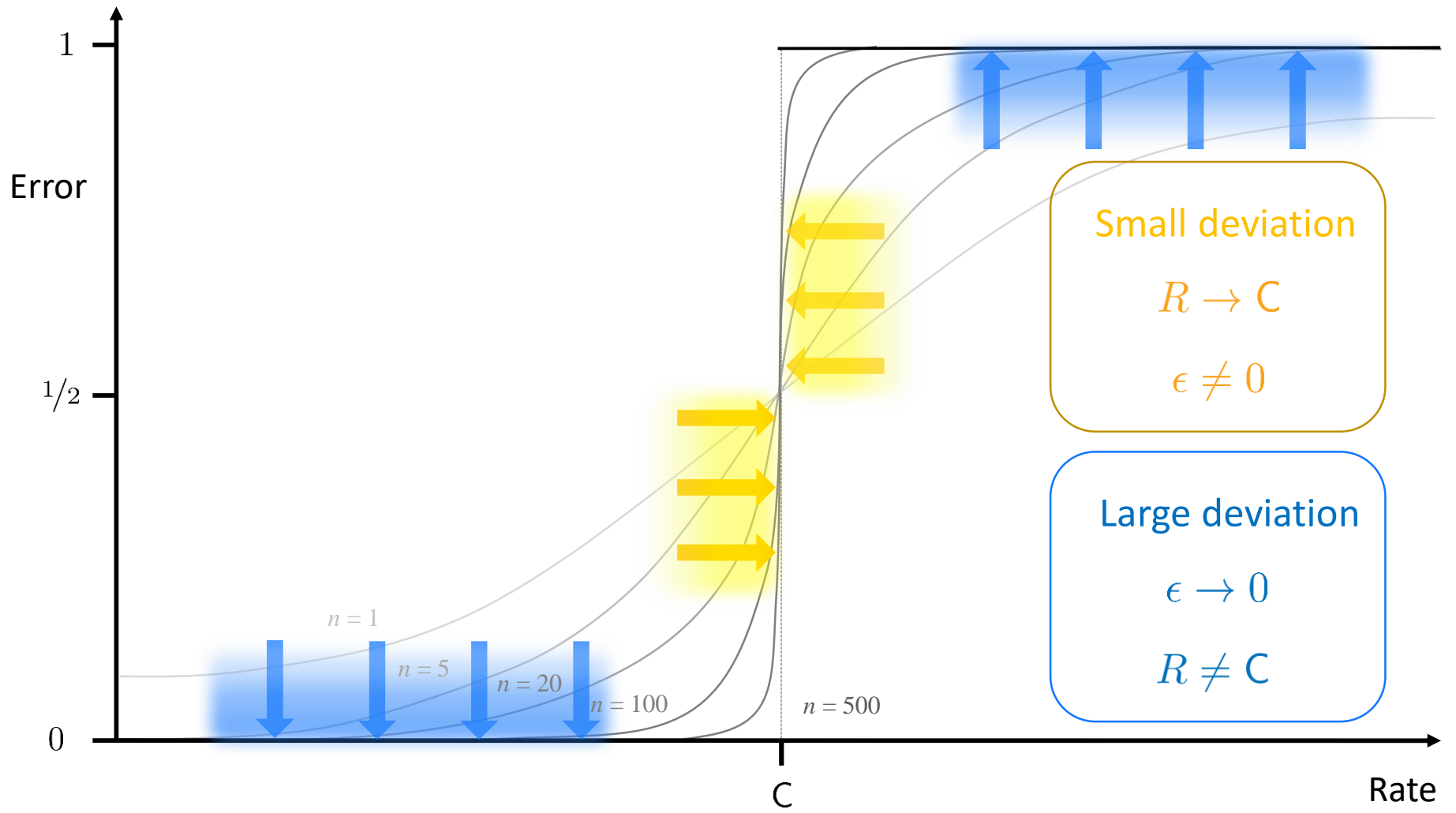
Different regimes



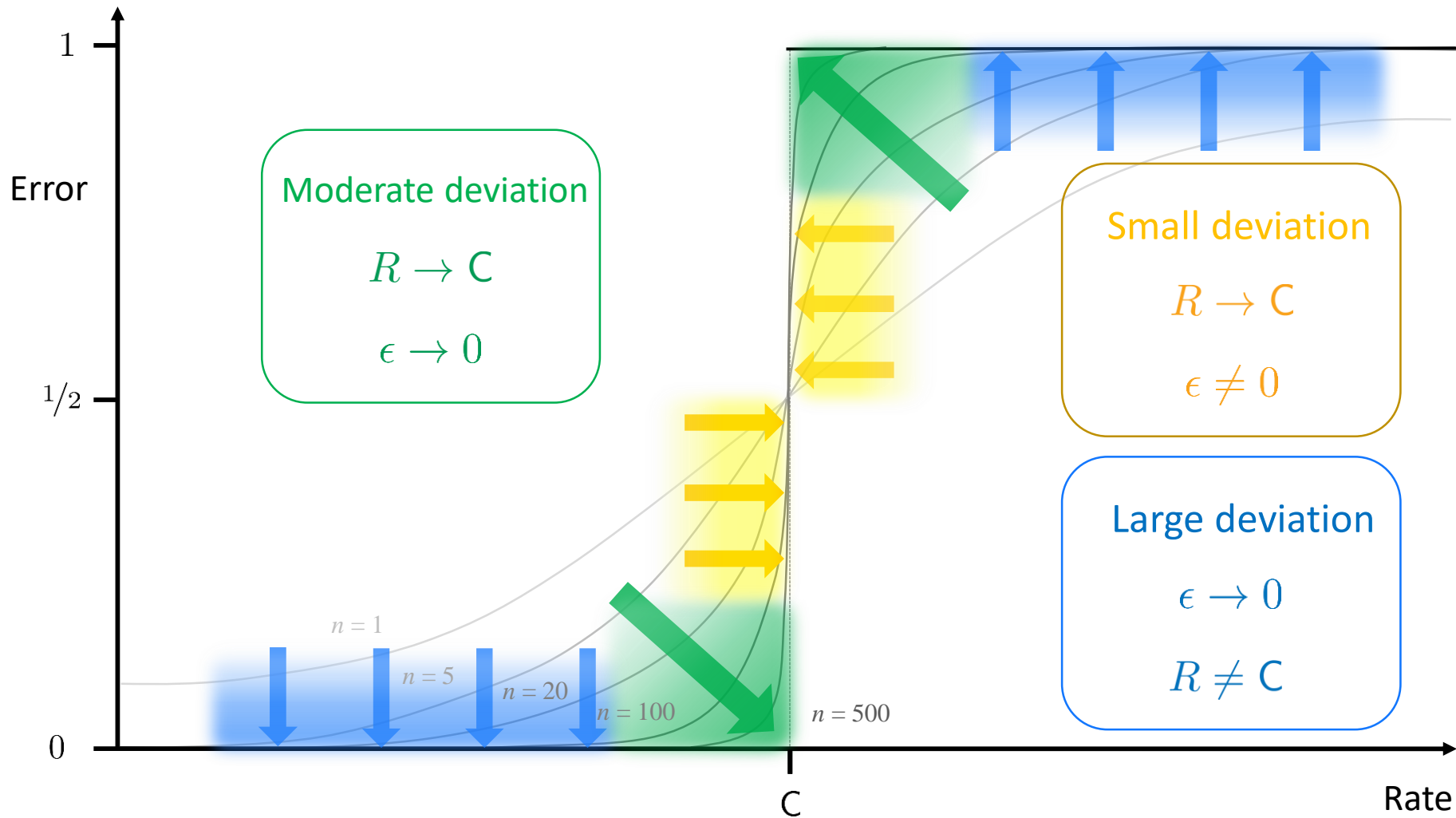
Different regimes



Different regimes



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From large deviation regime



Is the reliable communication possible as rate approaches capacity?

From large deviation regime



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▶ Let $R_n = C - a_n$, where $\left\{ \begin{array}{l} \text{(i) } \lim_{n \rightarrow \infty} a_n = 0 \\ \text{(ii) } \lim_{n \rightarrow \infty} \sqrt{n} a_n = \infty \end{array} \right.$

From large deviation regime



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$$\epsilon^*(n, R_n) = \exp \left\{ -\frac{na_n^2}{2V} + o(na_n^2) \right\} \rightarrow 0$$

Proof ideas

$$R^*(n, \epsilon_n) = C - \sqrt{2V}a_n + o(a_n)$$

$$\epsilon_n = \exp\{-na_n^2\}$$

Channel coding

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$$\alpha_n^* \rightarrow \exp\left\{-\frac{na_n^2}{2V}\right\},$$

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[Small deviation]

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Concentration inequalities

[Large deviation]

Moderate deviations for hypothesis testing

▶ Type-I, -II errors: $\alpha_n := \text{Tr}[(\mathbf{1} - A_n)\rho^{\otimes n}]$

$$\beta_n := \text{Tr}[A_n\sigma^{\otimes n}]$$

▶ Given $\beta_n \leq \exp\{-nR\}$

▶ Quantum Stein's lemma (Hiai and Petz 1991, Ogawa and Nagaoka 1999)

$$\alpha_n^* \rightarrow \begin{cases} 0, & R < D(\rho\|\sigma) \\ 1, & R > D(\rho\|\sigma) \end{cases}$$

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▶ Answer:

$$\alpha_n^* = \exp\left\{-\frac{na_n^2}{2V(\rho\|\sigma)} + o(na_n^2)\right\} \rightarrow 0$$

Proof idea

- ▶ Quantum Hoeffding bound ($\beta_n \leq \exp\{-nR\}$)

$$\alpha_n^* = \exp\{-nE(R) + o(n)\}$$

$$\sup_{0 < \alpha \leq 1} \frac{1 - \alpha}{\alpha} (D_\alpha(\rho \parallel \sigma) - R)$$

- ▶ Achievability (Audenaert *et al.* 2007, Hayashi 2007, Audenaert, Nussbaum, Szkola, Verstraete 2008)

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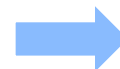
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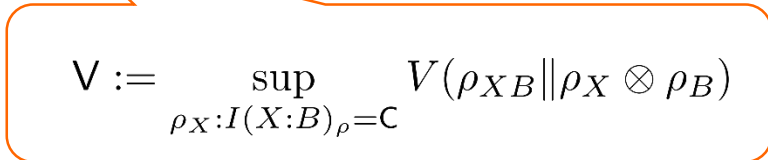
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Channel coding

- ▶ Goal: for $R_n = C - a_n$,

$$\Rightarrow \epsilon^*(n, R_n) = \exp \left\{ -\frac{na_n^2}{2V} + o(na_n^2) \right\}$$

Information variance


$$V := \sup_{\rho_X: I(X:B)_\rho = C} V(\rho_{XB} \| \rho_X \otimes \rho_B)$$

- ▶ Challenges:

- ▶ The optimal error exponent is still open
- ▶ Need a tight finite blocklength analysis for the optimal error probability

Achievability

- ▶ Hayashi 2007: $\epsilon^*(n, R) \leq 4 \exp \{ -n E_{\frac{1}{2}}^{\downarrow}(R) \}$

$$\max_{\frac{1}{2} \leq \alpha \leq 1} \frac{1 - \alpha}{\alpha} \left(D_{2 - \frac{1}{\alpha}}(\rho_{XB} \| \rho_X \otimes \rho_B) - R \right)$$

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Converse (Sphere-Packing Bound)

- ▶ Winter 1999:

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \epsilon^*(n, R) \leq \tilde{E}_{\text{sp}}(R) := \max_{\rho_X} \min_{\sigma_{XB}: \sigma_X = \rho_X} \{D(\sigma_{XB} \| \rho_{XB}) : I(X : B)_\sigma \leq R\}$$

- ▶ Dalai 2013:

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \epsilon^*(n, R) \leq E_{\text{sp}}(R) := \max_{\rho_X} \sup_{0 < \alpha \leq 1} \min_{\sigma_B} \frac{1 - \alpha}{\alpha} (D_\alpha(\rho_{XB} \| \rho_X \otimes \sigma_B) - R)$$

- ▶ Questions:

- ▶ What is the right exponent?
- ▶ Finite blocklength bound with tight prefactor?

Converse (Sphere-Packing Bound)

- ▶ Classical approach (Altug, Wagner 2014)

$$\epsilon^*(n, R_n) \geq \exp \left\{ -n\tilde{E}_{\text{sp}}(R) + o(na_n^2) \right\}$$

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- ▶ Result: a tight sphere-packing bound

$$\epsilon^*(n, R) \geq \frac{A}{(1 - \alpha^*) \sqrt{n}} \exp \{ -n E_{\text{sp}}(R) \}$$

Dalai: $\exp\{O(\sqrt{n})\}$

[arXiv:1704.05703]

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Conclusions

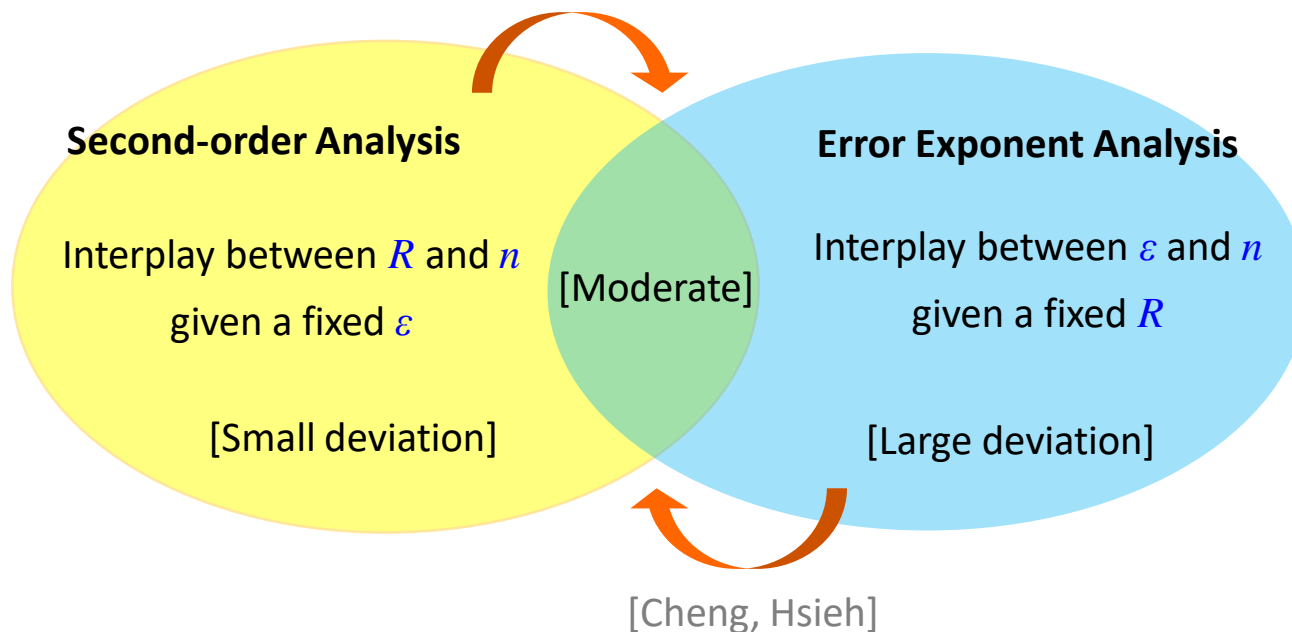
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 - ▶ How fast are the convergences $R \rightarrow C$ or $\epsilon \rightarrow 0$ as $n \rightarrow \infty$?

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$$\begin{cases} R^*(n, \epsilon_n) = C - \sqrt{2V}a_n + o(a_n), & \epsilon_n = \exp\{-na_n^2\} \\ \epsilon^*(n, R_n) = \exp\left\{-\frac{na_n^2}{2V} + o(na_n^2)\right\}, & R_n = C - a_n \end{cases}$$

[Chubb, Tan, Tomamichel]



[Cheng, Hsieh]

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- ▶ Extension to image-additive channels – What about other channels (entanglement-breaking) or capacities (entanglement-assisted)?
- ▶ Other applications – private communications, classical data compression with quantum side information, *etc.*

Different concentration regimes

Regimes	Channel Coding	Concentration
Small deviation	$\epsilon^* \left(n, C - \frac{A}{\sqrt{n}} \right) \sim \Phi \left(\frac{A}{\sqrt{V}} \right)$	$\Pr \left[\bar{X}_n \geq \frac{1}{\sqrt{n}} t \right] \sim 1 - Q \left(\frac{x}{\sqrt{V}} \right)$
Moderate deviation	$\epsilon^* (n, C - a_n) = e^{-\frac{na_n^2}{2V} + o(na_n^2)}$	$\Pr \left[\bar{X}_n \geq a_n t \right] = e^{-\frac{na_n^2}{2V} x + o(na_n^2)}$
Large deviation	$\epsilon^* (n, R) = e^{-nE(R) + o(n)}$	$\Pr \left[\bar{X}_n \geq t \right] = e^{-nI(x) + o(n)}$

