Faster ground state preparation and high-precision ground energy estimation with fewer qubits

Yimin Ge, J. Tura, J.I. Cirac
QIP 2018

arXiv:1712.03193*
“Nature isn’t classical, dammit, and if you want to make a simulation of nature, you’d better make it quantum mechanical, and by golly it’s a wonderful problem, because it doesn’t look so easy.”
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Quantum simulation

Quantum chemistry

Small quantum computers
Motivation

Quantum simulation

Many important applications: $|\psi_0\rangle$ ground state of another non-trivial Hamiltonian!

Ground state problems generally hard! But may not apply to natural systems
Motivation

Quantum simulation

\[ H, t \]

| \psi_0 \rangle

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| \psi_0 \rangle ground state of another non-trivial Hamiltonian!

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Hamiltonian simulation algorithm

\[ |\psi_0\rangle \]

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$H, t \rightarrow |\psi_0\rangle \rightarrow |\psi_t\rangle = e^{-iHt} |\psi_0\rangle$

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**Motivation**

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\[ H, t \]

\[ |\psi_0\rangle \]

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Many important applications: \(|\psi_0\rangle\) ground state of another non-trivial Hamiltonian!

Ground state problems generally hard!

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Motivation

Quantum linear systems algorithm with exponentially improved dependence on precision

Andrew Childs (U. Maryland)  
Robin Kothari (MIT)  
Rolando Somma (Los Alamos)

arXiv:1511.02786  
QIP 2016
Are there any killer applications of this??
1. General approaches for ground state preparation

2. Algorithms – details

3. Suitability for early quantum computers
Ground state preparation – approaches

Phase estimation

\[ |+\rangle \quad \cdots \quad |\phi\rangle \]

\[ U^2_0 \quad U^2_1 \quad \cdots \quad U^2_{n-1} \quad QFT^\dagger \]

Paradigm: First heuristic method, then phase estimation

Problem: Project given trial state \[ |\phi\rangle \] onto its ground state component
Ground state preparation – approaches

Phase estimation

\[|+\rangle \quad \cdots \quad |+\rangle \]

\[U^{2^0} \quad U^{2^1} \quad \cdots \quad U^{2^{n-1}}\]

\[QFT^\dagger\]

\[|\phi\rangle \quad |+\rangle \quad |+\rangle \quad |+\rangle \]

Problem: Project given trial state \(|\phi\rangle\) onto its ground state component.
Ground state preparation – approaches

Phase estimation

\[ |+\rangle \quad \ldots \quad U^2_n |\phi\rangle \approx |\text{ground state}\rangle \]

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Phase estimation

\[ |+\rangle \quad \cdots \quad |+\rangle \quad |+\rangle \quad |\phi\rangle \]

\[ U^0 \quad U^1 \quad \cdots \quad U^{n-1} \]

\[ QFT^\dagger \]

\[ |\text{ground state}\rangle \]

Adiabatic algorithms

Paradigm: First heuristic method, then phase estimation

This work: Improves part of phase estimation

Problem: Project given trial state \(|\phi\rangle\) onto its ground state component
Ground state preparation – approaches

Phase estimation

| + ⟩ ---- ⋯ ---- + ⟩
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QFT†

| φ ⟩ = U²⁰ ⋯ U²ⁿ⁻¹ | ground state ⟩?

Adiabatic algorithms

H(0) \stackrel{H(s)}{\longrightarrow} H(1)

Problem: Project given trial state | φ ⟩ onto its ground state component.
Ground state preparation – approaches

Phase estimation

\[ |+_\rangle \quad \cdots \quad |+_\rangle \quad |+_\rangle \]

\[ |\varphi\rangle = U^0 |+_\rangle \quad U^1 |+_\rangle \quad \cdots \quad U^{n-1} |+_\rangle \]

\[ QFT^\dagger \]

Adiabatic algorithms

\[
\begin{aligned}
H(0) & \xrightarrow{H(s)} H(1) \\
|\text{GS}(0)\rangle & \\
\end{aligned}
\]

Problem: Project given trial state \( |\varphi\rangle \) onto its ground state component

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trial state
Ground state preparation – approaches

Phase estimation

\[
|+\rangle \rightarrow \cdots \rightarrow |\phi\rangle = U^0 |\rangle U^1 |\rangle \cdots U^{n-1} |\rangle \approx |\text{ground state}\rangle?
\]

\[
\text{QFT}^\dagger
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\[
H(0) \xrightarrow{H(s)} H(1) |\text{GS(0)}\rangle
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Problem: Project given trial state \( |\phi\rangle \) onto its ground state component
Ground state preparation – approaches

Phase estimation

Adiabatic algorithms

$$H(0) \xrightarrow{H(s)} H(1)$$

$$|\text{GS(0)}\rangle$$

$$|\phi\rangle = U^{2^0} U^{2^1} \cdots U^{2^{n-1}} |\text{ground state}\rangle?$$

Phase estimation

Problem:

Project given trial state $$|\phi\rangle$$ onto its ground state component.
Ground state preparation – approaches

Phase estimation

\[ |\phi\rangle \quad U^0 \quad U^1 \quad \cdots \quad U^{n-1} \quad |\text{ground state}\rangle? \]

Adiabatic algorithms

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\begin{align*}
H(0) & \xrightarrow{H(s)} H(1) \\
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\end{align*}
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Ground state preparation – approaches

Phase estimation

\[ |+\rangle \quad \ldots \quad |+\rangle \quad QFT^{\dagger} \quad |\text{ground state}\rangle \]  

Adiabatic algorithms

\[ H(0) \xrightarrow{H(s)} H(1) \quad |\text{GS}(0)\rangle \approx |\text{GS}(1)\rangle \]  

\[ \begin{array}{c}
\text{E}_0(s) \\
\Delta_{\text{min}} \\
\text{E}_1(s)
\end{array} \]

Problem: Project given trial state \( |\phi\rangle \) onto its ground state component.

Paradigm: First heuristic method, then phase estimation.

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 Ground state preparation – approaches

Phase estimation

\[ |\psi\rangle = U^0 |+\rangle + U^1 |+\rangle + \cdots + U^{n-1} |+\rangle \]

|ground state⟩?

Adiabatic algorithms

\[ H(0) \xrightarrow{H(s)} H(1) \]

\[ |\text{GS}(0)\rangle \quad |\phi\rangle \]

\[ E \]

\[ 0 \quad 1 \]

\[ \Delta_{\text{min}} \]

\[ E_0(s) \quad E_1(s) \]

Phase estimation

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Problem: Project given trial state \(|\phi\rangle\) onto its ground state component
Ground state preparation – approaches

Phase estimation

\[ |\phi\rangle = U^0 \cdot |\phi\rangle \]

|\phi\rangle = U^0 \cdot U^1 \cdot \ldots \cdot U^{n-1} |\text{ground state}\rangle?

Adiabatic algorithms

\[ H(0) \xrightarrow{H(s)} H(1) \]

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\[ E_1(s) \]

Paradigm: First heuristic method, then phase estimation
Phase estimation

\[ |+\rangle \rightarrow \cdots \rightarrow |\phi\rangle \]

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\[ |\phi\rangle = U^2_0 U^1 \cdots U^{n-1} \text{ |ground state}\]?

Adiabatic algorithms

\[ H(0) \rightarrow^{H(s)} H(1) \]

\[ |\text{GS}(0)\rangle \rightarrow |\phi\rangle \]

\[ E \rightarrow \Delta_{\text{min}} E_1(s) \]

\[ E_0(s) \]

\[ 0 \rightarrow 1 \]

\[ \text{Phase estimation} + \]

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Ground state preparation – approaches

**Phase estimation**

\[
|+\rangle, \ldots, |+\rangle, |\phi\rangle = U^0 \cdots U^{n-1} |\text{ground state}\rangle
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**Adiabatic algorithms**

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**Paradigm:** First heuristic method, then phase estimation

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**Problem:** Project given trial state \(|\phi\rangle\) onto its ground state component
Setup

- Hamiltonian $H$, spectrum in $[0, 1]$
  - Eigenstates $|\lambda_i\rangle$
  - Ground energy $\lambda_0$, ground state $|\lambda_0\rangle$
  - All other eigenvalues: $\lambda_i \geq \lambda_0 + \Delta$
  - Can efficiently perform time evolution of $H$ (e.g., sparse & oracle access, linear combination of easy unitaries, etc [BCK15, BCCKS15, LC16, LC17])

Circuit $C_\phi$, prepares trial state $|\phi\rangle$

- $\phi_0 := \langle \lambda_0 | \phi \rangle$ (generally unknown)

- Known lower bound $\chi \leq |\phi_0|$

- Trivial assumption: $\chi = e^{-O(\log N)}$

Aim: Extract state $|\lambda'_0\rangle$ such that $\| |\lambda'_0\rangle - |\lambda_0\rangle \| < \epsilon$ for given $\epsilon$
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\( N \times N \) Hamiltonian \( H \), spectrum in \([0, 1]\)

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Ground state preparation

N = total dimension of \( H \)

\( \Delta \) = known lower bound on spectral gap

\( \epsilon \) = allowed error

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\( \phi_0 \) = overlap of trial state with ground state

\( \chi \) = known lower bound on \( |\phi_0| \)

\( \Lambda \) = base cost of Hamiltonian simulation

\( \Phi \) = cost of preparing trial state
Ground state preparation
Ground state preparation

Ground energy known
# Results & Comparisons

## Ground state preparation

Ground energy known

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Results & Comparisons

Ground energy estimation

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* for $\xi \ll \Delta$

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$\phi_0$ = overlap of trial state with ground state

$\chi$ = known lower bound on $|\phi_0|$
Algorithm

Idea:

1. Approximate ground state projector

\[ \cos^2 m H' |\phi\rangle \propto \sim |\lambda_0\rangle \]

for \( m \approx 1/\Delta^2 \)

2. Approximate as linear combination of easy unitaries

\[ \cos^2 m H' = \sum_{k=-m}^m \alpha_k e^{-2iH'k} \]

\( \alpha_k = \frac{1}{2} m \left( \frac{2m^2 + k}{2m} \right) \)

\( m_0 \approx \sqrt{m} \)

3. Use LCU Lemma

Alternative:

1. \( (1 - H'^2)^2 m \) as approximate ground state projector

2. Expand in Chebyshev polynomials

3. Quantum walks

Implementing linear combination of unitaries eg [CKS'15]

LCU Lemma:

Able to perform unitaries \( U_k \Rightarrow \) can perform \( V := \sum_k \alpha_k U_k \)

1. Implement \( V \) with some amplitude \( B |0\rangle = \sum_k \sqrt{\alpha_k} |k\rangle \), \( \alpha = \sum_k |\alpha_k|^2 \)

\[ B^\dagger U_k |0\rangle = |\phi\rangle \langle 0| V |\phi\rangle \langle \phi| \]

Postselection on ancilla: implement deterministically

2. Amplitude amplification:

\[ \| \alpha |0\rangle V |\psi\rangle \| \rightarrow 1 \]
Idea:

1. Approximate ground state projector
Algorithm

Idea:

1. Approximate ground state projector

2. Approximate as linear combination of easy unitaries

Alternative:

1. \((1 - H' / 2)^2 m\) as approximate ground state projector

2. Expand in Chebyshev polynomials

3. Quantum walks

Implementing linear combination of unitaries eg [CKS'15]

LCU Lemma: Able to perform unitaries \(U_k \Rightarrow \) can perform \(V := \sum_k \alpha_k U_k\)

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\[ |0\rangle = \frac{1}{\sqrt{\alpha}} \sum \sqrt{\alpha_k} |k\rangle \], \(\alpha = \sum |\alpha_k|^2\)"
Algorithm

Idea:

1. Approximate ground state projector

2. Approximate as linear combination of easy unitaries

3. Use LCU Lemma
Algorithm

Implementing linear combination of unitaries

LCU Lemma: Able to perform unitaries $U_k$ $\Rightarrow$ can perform $V := \sum_k \alpha_k U_k$

eg [CKS’15]
Algorithm

Implementing linear combination of unitaries

LCU Lemma: Able to perform unitaries $U_k \Rightarrow$ can perform $V := \sum_k \alpha_k U_k$

1. Implement $V$ with some amplitude
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Implementing linear combination of unitaries

LCU Lemma: Able to perform unitaries $U_k$  $\Rightarrow$ can perform $V := \sum_k \alpha_k U_k$

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|0⟩ ——— B ——— Š ——— ⟨0|  

|φ⟩ ——— U_k ——— V |φ⟩
**Algorithm**

**Implementing linear combination of unitaries**

**LCU Lemma:** Able to perform unitaries $U_k \Rightarrow$ can perform $V := \sum_k \alpha_k U_k$

1. **Implement $V$ with some amplitude**

   $$B |0\rangle = \frac{1}{\sqrt{\alpha}} \sum \sqrt{\alpha_k} |k\rangle, \quad \alpha = \sum |\alpha_k|$$

   $\begin{array}{c}
   \text{Postselection on ancilla: implement $V$ deterministically}
   \end{array}$
Implementing linear combination of unitaries

**Algorithm**

1. Approximate ground state projector

\[ \cos^2 m H' \approx |\lambda_0 \rangle \]

for \( m \approx 1/\Delta^2 \)

2. Approximate as linear combination of easy unitaries

\[
\cos^2 m H' = \sum_{k=-m}^{m} \alpha_k e^{-2iH'k}, \quad \alpha_k = \frac{1}{2m} \left( \frac{2m}{m} + k \right)
\]

3. Use LCU Lemma

**LCU Lemma:** Able to perform unitaries \( U_k \) \( \Rightarrow \) can perform \( V := \sum_k \alpha_k U_k \)

1. Implement \( V \) with some amplitude

\[
B \langle 0 | = \frac{1}{\sqrt{\alpha}} \sum \sqrt{\alpha_k} | k \rangle, \quad \alpha = \sum |\alpha_k|
\]

\[ |0\rangle \rightarrow B \]

\[ |\phi\rangle \rightarrow U_k \]

\[ |\ast\rangle = \frac{1}{\alpha} |0\rangle V |\phi\rangle + \sqrt{1 - \frac{1}{\alpha^2}} |R\rangle, \quad \langle 0| R \rangle = 0
\]

**eg [CKS'15]**
Algorithm

Implementing linear combination of unitaries

**LCU Lemma**: Able to perform unitaries $U_k$ $\Rightarrow$ can perform $V := \sum_k \alpha_k U_k$

1. Implement $V$ with some amplitude

$$B |0\rangle = \frac{1}{\sqrt{\alpha}} \sum \sqrt{\alpha_k} |k\rangle,$$

$$\alpha = \sum |\alpha_k|$$

2. Amplitude amplification:

$$\left\| \frac{1}{\alpha} |0\rangle V |\psi\rangle \right\| \rightarrow 1$$
Algorithm

Idea:

1. Approximate ground state projector

2. Approximate as linear combination of easy unitaries

3. Use LCU Lemma
Assume: ground energy known. \( H' := H - \lambda_0 \)

1. Approximate ground state projector

2. Approximate as linear combination of easy unitaries

3. Use LCU Lemma
Algorithm

Assume: ground energy known. \( H' := H - \lambda_0 \)

1. Approximate ground state projector
   \[ \cos^{2m} H' \]

2. Approximate as linear combination of easy unitaries

3. Use LCU Lemma
Algorithm

Assume: ground energy known. $H' := H - \lambda_0$

1. Approximate ground state projector
   \[ \cos^{2m} H' |\phi\rangle \otimes |\lambda_0\rangle \quad \text{for} \quad m \approx 1/\Delta^2 \]

2. Approximate as linear combination of easy unitaries

3. Use LCU Lemma
Algorithm

Assume: ground energy known. $H' := H - \lambda_0$

1. Approximate ground state projector

$$\cos^2 m H' \ket{\phi} \propto \ket{\lambda_0} \quad \text{for } m \approx 1/\Delta^2$$

2. Approximate as linear combination of easy unitaries

$$\cos^2 m H' = \sum_{k=-m}^{m} \alpha_k e^{-2iH'k} \quad \alpha_k := \frac{1}{2^{2m}} \binom{2m}{m + k}$$

3. Use LCU Lemma
Algorithm

Assume: ground energy known. $H' := H - \lambda_0$

1. Approximate ground state projector

$$\cos^{2m} H' \ket{\phi} \cong \ket{\lambda_0} \quad \text{for } m \approx 1/\Delta^2$$

2. Approximate as linear combination of easy unitaries

$$\cos^{2m} H' \approx \sum_{k=-m_0}^{m_0} \alpha_k e^{-2iH'k}, \quad \alpha_k := \frac{1}{2^{2m}} \binom{2m}{m+k}, \quad m_0 \approx \sqrt{m}$$

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Alternative:
1. $(1 - H'^2)^{2m}$ as approximate ground state projector
2. Expand in Chebyshev polynomials
3. Quantum walks
Algorithm – ground energy unknown

Previous algorithm:
• Requires knowing ground energy up to precision $\tilde{O}(\Delta)$
• Smaller values OK, but exponentially small prob of success

Naive approach: run with increasing values for $\lambda_0$, step size $\tilde{O}(\Delta)$, stop when successful → overall runtime factor $\tilde{O}(1/\Delta)$.

Quantum search: $\tilde{O}(1/\sqrt{\Delta})$.

Lemma (Minimum label finding)
• $L$ unitaries $U_j |0\rangle |0\rangle = |0\rangle |\Phi_j\rangle + |R_j\rangle$, $\langle 0 | R_j \rangle = 0$
• $|\Phi\rangle := \frac{1}{\sqrt{L}} \sum_j |0\rangle |j\rangle |\Phi_j\rangle + |R\rangle$, $\langle 0 | R \rangle = 0$

⇒ Given $\chi$, can approximately find smallest $j$ s.t. $\| |\Phi_j\rangle\| \geq \chi$ using $\tilde{O}(\sqrt{L}/\chi)$ calls to $U = \sum_j |j\rangle \langle j| \otimes U_j I$. 

Idea: Binary search on label ancilla using amplitude amplification
Algorithm – ground energy unknown

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- Requires knowing ground energy up to precision $\tilde{O}(\Delta)$
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$\Rightarrow$ Given $\chi$, can approximately find smallest $j$ s.t. $\|\Phi_j\| \geq \chi$

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Lemma (Minimum label finding)

- $L$ unitaries $U_j |0\rangle|0\rangle = |0\rangle|\Phi_j\rangle + |R_j\rangle$, $\langle 0|R_j \rangle = 0$
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$\Rightarrow$ Given $\chi$, can approximately find smallest $j$ s.t. $||\Phi_j|| \geq \chi$

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Idea: Binary search on label ancilla using amplitude amplification
Lemma (Minimum label finding)

- \( L \) unitaries \( U_j |0\rangle|0\rangle = |0\rangle|\Phi_j\rangle + |R_j\rangle, \quad \langle 0|R_j \rangle = 0 \)
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\[ \Rightarrow \text{Given } \chi, \text{ can approximately find smallest } j \text{ s.t. } \|\Phi_j\| \geq \chi \]

using \( \tilde{O}(\sqrt{L}/\chi) \) calls to \( U = \sum_j |j\rangle\langle j| \otimes U_j \)
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• \( L \) unitaries \( U_j |0\rangle|0\rangle = |0\rangle|\Phi_j\rangle + |R_j\rangle, \quad \langle 0|R_j \rangle = 0 \)

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\implies \text{Given } \chi, \text{ can approximately find smallest } j \text{ s.t. } |||\Phi_j||| \geq \chi

\text{using } \tilde{O}(\sqrt{L/\chi}) \text{ calls to } U = \sum_j |j\rangle\langle j| \otimes U_j

• \( U_j \) = previous algorithm, assuming ground energy is \( E_j \propto j\Delta \)

• \( U \) essentially same cost as \( U_j \) \implies \text{overall runtime factor } \sqrt{L} \approx \frac{1}{\sqrt{\Delta}}

• Runtime dependence on \( \chi \), not \(|\phi_0|\)
Lemma (Minimum label finding)

- L unitaries $U_j |0\rangle |0\rangle = |0\rangle |\Phi_j\rangle + |R_j\rangle$, $\langle 0|R_j\rangle = 0$
- $|\Phi\rangle := \frac{1}{\sqrt{L}} \sum_j |0\rangle |j\rangle |\Phi_j\rangle + |R\rangle$, $\langle 0|R\rangle = 0$

$\Rightarrow$ Given $\chi$, can approximately find smallest $j$ s.t. $||\Phi_j|| \geq \chi$ using $\tilde{O}(\sqrt{L}/\chi)$ calls to $U = \sum_j |j\rangle \langle j| \otimes U_j$

- $U_j =$ previous algorithm, assuming ground energy is $E_j \propto j\Delta$
- $U$ essentially same cost as $U_j \Rightarrow$ overall runtime factor $\sqrt{L} \approx \frac{1}{\sqrt{\Delta}}$
- Runtime dependence on $\chi$, not $|\phi_0|$

Bonus: Also find ground energy to precision $\tilde{O}(\Delta)$
- $\Delta$ only required to be lower bound on gap
  $\Rightarrow$ general ground energy estimation algorithm for high precisions
Algorithm – ground energy unknown

Lemma (Minimum label finding)

- \( L \) unitaries \( U_j |0\rangle|0\rangle = |0\rangle|\Phi_j\rangle + |R_j\rangle, \quad \langle 0|R_j\rangle = 0 \)
- \( |\Phi\rangle := \frac{1}{\sqrt{L}} \sum_j |0\rangle|j\rangle|\Phi_j\rangle + |R\rangle, \quad \langle 0|R\rangle = 0 \)

=> Given \( \chi \), can approximately find smallest \( j \) s.t. \( \|\Phi_j\| \geq \chi \) using \( \tilde{O}(\sqrt{L}/\chi) \) calls to \( U = \sum_j |j\rangle\langle j| \otimes U_j \)

- \( U_j \) = previous algorithm, assuming ground energy is \( E_j \propto j\Delta \)
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Bonus: Also find ground energy to precision \( \tilde{O}(\Delta) \)
- \( \Delta \) only required to be lower bound on gap
  \( \Rightarrow \) general ground energy estimation algorithm for high precisions

Alternative: first use PEA to find ground energy
  \( \rightarrow \) better scaling in \( \Delta \) but worse scaling in overlap
Early quantum computers

Adaption for early quantum computers:

- Amplitude amplification
- Repeated measurements

NISQ: devices with $\approx 100$ qubits, $\approx 10^4 - 10^5$ gates reliably

Limiting factor: number of gates coherently in single-run, not total runtime!

Ground state preparation algorithms, ground energy known

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This work $\tilde{O}(\Lambda | \phi_0 | \Delta \epsilon + \Phi | \phi_0 | 2)$

$N$ = total dimension of $H$

$\Delta = \text{known lower bound on spectral gap}$

$\epsilon = \text{allowed error}$

$|\phi_0\rangle = \text{overlap of trial state with ground state}$

$\Lambda = \text{base cost of Hamiltonian simulation}$

$\Phi = \text{cost of preparing trial state}$

$|\phi_0\rangle$
Early quantum computers

Adaption for early quantum computers:

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This work

\[\tilde{O}(\Lambda |\phi_0| \Delta \epsilon + \Phi |\phi_0|)\]

\[\tilde{O}(\Lambda |\phi_0| \Delta + \Phi |\phi_0|)\]

\[\tilde{O}(1 |\phi_0|)\]

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Early quantum computers

Adaption for early quantum computers:
Amplitude amplification
Early quantum computers

Adaption for early quantum computers:

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NISQ: devices with \( \approx 100 \) qubits, \( \approx 10^4 \)–\( 10^5 \)(?) gates reliably

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Early quantum computers

Adaption for early quantum computers:

Amplitude amplification  Repeated measurements

NISQ: devices with $\approx 100$ qubits
Early quantum computers

Adaption for early quantum computers:

- **Amplitude amplification**
- **Repeated measurements**

NISQ: devices with \( \approx 100 \) qubits, \( \approx 10^4 - 10^5 \) (?) gates reliably

Limiting factor: number of gates coherently in *single-run*, **not** *total* runtime!
Early quantum computers

Adaption for early quantum computers:
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Ground state preparation algorithms, ground energy known

| Algorithm                  | Gates \( N \) = total dimension of \( H \) | Gates \( \Delta \) = known lower bound on spectral gap | Gates \( \epsilon \) = allowed error | Gates \( \phi_0 \) = overlap of trial state with ground state | Gates \( \Lambda \) = base cost of Hamiltonian simulation | Gates \( \Phi \) = cost of preparing trial state \( |\phi\rangle \) |
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| Multicopy PEA              |                                           |                                               |                                  |                                      |                                           |                                     |
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Early quantum computers

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Early quantum computers

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Ground state preparation algorithms, ground energy known

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\( N = \) total dimension of \( H \)
\( \Delta = \) known lower bound on spectral gap
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\( \phi_0 = \) overlap of trial state with ground state
\( \Lambda = \) base cost of Hamiltonian simulation
\( \Phi = \) cost of preparing trial state \( |\phi\rangle \)
Early quantum computers

Adaption for early quantum computers:
- **Amplitude amplification**
- **Repeated measurements**

NISQ: devices with $\approx 100$ qubits, $\approx 10^4 - 10^5$ (?) gates reliably

Limiting factor: number of gates coherently in *single-run*, not *total* runtime!

Ground state preparation algorithms, ground energy known

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Gates Ampl amplif</th>
<th>Gates Repeated mmt</th>
<th>Gates single run</th>
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<td>Multicopy PEA</td>
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Summary

Ground state preparation algorithm
• Faster than naive phase estimation
• Fewer qubits than improved phase estimation
• Known and unknown ground energy
• Estimates ground energy to high precision

Applications
• Quantum simulation of many-body systems (quenches!)
• Quantum chemistry
• Single-copy tomography, QMA witnesses, optimisation problems, quantum machine learning, . . .

Potential applications for early quantum computers!
### Summary

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**Potential applications for early quantum computers!**