Classical boson sampling algorithms and the outlook for experimental boson sampling

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LETTERS

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Classical boson sampling algorithms with superior performance to near-term experiments

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The Classical Complexity of Boson Sampling

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Proc. SODA (2018)
arXiv:1706.01260

Outline

- Boson sampling
  - Mathematical problem
  - Linear optical experiment
- Previous expectations
- Our results
  - (Approximate) Markov chain Monte Carlo sampler
  - Exact sampler
- Experimental time "complexity"
  - Photon loss
- Revised expectations
- Algorithm outlines
- Bonus – estimated mixing time
- Rejection sampling?

Boson sampling

Input:

$$U = \begin{pmatrix} U_{11} & \dots & U_{1n} & \dots & U_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ U_{m1} & \dots & U_{mn} & \dots & U_{mm} \end{pmatrix} \quad \begin{array}{c} m \gg n \\ (m = n^2) \end{array}$$

$$A = \begin{pmatrix} U_{11} & \dots & U_{1n} \\ \vdots & \ddots & \vdots \\ U_{m1} & \dots & U_{mn} \end{pmatrix}$$

How many copies of each row

$$S = (s_1, \dots, s_m) \in \Phi_{m,n}$$

A_S : $n \times n$ matrix made from taking rows of A according to S

Output: Sample S with prob.

$$\Pr(S) = \frac{|\mathrm{Per}(A_S)|^2}{s_1! \dots s_m!}$$

$s_i \in \{0, 1\} \longleftarrow$ Collision Free Subspace (CFS)

AA: Efficient classical algorithm for (even approximate) boson sampling would have surprising complexity theoretic consequences

Aaronson & Arkhipov, arXiv:1011.3245 (2010)

Boson sampling – example (n=3)

$$U = \begin{pmatrix} \boxed{\begin{matrix} U_{11} \\ U_{21} \\ U_{31} \\ \vdots \\ U_{m1} \end{matrix}} & \boxed{\begin{matrix} U_{12} \\ U_{22} \\ U_{32} \\ \vdots \\ U_{m2} \end{matrix}} & \boxed{\begin{matrix} U_{13} \\ U_{23} \\ U_{33} \\ \vdots \\ U_{m3} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \ddots \\ \cdots \end{matrix} & \begin{matrix} U_{1m} \\ U_{2m} \\ U_{3m} \\ \vdots \\ U_{mm} \end{matrix} \end{pmatrix}$$

$$A = \begin{pmatrix} \boxed{U_{11} \quad U_{12} \quad U_{13}} \\ U_{21} \quad U_{22} \quad U_{23} \\ \boxed{U_{31} \quad U_{32} \quad U_{33}} \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ \boxed{U_{m1} \quad U_{m2} \quad U_{m3}} \end{pmatrix}$$

$$S = (1, 0, 1, \ldots, 1)$$

$$A_S = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{31} & U_{32} & U_{33} \\ U_{m1} & U_{m2} & U_{m3} \end{pmatrix}$$

$$\Pr(S) = |\mathrm{Per}(A_S)|^2$$

Boson sampling – Matrix permanents

$$X \in \mathbb{C}^{n\times n}$$

$$\mathrm{Per}(X) = \sum_{\sigma\in\mathcal{S}_n}\prod_{i=1}^{n} x_{i,\sigma(i)}$$

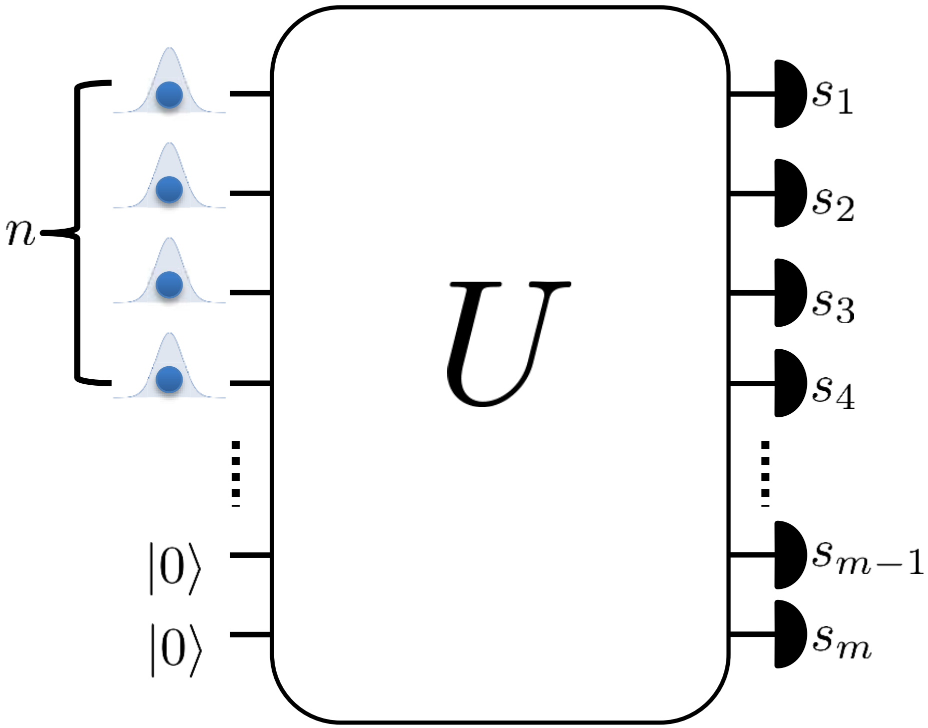
$$\left(\mathrm{Det}(X) = \sum_{\sigma\in\mathcal{S}_n}\mathrm{sgn}(\sigma)\prod_{i=1}^{n} x_{i,\sigma(i)}\right)$$

$\#\mathrm{P}$-hard

Time complexity: $\mathcal{O}(n2^n)$
(Ryser or Glynn)

(Time complexity: $\mathcal{O}(n^{2.373})$
(Fast matrix multiplication))

Boson sampling – Linear optical experiment



multiphoton transformation

$$|\langle S|\mathcal{U}|1\ldots10\ldots0\rangle|^2$$

$$=$$

$$\frac{|\mathrm{Per}(A_S)|^2}{s_1!\ldots s_m!}$$

The experiment "naturally" performs boson sampling

No feed-forward required

Aaronson & Arkhipov, arXiv:1011.3245 (2010)

Previous expectations – quantum computational supremacy

Aaronson & Arkhipov,
arXiv:1011.3245 (2010)

If one could implement our experiment with (say) $20 \leq n \leq 30$, then certainly a classical computer could verify the answers—but at the same time, one would be getting direct evidence that a quantum computer could efficiently solve an "interestingly difficult" problem, one for which the best-known classical algorithms require many millions of operations. While *disproving* the Extended Church-Turing Thesis is formally impossible, such an experiment would arguably constitute the strongest evidence against the ECT to date.

Aaronson & Arkhipov,
arXiv:1309.7460 (2013)

the first steps. The eventual goal would be to demonstrate BosonSampling with (say) $n = 20$ or $n = 30$ photons: a regime where the quantum experiment probably *would* outperform its fastest classical simulation, if not by an astronomical amount. In our view, this would be an exciting proof-of-principle for quantum computation.

Preskill,
arXiv:1203.5813 (2012)

though, this linear optics experiment is still not at all easy — to reach the regime where digital simulation is currently infeasible one should detect a coincidence of about 30 photons, whose paths through the interferometer can interfere. Further-

Goldstein et al.,
Phys. Rev. B **95** (2017)
Barkhofen et al.,
Phys. Rev. Lett **118** (2017)

extending to N of order 20 with currently available coherence times, clearly growing beyond the capabilities of modern classical supercomputing. We note that the fidelity will not

input modes [3–11]. However, it remains a challenge to scale up the devices to 20–30 photons [1] traversing a correspondingly large network, a regime in which a quantum boson sampling machine is expected to outperform classical computers.

Latmiral et al.,
New J. Phys **18** (2016)

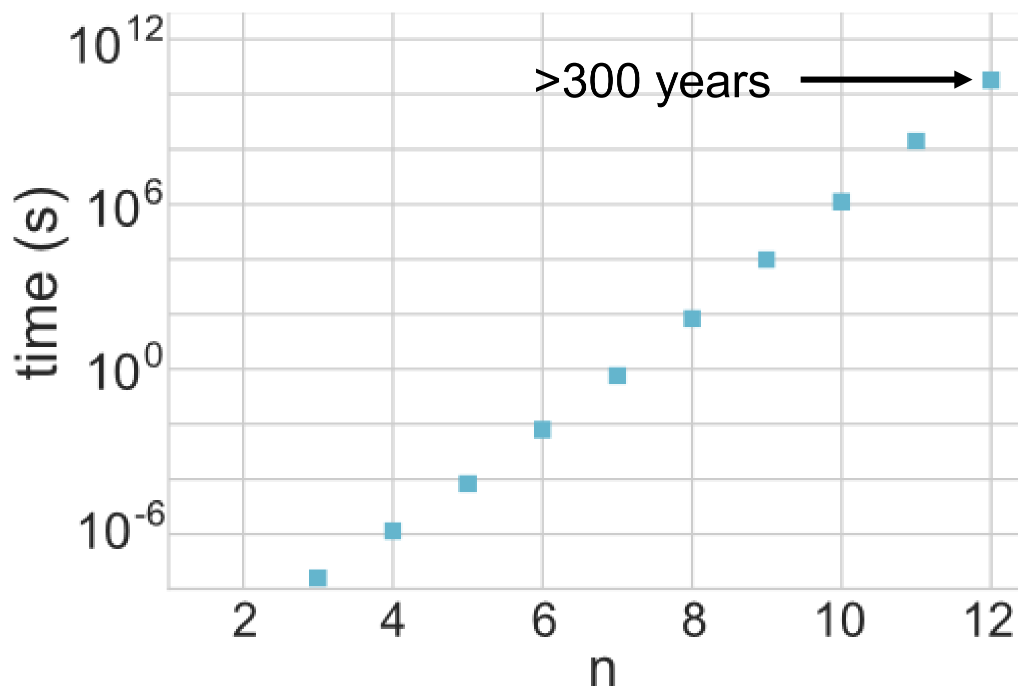
cavities decoherence time. The final theoretical result leads to a significant improvement in the efficiency and an additional step towards quantum supremacy which can be achieved with a 7 photons in 50 modes experiment.

Brute force sampling

1. Compute $\Pr(S)\,\forall\, S$
2. While more samples required
   2.1. Sample S with prob. $\Pr(S)$

$\binom{m}{n}$ $n \times n$ matrix permanents in the CFS

$$\binom{n^2}{n} \gg n^n \gg n2^n$$

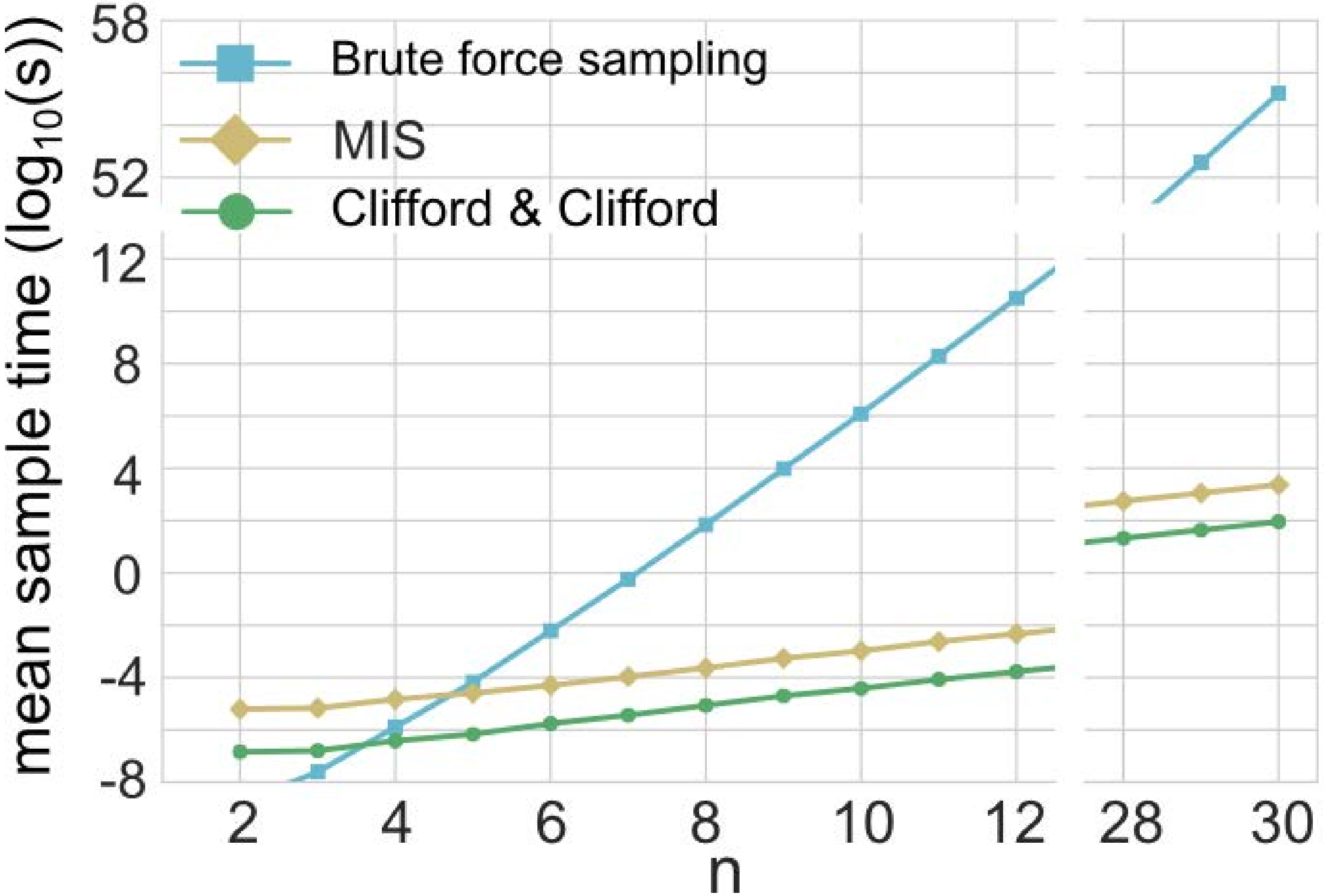


Our results – classical sampling algorithms

- **<u>Metropolised independence sampling (MIS)</u>**
  Neville et al. *Nature Physics (2017)*
  - Markov chain Monte Carlo
  - Collision free subspace
  - Experimentally verified approximate sampling
  - $\sim$ 100 $n \times n$ matrix permanents per sample for $n \leq 30$ (and probably beyond)

- **<u>Exact boson sampling</u>**
  Clifford & Clifford, Proc. SODA (2018)
  - Full space with collisions
  - Rigorously proven exact sampling
  - Equivalent of $\sim$ 2 $n \times n$ matrix permanents per sample

Our results – sampler timing (on a laptop)



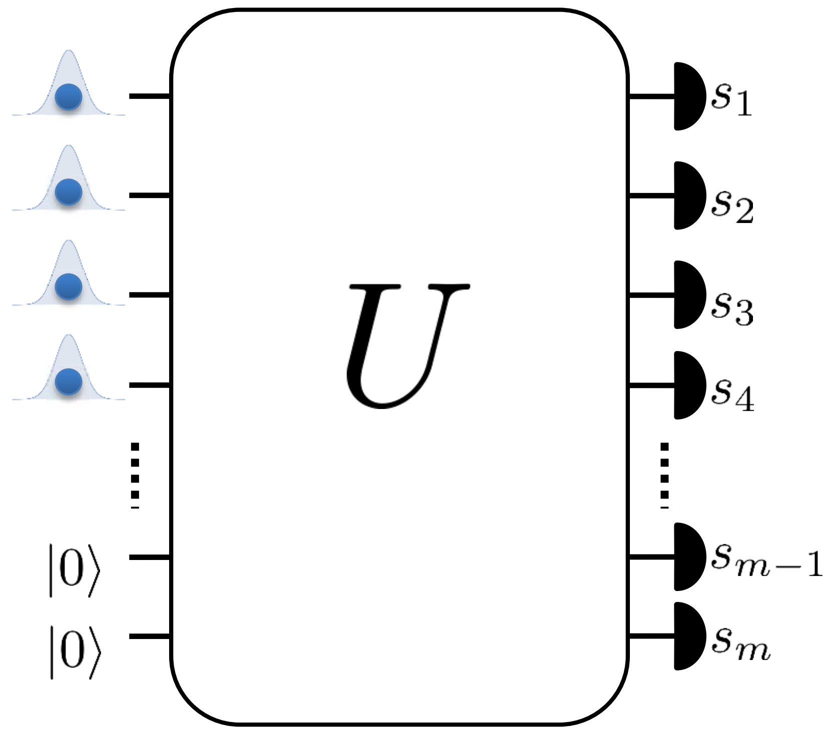
Experimental time "complexity" – photon loss

$\Pr(\text{an input photon survives}) = \eta \ (< 1)$

$\Pr(\text{n input photons survive}) = \eta^n$

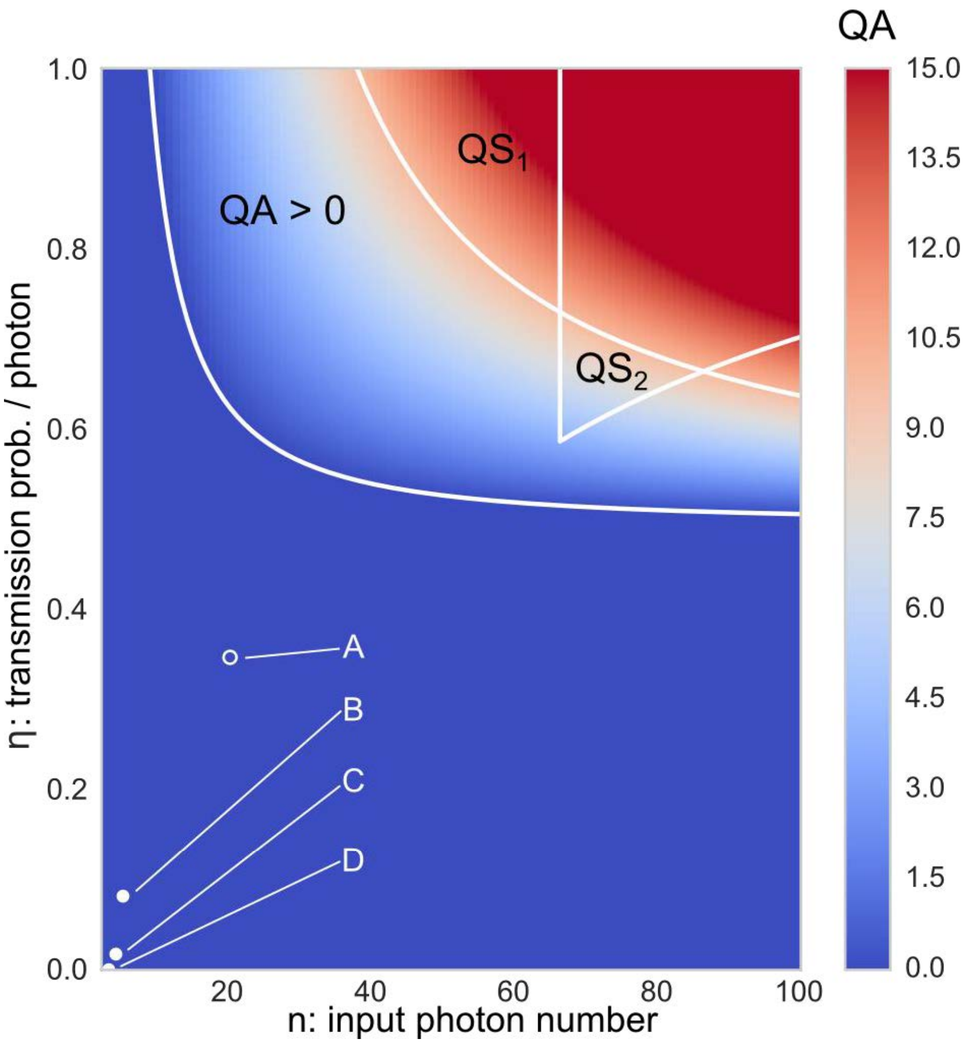
$\mathbb{E}(\text{time per n photon output}) = \mathcal{O}\left((\frac{1}{\eta})^n \right)$

If η decreases with n (without some dominating boost in survival probability), there is no asymptotic advantage



Typical implementation
(each photon sees $\mathcal{O}(n^2)$ beamsplitters): $\frac{1}{\eta} \propto c^{n^2}$ (!!!)

Revised expectations



$m = n^2$

Source rep. rate: $R(n) = 10 \text{ GHz}$

$$\boxed{q_t(n) = (R(n) P_{\text{CFS}} \eta^n)^{-1}}$$

$$\boxed{c_t(n) = 8 \times 10^{-15} n 2^n}$$

$$\text{QA}(n, \eta) = \max\left[0, \log_{10}\left(\frac{c_t}{q_t}\right)\right]$$

$$\text{QS}_1 : \text{QA} > 10$$

$$\text{QS}_2 : q_t < 1 \text{ week}, c_t > 100 \text{ years}.$$

1 day of supercomputer time:
58 photon boson sampling

A,B: Wang et al. *Nature Photonics (2017)*

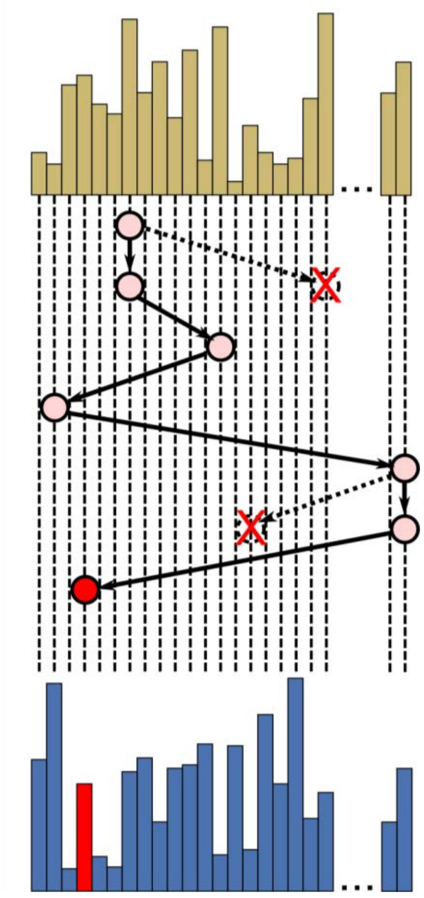
MCMC - Metropolised independence sampling (MIS)

- Markov chain: $\Pr(S^{(i)}|S^{(i-1)}, ..., S^{(1)}) = T(S^{(i)}|S^{(i-1)})$

- Proposal distribution: Distinguishable particle distribution

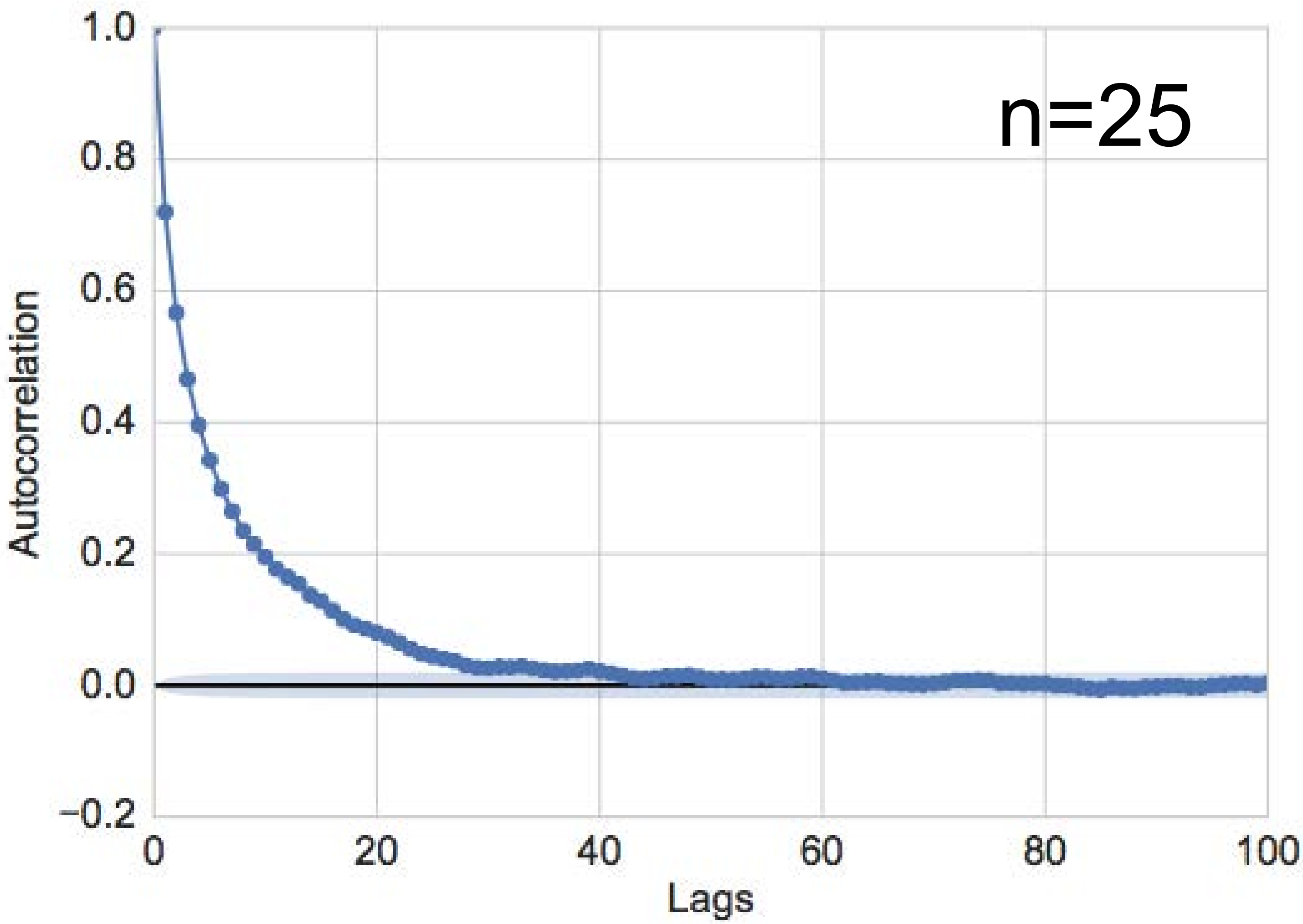
$\Pr_{\mathrm{D}}(S) = \mathrm{Per}(|A_S|^2) \longleftarrow$ Elementwise $|\cdot|^2$

1. Sample initial state S with prob. $\Pr_D(S)$
2. While more samples required
    2.1. Sample S' with prob. $\Pr_D(S')$
    2.2. $\Pr(S \to S') = \min\left(1, \dfrac{\Pr(S')}{\Pr(S)} \dfrac{\Pr_D(S)}{\Pr_D(S')}\right)$



Neville et al. *Nature Physics (2017)*

Metropolised independence sampling – Thinning



Neville et al. *Nature Physics (2017)*

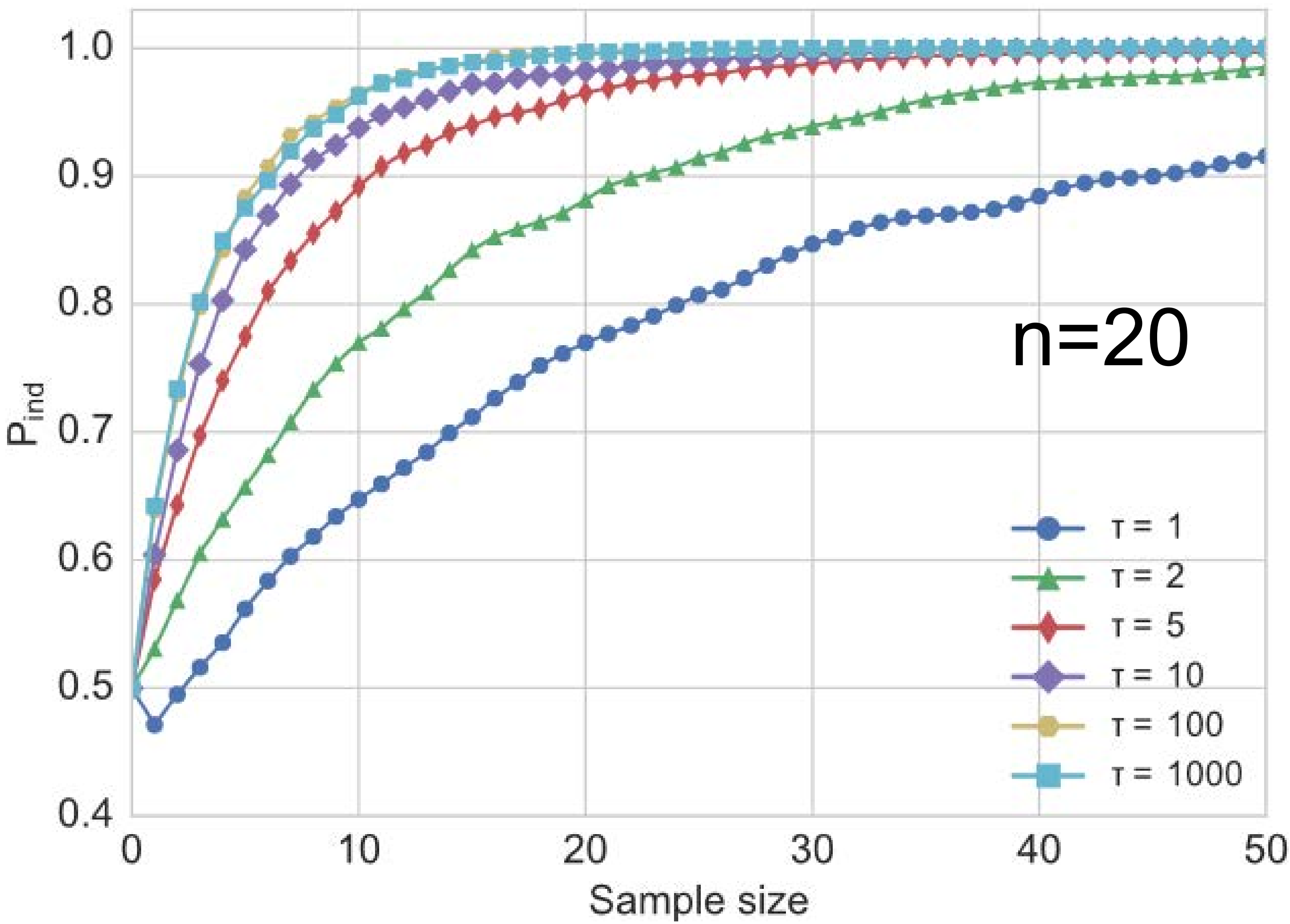
MIS – "Verification" overview

- **Likelihood ratio test**
  - Alternative hypothesis: distinguishable particle sampling
  - Used by experimentalists – photon distinguishability is a source of error
  - Relevant to MIS – proposals from distinguishable particle distribution
  - $n = 7, 12, 20, 25$

- **2-sample Bootstrap Kolmogorov-Smirnov (KS) test**
  - Null hypothesis: 2 samples are drawn from same distribution
  - Comparison samplers: <u>Brute force</u>, <u>rejection</u>, <u>alternative MIS (longer thinning interval)</u>, <u>distinguishable particles</u>
  - $n = 7, 12, 20$

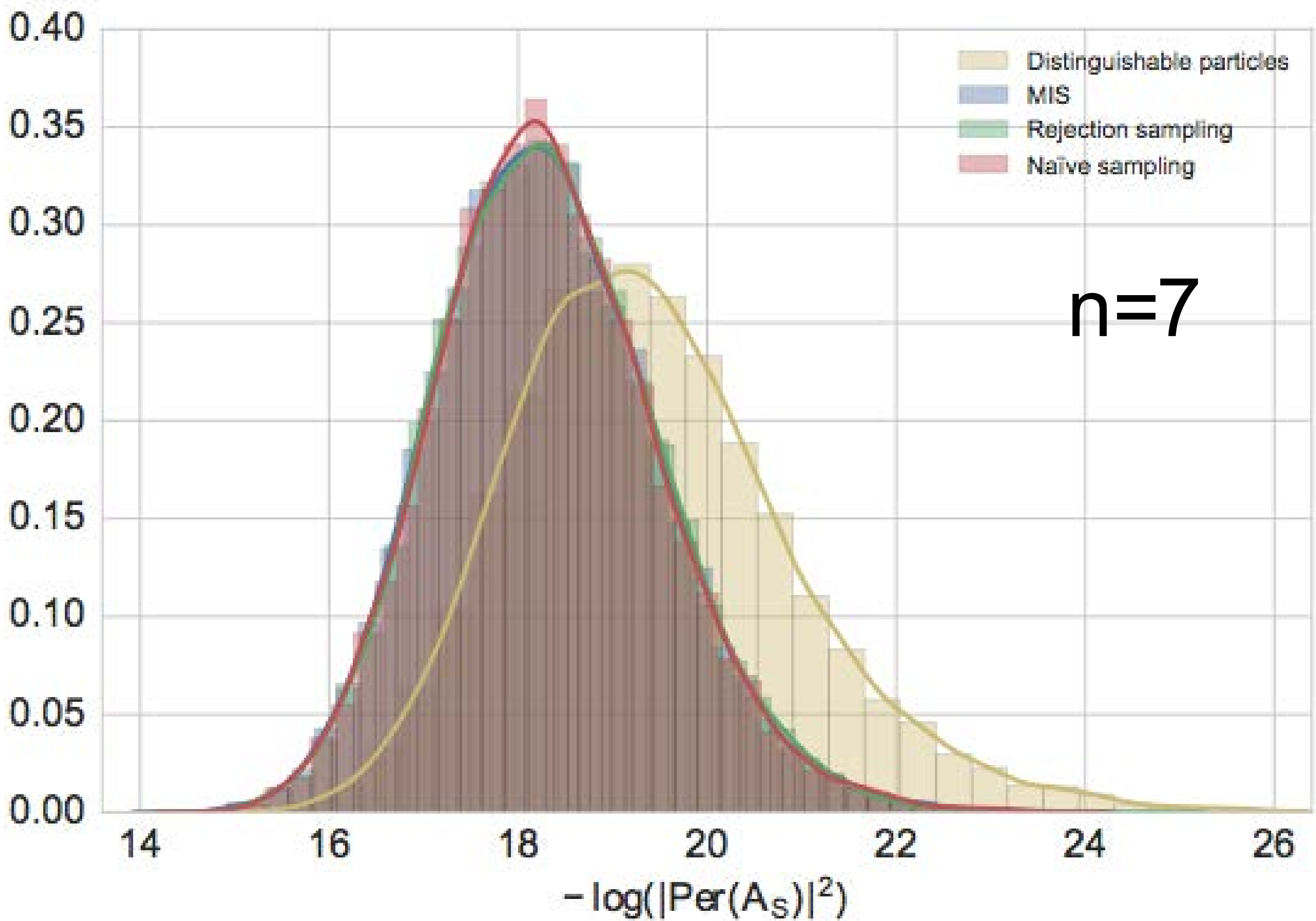
Neville et al. *Nature Physics (2017)*

MIS – "Verification" part 1



Neville et al. *Nature Physics (2017)*

MIS – “Verification” part 2



Neville et al. *Nature Physics (2017)*

Exact classical algorithms for boson sampling

Rough idea:

1. Expand the sample space
2. Compute marginal probabilities for row and column choices
3. Incrementally sample rows and columns using probabilistic chain rule
4. Exploit Laplace expansion of the permanent for further speedup

Clifford & Clifford, Proc SODA (2018)

Clifford & Clifford Algorithm A

Recall: $S = (s_1, \dots, s_m)$ How many of each row to include.

Rewrite: $\mathbf{z} = (z_1, \dots, z_n)$ Which rows to include.

$$z_1 \leq z_2 \leq \cdots \leq z_n$$

Expand the space by relaxing the ordering requirement:

$$\mathbf{r} = (r_1, \dots, r_n) \in [m]^n$$

Sample from $p(\mathbf{r}) = \frac{1}{n!} |\mathrm{Per} A_{\mathbf{r}}|^2$

(The permanent is invariant under permutations of rows)

Clifford & Clifford, Proc SODA (2018)

Clifford & Clifford Algorithm A

Marginals:

$$p(r_1, \ldots, r_k) = \frac{(n-k)!}{n!} \sum_{c \in \mathcal{C}_k} \left| \mathrm{Per} A^c_{r_1,\ldots,r_k} \right|^2$$

$\mathcal{C}_k$ - Set of k-combinations from $[n]$

$A^c_{r_1,\ldots,r_k}$ - Submatrix of A with rows $r_1, \ldots, r_k$ and columns c

Chain rule:

$$p(\mathbf{r}) = p(r_1)p(r_2|r_1)p(r_3|r_1,r_2)\ldots p(r_n|r_1,\ldots,r_{n-1})$$

Clifford & Clifford, Proc SODA (2018)

Clifford & Clifford Algorithm A

1. Sample r_1 from $p(r_1)$

2. Sample r_2 from $p(r_2|r_1) \propto p(r_1, r_2)$

…

n. Sample r_n from $p(r_n|r_1, \ldots, r_{n-1}) \propto p(r_1, \ldots, r_n)$

Running time: $$m \sum_{k=1}^{n} k 2^k \binom{n}{k} = \mathcal{O}(mn3^n)$$

Clifford & Clifford, Proc SODA (2018)

Clifford & Clifford Algorithm B

Further **expand** the space:

Introduce $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \pi[n]$ - auxiliary column labels (permutation)

Define a new **marginal** pmf:

$$\phi(r_1, \ldots, r_k | \boldsymbol{\alpha}) = \frac{1}{k!} \left| \mathrm{Per} A^{\alpha_1, \ldots, \alpha_k}_{r_1, \ldots, r_k} \right|^2$$

$$\boxed{\text{Sample from } p(r_1, \ldots, r_n) = \mathbb{E}_{\boldsymbol{\alpha}} \{ \phi(r_1, \ldots, r_n | \boldsymbol{\alpha}) \}}$$

(The permanent is invariant under permutations of columns)

Clifford & Clifford, Proc SODA (2018)

Clifford & Clifford Algorithm B

0. Sample $\boldsymbol{\alpha}$ from $\mathrm{Unif}\{\pi[n]\}$

1. Sample r_1 from $\phi(r_1|\boldsymbol{\alpha})$

2. Sample r_2 from $\phi(r_2|r_1, \boldsymbol{\alpha}) \propto \phi(r_1, r_2|\boldsymbol{\alpha})$

…

n. Sample r_n from $\phi(r_n|r_1, \dots, r_{n-1}, \boldsymbol{\alpha}) \propto \phi(r_1, \dots, r_n|\boldsymbol{\alpha})$

Running time: $\sum_{k=1}^{n} mk2^k = \mathcal{O}(mn2^n)$?

Clifford & Clifford, Proc SODA (2018)

Clifford & Clifford Algorithm B

Trick:

$$\mathrm{Per}(X) = \sum_{j=1}^{n} x_{nj} \mathrm{Per} X_{1,\dots,n-1}^{[n]\setminus j}$$

(Laplace expansion of the permanent)

+

$$\mathrm{Per}(X) = \frac{1}{2^{n-1}} \left[\sum_{\delta} \left(\prod_{k=1}^{n} \delta_k \right) \prod_{j=1}^{n} \sum_{i=1}^{n} \delta_i x_{ij} \right]$$

(Glynn formula for the permanent (Gray code ordered))

$$\Downarrow$$

Evaluate $\phi(r_k | r_1, \dots, r_{k-1}, \boldsymbol{\alpha}) \; \forall \; r_k \in [m]$ in $\mathcal{O}(k2^k + mk)$ time

Clifford & Clifford, Proc SODA (2018)

Clifford & Clifford Algorithm B

0. Sample $\boldsymbol{\alpha}$ from $\mathrm{Unif}\{\pi[n]\}$

1. Sample r_1 from $\phi(r_1|\boldsymbol{\alpha})$

2. Sample r_2 from $\phi(r_2|r_1, \boldsymbol{\alpha}) \propto \phi(r_1, r_2|\boldsymbol{\alpha})$

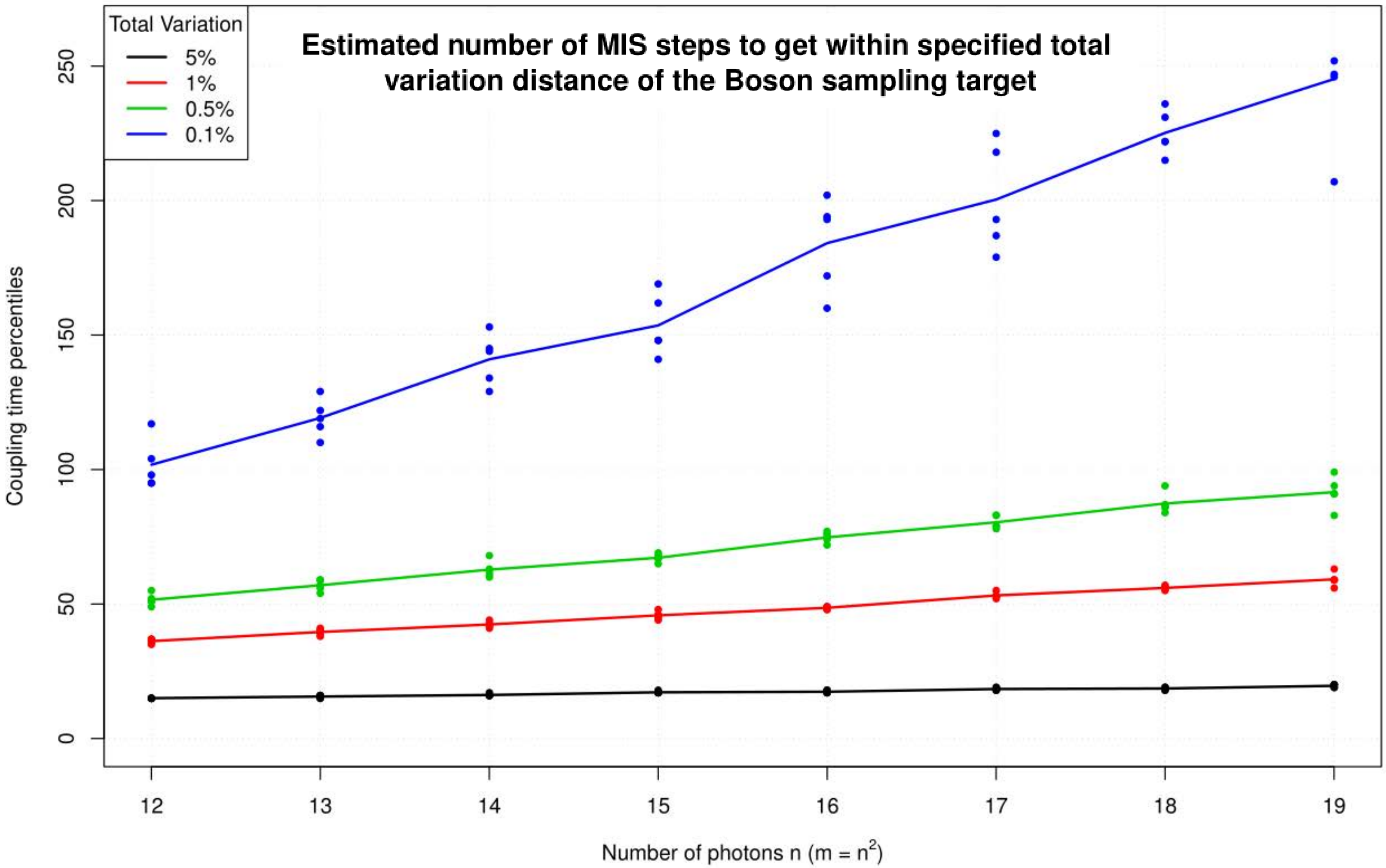
…

n. Sample r_n from $\phi(r_n|r_1, \ldots, r_{n-1}, \boldsymbol{\alpha}) \propto \phi(r_1, \ldots, r_n|\boldsymbol{\alpha})$

Running time: $\sum_{k=1}^{n} (k2^k + mk) = \mathcal{O}(n2^n + mn^2)$

Clifford & Clifford, Proc SODA (2018)

Bonus: MIS mixing time



- For $n = 20$ suggests that 50 burn in states required to sample from distribution at most ~1% close to target

Jun S. Liu, *Monte Carlo strategies in scientific computing* (2001)

Could approximate rejection sampling work?

With thanks to the QIP PC

Corollary 24 (Brandao) *Let $m \geq n^{5.1}$, and let $A \in \mathbb{C}^{m \times n}$ be a Haar-random* BosonSampling *matrix. Then for all $\varepsilon, \delta > 0$, with probability at least $1-\delta$ over A, there exists a distribution $\mathcal{D}'$ over $\Lambda_{m,n}$ such that $\|\mathcal{D}' - \mathcal{D}_A^*\| \leq \varepsilon$ and*

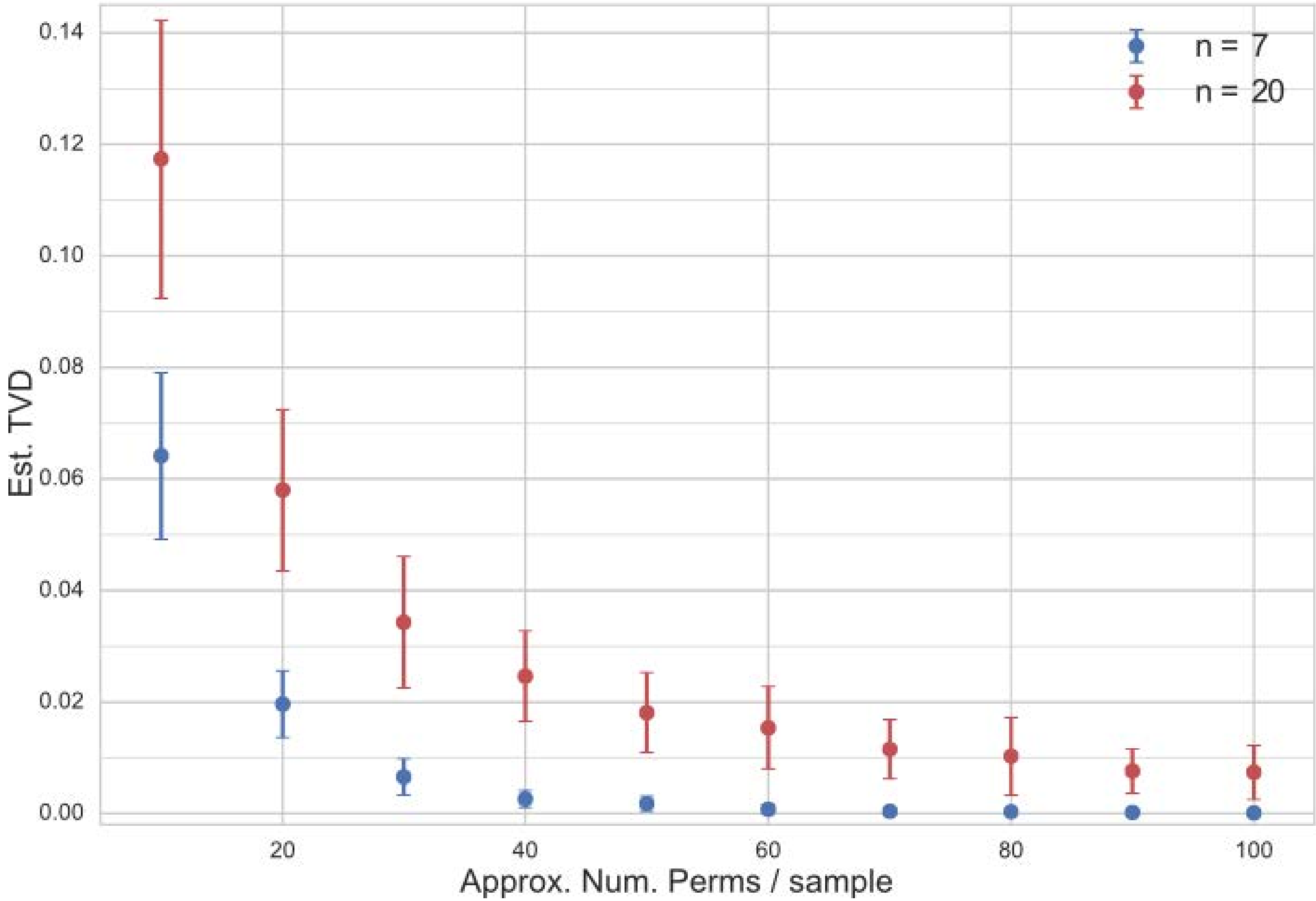
$$H_{\min}(\mathcal{D}') \geq \log_2 \binom{m}{n} - \log_2 \frac{n}{\varepsilon\delta} - O(1).$$

Translation: Rejection sampling with uniform proposal *could* be good* to approximately sample from a distribution at least ϵ close to the target distribution with probability $1-\delta$ over boson sampling matrices with $m = n^2$ and small n

*with $\mathcal{O}(\frac{n}{\epsilon\delta})$ accepts per reject

Aaronson & Arkhipov, arXiv:1309.7460 (2013)

Rejection sampling - How many permanents per sample?



- For $n = 20$ need > 100 permanent computations per sample for reliable TVD $< 1\%$
- Seems slower than MIS for same n, but a comparison requires subtle considerations

Rejection sampling - summary

- We tested to see whether rejection sampling could perform approximate sampling for small n and with small TVD to the boson sampling distribution

- It seems to work remarkably well, but still probably slower than MIS

- Clifford and Clifford sampler **faster** than both and **samples exactly** for **all input unitaries**

Future work

- **Improve classical algorithms as much as possible**
  - E.g. Include realistic experimental error
    (García-Patrón et al. arXiv:1712.10037, Renema et al. arXiv:1707.02793)
- **Verification schemes**
  - Using the exact sampler
  - Without computing additional permanents?
    (Agresti et al. arXiv:1712.06863)
- **Reduce linear optical circuit depth?**
- **Loss tolerance scheme?**

Conclusion

- Classical boson sampling algorithms vastly outperform current state of the art quantum experiments
- Increased lower bound for the number of photons required to demonstrate quantum computational supremacy by boson sampling
- Decreased upper bound on quantum advantage by boson sampling as a function of n and η
- **It's important to optimise classical algorithms as much as possible – 2018 may be the year of the first quantum computational supremacy claim, it would be sad if it had to be retracted**

Thank you