ACHIEVING THE HEISENBERG LIMIT IN QUANTUM METROLOGY USING QUANTUM ERROR CORRECTION

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SZ, M. Zhang, J. Preskill, L. Jiang. *Nature Communications* 9:78 (2018)

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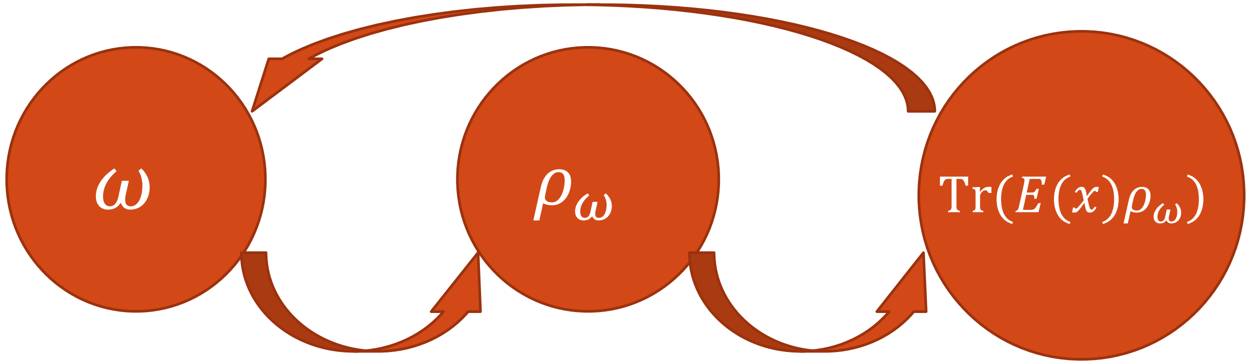
PARAMETER ESTIMATION

Estimator: $\widehat{\omega}: x \mapsto \widehat{\omega}(x)$



Probing

POVM $\{E(x)\}$

For example, ω is the strength of the Hamiltonian, $H = \omega \cdot G$.

Resource:

τ: probing time

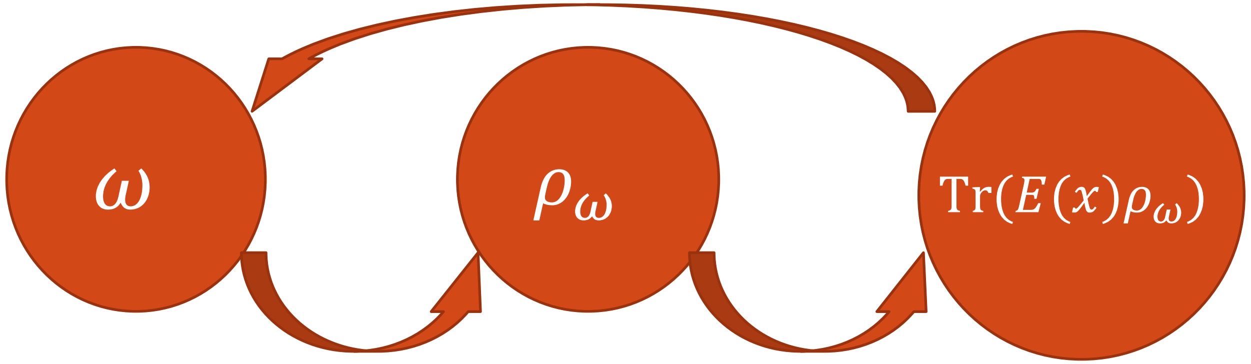
N: number of probes

Goal:

Minimize the total probing time $t = N\tau$ required to obtain certain estimation precision $\delta\omega$

PARAMETER ESTIMATION

Estimator: $\widehat{\omega}: x \mapsto \widehat{\omega}(x)$



Probing POVM $\{E(x)\}$

For example, ω is the strength of the Hamiltonian, $H = \omega \cdot G$.

Quantum Cramér-Rao Bound:

$$\delta\omega \geq \frac{1}{\sqrt{F(\rho_\omega)}}$$

$F(\rho_\omega)$: Quantum Fisher Information

Goal:

Maximize $F(\rho_\omega)$ over t

Braunstein & Caves. PRL 72(22):3439 (1994)

QUANTUM FISHER INFORMATION

Quantum Fisher Information:

$$F(\rho) = \mathrm{Tr}\left(\rho L_\rho^2\right)$$

where L_ρ (symmetric logarithmic derivative) is the unique Hermitian operator satisfying $\frac{\partial \rho}{\partial \omega} = \frac{1}{2}(\rho L_\rho + L_\rho \rho)$.

For a unitary channel, $\rho = |\psi(t)\rangle\langle\psi(t)|$ where $|\psi(t)\rangle = e^{-i\omega G t}|\psi(0)\rangle$,

$$F(\rho) = 4t^2(\langle\psi(0)|G^2|\psi(0)\rangle - \langle\psi(0)|G|\psi(0)\rangle^2) = 4t^2 \Delta G^2$$

$$\boxed{\delta\omega \propto \frac{1}{t}}$$

the Heisenberg limit

Classically, $t \propto$ number of experiments

$$\boxed{\delta\omega \propto \frac{1}{\sqrt{t}}}$$

the standard quantum limit

SPIN ½ SYSTEM

Hamiltonian: $H = \omega \frac{\sigma_z}{2}$

$$|\psi(0)\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{\text{probing time } t} |\psi(t)\rangle = \frac{e^{-i\omega t/2}|0\rangle + e^{i\omega t/2}|1\rangle}{\sqrt{2}}$$

$$p = |\langle\psi(t)|\psi(0)\rangle|^2 = \frac{1}{2} + \frac{1}{2}\cos\omega t \rightarrow \boxed{\delta\omega \propto \frac{1}{t}}$$

$$|\psi(0)\rangle = \frac{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}}{\sqrt{2}} \xrightarrow{\text{probing time } \tau = \frac{t}{N}} |\psi(\tau)\rangle = \frac{e^{-i\omega N\tau/2}|0\rangle^{\otimes N} + e^{i\omega N\tau/2}|1\rangle^{\otimes N}}{\sqrt{2}}$$

$$p = |\langle\psi(\tau)|\psi(0)\rangle|^2 = \frac{1}{2} + \frac{1}{2}\cos\omega N\tau \rightarrow \boxed{\delta\omega \propto \frac{1}{t}}$$

SPIN ½ SYSTEM

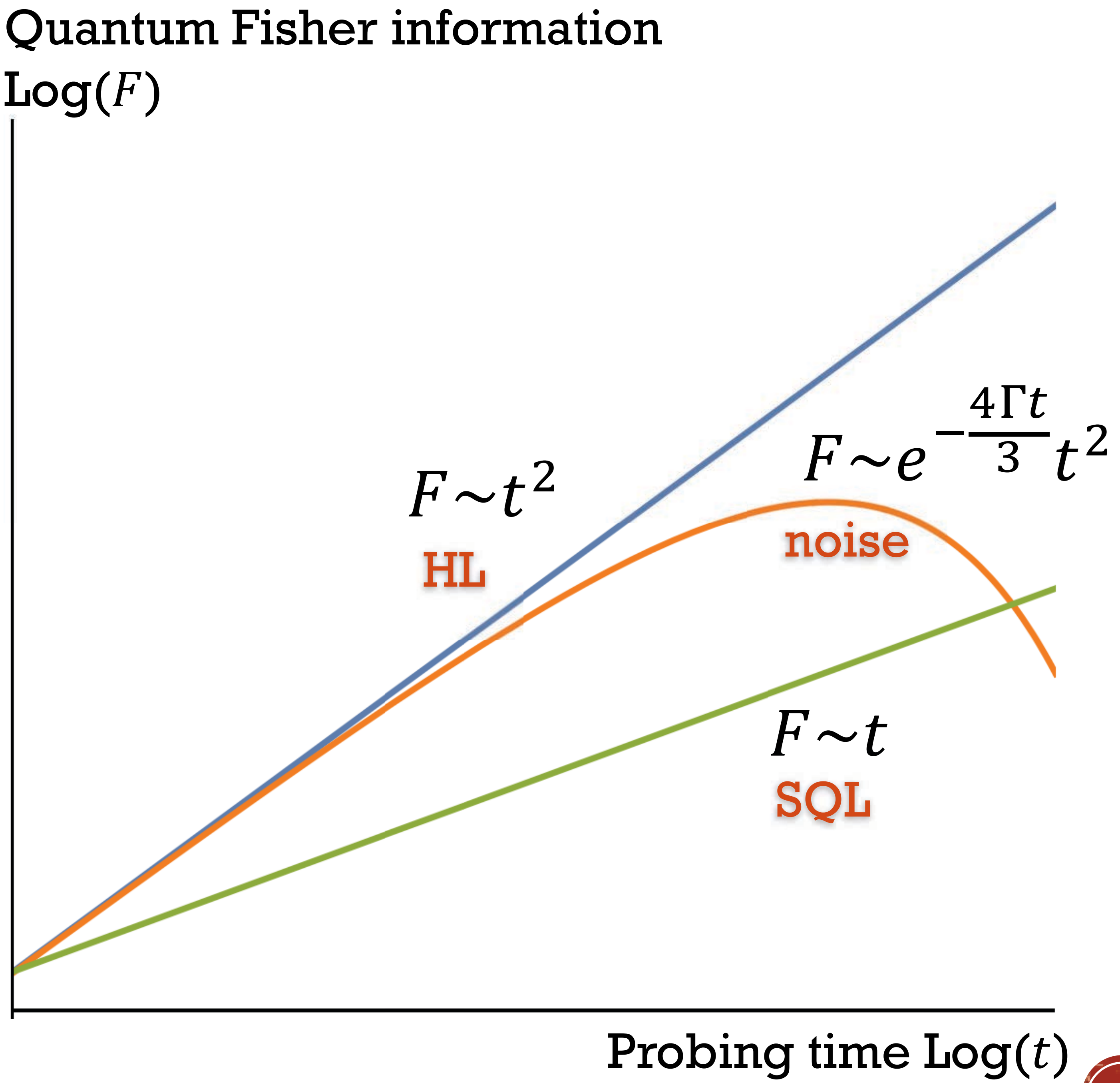
When noise exists

(e.g. depolarizing noise):

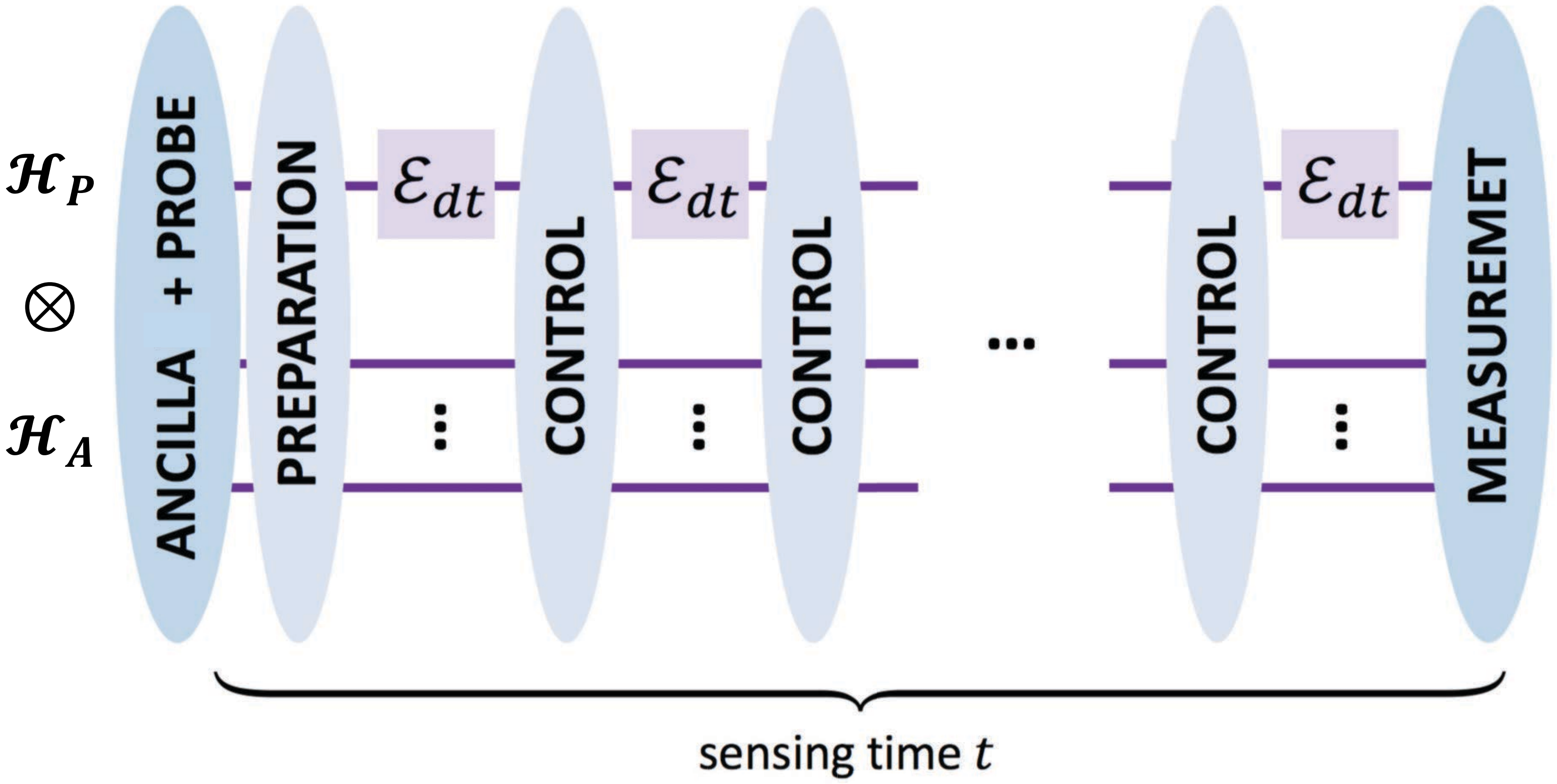
$$p = \frac{1}{2} + \frac{1}{2} e^{-\frac{2\Gamma}{3}t} \cos \omega t,$$

Γ is the noise rate.

The Heisenberg limit is no longer achievable.



METROLOGY PROTOCOL -- THE SEQUENTIAL SCHEME



Sequential scheme: the most general metrology protocol

Assume access to

(1) Noiseless ancilla
(2) Fast & accurate quantum control

Demkowicz-Dobrzański *et al.* PRL. 113, 250801 (2014)

MARKOVIAN NOISE

Quantum master equation ($H = \omega G$):

$$\frac{d\rho}{dt} = -i[\omega G, \rho] + \sum_{k=1}^{r} \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

The quantum channel ($dt \to 0$):

$$\mathcal{E}_{dt}(\rho) = \rho - i[\omega G, \rho]dt + \sum_{k=1}^{r} \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) dt + O(dt^2)$$

$$= \sum_{k=0}^{r} K_k \rho K_k^\dagger + O(dt^2),$$

where $K_0 = 1 + \left(-iH - \frac{1}{2}\sum_{k=1}^{r} L_k^\dagger L_k\right) dt$ and $K_k = L_k\sqrt{dt}$.

Breuer & Petruccione. *The theory of open quantum systems* (2002)

THE HAMILTONIAN-NOT-IN-LINDBLAD-SPAN (HNLS) CRITERION

For a finite-dimensional probe, ω can be estimated with Heisenberg-limited precision (asymptotically) if and only if

$$\boxed{G \notin \textbf{Lindblad Span } \mathcal{S} = \mathrm{span}\{I, L_j, L_k^\dagger L_j, \forall k, j\}}$$

- The necessity is proven by upper bounding quantum Fisher information;
- The sufficiency is proven by construction of a quantum error correction code to recover the unitary channel.

Demkowicz-Dobrzański *et al.* PRX. 7(4):041009 (2017)
SZ, *et al.* NComms 9:78 (2018)

QUANTUM ERROR CORRECTION

Given quantum channel $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$ and $\rho = \Pi_C \rho \Pi_C$,

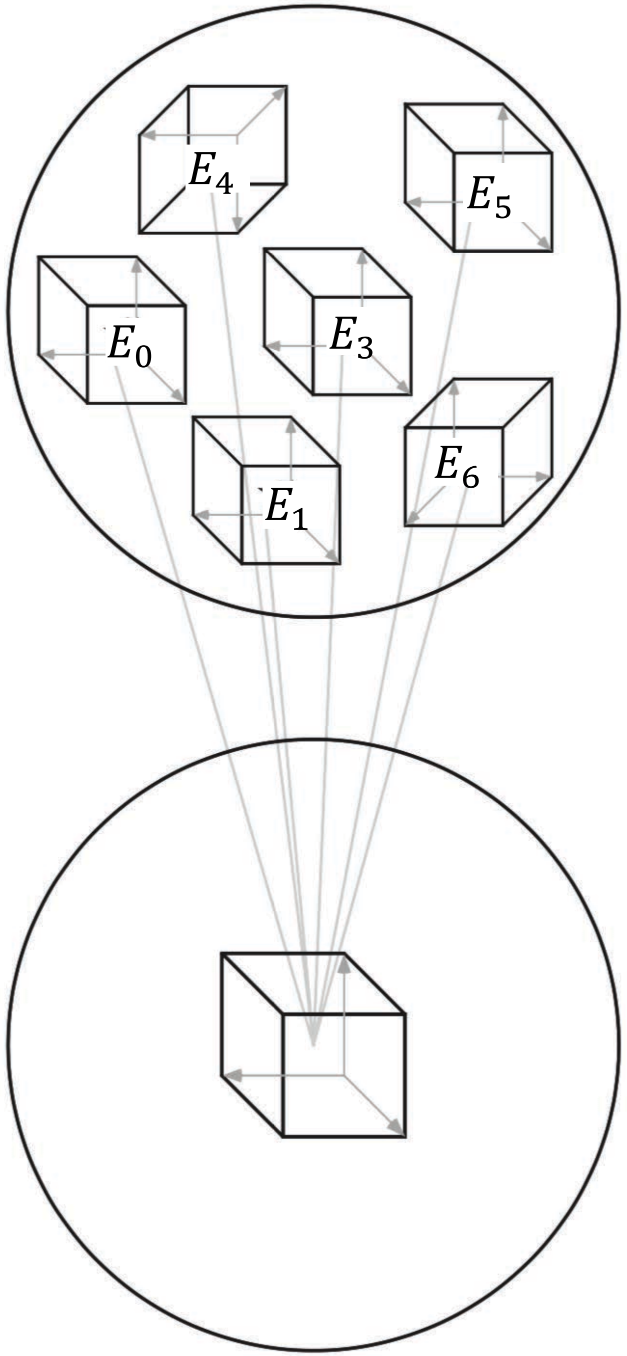
$\exists$ A recovery channel $\mathcal{R}$, s.t.

$$\rho = \mathcal{R} \circ \mathcal{E}(\rho)$$

$$\Leftrightarrow \boxed{\Pi_C E_j^\dagger E_k \Pi_C = \alpha_{jk} \Pi_C}$$

where Π_C is the projection onto the code space.

Bennett, *et al*. PRA 54(5), 3824 (1996)
Knill & Laflamme. PRA 55(2), 900 (1997)



Nielsen & Chuang, 2010

QUANTUM ERROR CORRECTION FOR SENSING

$$(1)\ \Pi_C(L_j \otimes I)\Pi_C = \lambda_j \Pi_C, \forall j$$

$$(2)\ \Pi_C(L_j^\dagger L_k \otimes I)\Pi_C = \mu_{jk} \Pi_C, \forall j, k$$

Correct noise

$$(3)\ \Pi_C(G \otimes I)\Pi_C \neq \text{constant} \cdot \Pi_C$$

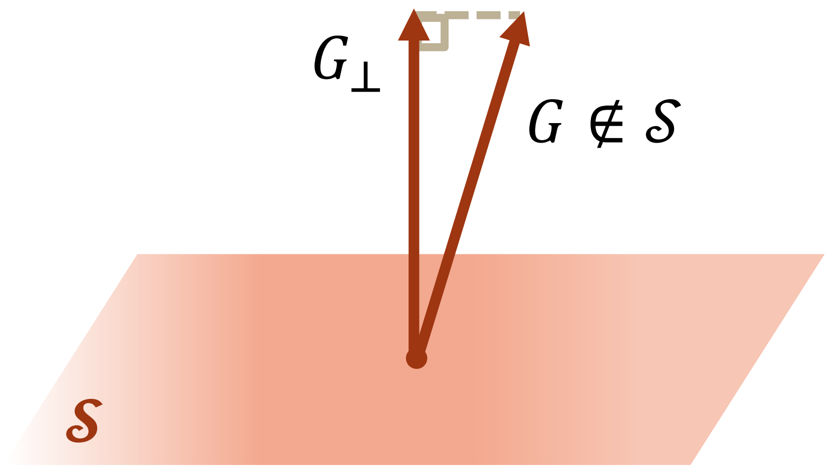
Detect signal

Effective Hamiltonian:

$$\mathcal{R} \circ \mathcal{E}_{dt}(\rho) = \rho - i[H_{\text{eff}}, \rho]dt + O(dt^2)$$

$$H_{\text{eff}} = \omega \Pi_C(G \otimes I)\Pi_C$$

CONSTRUCTION OF THE CODE



Lindblad Span: $\mathcal{S} = \text{span}\{I, L_j, L_k^\dagger L_j, \forall k, j\}$.

$$G = G_\parallel + G_\perp,$$

where $G_\parallel \in \mathcal{S}$ and $G_\perp \in \mathcal{S}^\perp$ is a non-zero traceless Hermitian matrix.

$$G_\perp = \frac{1}{2}\text{Tr}(|G_\perp|)(\rho_0 - \rho_1),$$

where $\rho_{0,1}$ are positive matrices with trace one. Then we choose $|C_0\rangle$ and $|C_1\rangle$ to be purification of ρ_0 and ρ_1 with orthogonal support in the ancillary space $\mathcal{H}_A$. For example, $\rho_0 = \sum_i \lambda_i |i\rangle_P\langle i|_P \Rightarrow |C_0\rangle = \sum_i \sqrt{\lambda_i}|i\rangle_P|i\rangle_A$.

CONSTRUCTION OF THE CODE

- $\langle C_0|O \otimes I|C_1\rangle = 0$, for any operator in the probe space $\mathcal{H}_P$.
- $\langle C_0|O \otimes I|C_0\rangle - \langle C_1|O \otimes I|C_1\rangle \propto \mathrm{Tr}(OG_\perp)$. When $O \in \mathcal{S}$, it is zero; when $O = G$, it is non-zero.

$$(1)\ \Pi_C(L_j \otimes I)\Pi_C = \lambda_j \Pi_C, \forall j$$

$$(2)\ \Pi_C(L_j^\dagger L_k \otimes I)\Pi_C = \mu_{jk} \Pi_C, \forall j, k$$

$$(3)\ \Pi_C(G \otimes I)\Pi_C \neq \text{constant} \cdot \Pi_C$$

EXAMPLE: QUBIT PROBE

- $H = \frac{\omega}{2}\sigma_z$
- $L = \sigma_x$

The optimal QFI: $F(\rho) = t^2$

QFI without noise: $F(\rho) = t^2$

The optimal code: $|C_0\rangle = |0\rangle_P \otimes |0\rangle_A; |C_1\rangle = |1\rangle_P \otimes |1\rangle_A$

Kessler, *et al*. PRL 112, 150802 (2014)
Arrad, *et al*. PRL 112, 150801 (2014)
Dür, *et al*. PRL 112, 080801 (2014)
Ozeri. arXiv:1310.3432 (2013)
Unden, *et al*. PRL 116, 230502 (2016)

CODE OPTIMIZATION

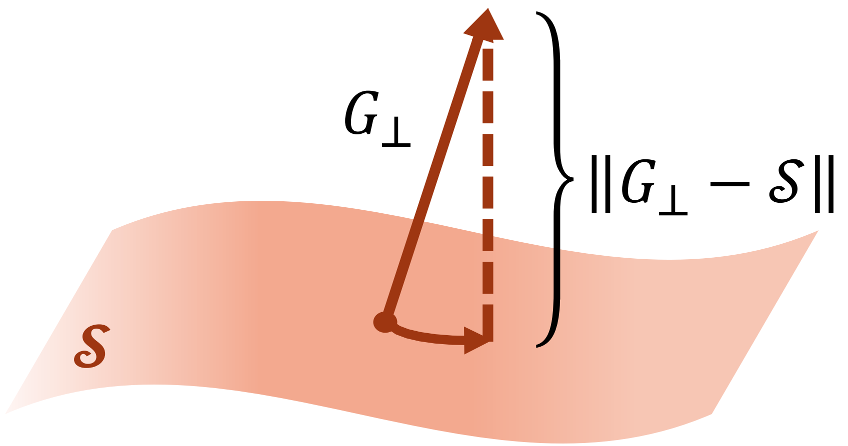
Optimization problem: $\tilde{G} = \widetilde{\rho_0} - \widetilde{\rho_1}$

- maximize $\mathrm{Tr}(\tilde{G}G)$
- subject to $\mathrm{Tr}(|\tilde{G}|) \leq 2$ and $\mathrm{Tr}(\tilde{G}O) = 0, \forall O \in \mathcal{S}$

The optimal quantum Fisher information:

$$F(\rho) = 4t^2 \|G_\perp - \mathcal{S}\|^2 = 4t^2 \|G - \mathcal{S}\|^2,$$

$\|\cdot\|$ is the operator norm.



EXAMPLE: KERR EFFECT WITH PHOTON LOSS

- $H = \omega\left(a^\dagger a\right)^2$
- $L = a$

Assume photon number is bounded by $\bar{n}$ (even).

The optimal QFI: $F(\rho) = t^2\bar{n}^4/16$

QFI without noise: $F(\rho) = t^2\bar{n}^4$

The optimal code: $|C_0\rangle = |\bar{n}/2\rangle; |C_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |\bar{n}\rangle)$

Michael, *et al*. PRX, 2016, 6(3): 031006 (2016)

CONCLUSION & OUTLOOK

- The HNLS Criterion: $G \notin \mathcal{S} = \mathrm{span}\{I, L_j, L_k^\dagger L_j, \forall k, j\}$
    - When HNLS is violated, SQL scaling cannot be surpassed.
    - When HNLS is satisfied, a QEC code can be constructed which achieves HL scaling.
- Generalization to infinite dimension
- Relaxation of the assumptions (noiseless ancilla, fast and accurate quantum control)
- Non-Markovian noise

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Alfred P. Sloan Foundation

GORDON AND BETTY MOORE FOUNDATION

NON-ACHIEVABILITY OF THE HEISENBERG LIMIT

The upper bound of QFI:

$$F\big(\rho(t)\big) \leq 4\frac{t}{dt}\|\alpha_{dt}\| + 4\left(\frac{t}{dt}\right)^2 \|\beta_{dt}\|\left(\|\beta_{dt}\| + 2\sqrt{\|\alpha_{dt}\|}\right),$$

$\|\cdot\| = \max_{|\psi\rangle} |\langle\psi| \cdot |\psi\rangle|$ is the operator norm.

$$\alpha_{dt} = \big(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\big)^{\dagger}\big(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\big)$$

$$\beta_{dt} = i\big(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\big)^{\dagger}\boldsymbol{K}$$

$\boldsymbol{K} \coloneqq (K_0, K_1, \cdots K_r)^T$, $\dot{\boldsymbol{K}} \coloneqq \partial\boldsymbol{K}/\partial\omega$, h is any Hermitian matrix.

Sekastski et al. arXiv:1603.08944

NON-ACHIEVABILITY OF THE HEISENBERG LIMIT

Expand in series of $\sqrt{dt}$,

$$h = h^{(0)} + h^{(1)}\sqrt{dt} + h^{(2)}dt + \cdots$$

$$\alpha_{dt} = \left(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\right)^{\dagger}\left(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\right) = \alpha^{(0)} + \alpha^{(1)}\sqrt{dt} + \alpha^{(2)}dt + O(dt^{3/2})$$

$$\beta_{dt} = i\left(\dot{\boldsymbol{K}} - ih\boldsymbol{K}\right)^{\dagger} = \beta^{(0)} + \beta^{(1)}\sqrt{dt} + \beta^{(2)}dt + \beta^{(3)}dt^{3/2} + O(dt^2)$$

When $G \in \mathcal{S}$, we can choose h such that $\alpha_{dt} = O(dt)$ and $\beta_{dt} = O(dt^2)$, so

$$F\big(\rho(t)\big) \leq 4\left\|\alpha^{(2)}\right\|t + O(\sqrt{dt})$$

SEQUENTIAL SCHEME VS PARALLEL SCHEME