Optimal Port-based Teleportation in Arbitrary Dimension

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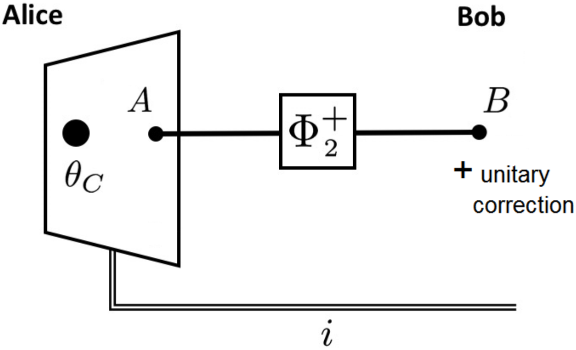
QIP2018, Delft 19.01.2018

Quantum Teleportation

Quantum Teleportation: *C.H. Bennett et al. PRL* ***70****, 1895-1899 (1993)*



$$\Phi_2^+ = |\psi_2^+\rangle\langle\psi_2^+|, \quad |\psi_2^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Port-based Teleportation (PBT)

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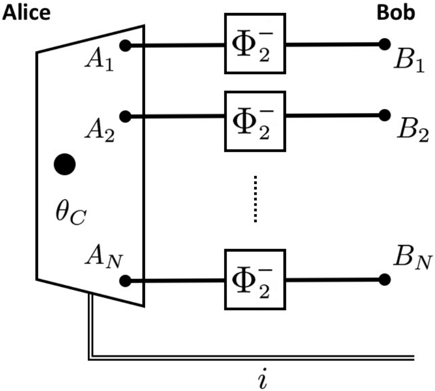
S. Ishizaka, T. Hiroshima, PRL **101***, 240501 (2008)*



$$\Phi_2^- = |\psi_2^-\rangle\langle\psi_2^-|, \quad |\psi_2^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Probabilistic and Deterministic PBT

Deterministic Scheme

- Set of measurements $\{\Pi_i\}_{i=1}^{N}$
- The state θ_C is always teleported
- Performance is described by the *entanglement fidelity* F

Probabilistic Scheme

- Set of measurements $\{\Pi_i\}_{i=0}^{N}$
- POVM Π_0 corresponds to failure
- The state θ_C is teleported perfectly
- Performance is described by the *probability of success* p

All information are encoded in **Port-based Operator**

$$\rho = \frac{1}{2^N}\sum_{i=1}^{N} P^{-}_{A_iB} \otimes \mathbf{1}_{\overline{A_i}}$$

Applications of PBT

- New architecture for the universal programmable quantum processor
  S. Ishizaka, T. Hiroshima, PRA **79**, *042306 (2009)*

- Efficient attacks for position based cryptography
  S.Beigi, R. König, NJP **13**, *093036 (2011)*

- Violation of Bell inequality from any large quantum advantage
  H. Buhrman et al. PNAS **113**, *3191-3196 (2016)*

Results for qubits ($d = 2$)

Known Results: Deterministic Case

S. Ishizaka, T. Hiroshima, PRA ***79****, 042306 (2009)*

- Average fidelity f vs. entanglement fidelity F

$$f = \frac{Fd+1}{d+1} \quad \text{here} \quad d = 2$$

- Maximally entangled states as a resource state

$$F = \frac{1}{2^{N+3}} \sum_{k=0}^{N} \left(\frac{N-2k-1}{\sqrt{k+1}} + \frac{N-2k+1}{\sqrt{N-k+1}} \right) \binom{N}{k}$$

$$f \sim 1 - O(1/N)$$

- Optimization over Alice's measurements and resource state

$$F = \cos^2 \frac{\pi}{N+2} \qquad f \sim 1 - O(1/N^2)$$

Known Results: Deterministic Case

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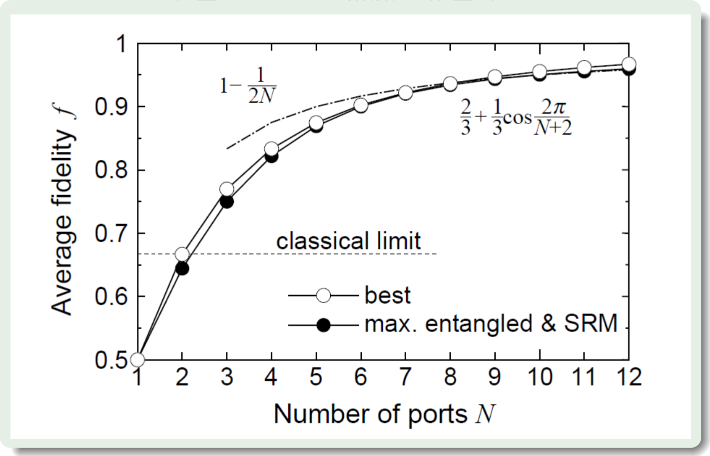
- Average fidelity f vs. entanglement fidelity F

$$f = \frac{Fd+1}{} \quad \text{here} \quad d = 2$$

- Max

- Opt



Known Results: Probabilistic Case

S. Ishizaka, T. Hiroshima, PRA **79**, *042306 (2009)*

- Maximally entangled states as a resource state

$$p = \frac{1}{2^N} \sum_{s=s_{\min}}^{(N-1)/2} \frac{(2s+1)^2 N!}{\left(\frac{N-1}{2} - s\right)! \left(\frac{N+3}{2} + s\right)!}$$

$$p \to 1 - \sqrt{\frac{8}{\pi N}} \quad \text{for} \quad N \to \infty$$

- Optimization over Alice's measurements and resource state

$$p = 1 - \frac{3}{N+3}$$

Known Results: Probabilistic Case

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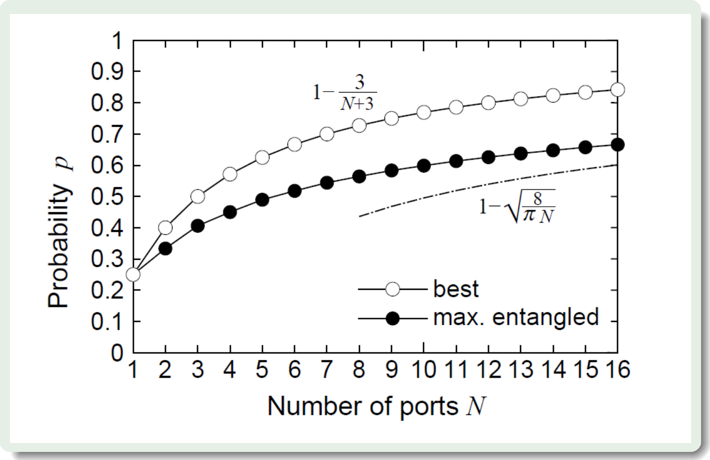

- Opt

Mathematical tools in the qubit case

S. Ishizaka, T. Hiroshima, PRA **79***, 042306 (2009)*
S. Ishizaka, T. Hiroshima, PRL **101***, 240501 (2008)*

- Correspondence between qubits and spins 1/2

$$|0\rangle,|1\rangle \leftrightarrow |1/2,-1/2\rangle,|1/2,1/2\rangle$$

- Each qubit is 1/2 spin → basis of $SU(2)$
- In the protocol we have $SU(2)^{\otimes N}$ symmetry → representation theory, theory of angular momentum
- Main tolls here: Clebsch-Gordan coefficients + SDP methods

Known Results for $d > 2$

Z.-W. Wang, S.L. Braunstein, Sci. Reps **6***, 33004 (2016)*

- Solution for an arbitrary d, but $N = 2, 3, 4$
- Solution obtained by using a graphical variant of **Temperely-Lieb algebra**

Known Results for $d > 2$

Z.-W. Wang, S.L. Braunstein, Sci. Reps ***6****, 33004 (2016)*

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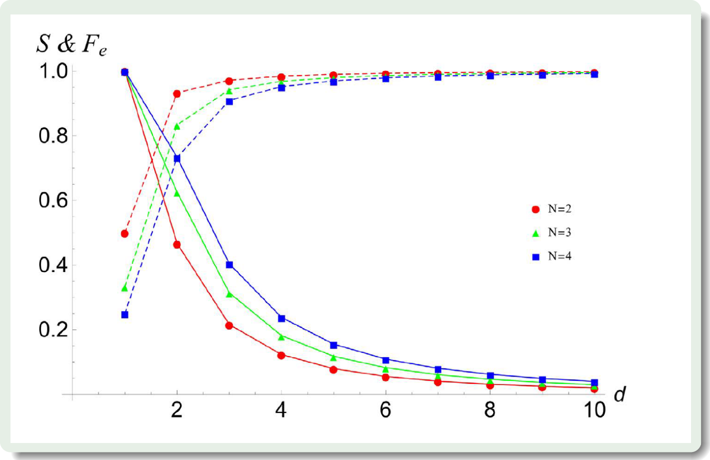


Results for $d \geq 2$

Results based on:

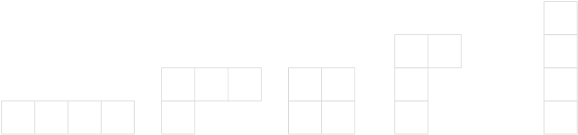
1) *M. Studziński, S. Strelchuk, M. Mozrzymas, M. Horodecki, Sci. Rep.* ***7****, 10871 (2017)*
2) *M. Mozrzymas, M. Studziński, S. Strelchuk, M. Horodecki, arXiv: 1707.08456*

Representation Theory of $S(n)$ in a Nutshell

- Let us take permutation group $S(n)$
- For natural number n we define **partition** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$

$$\forall i \ \lambda_i \geq 0, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \quad \sum_{i=1}^{r} \lambda_i = n$$

- Every sequence can be represented graphically $\leftrightarrow$ **Young diagrams**



(4) $(3,1)$ $(2,2)$ $(2,1,1)$ $(1,1,1,1)$

- We denote **irreps** of $S(n)$ by Greek letters α, β, μ, ν etc.

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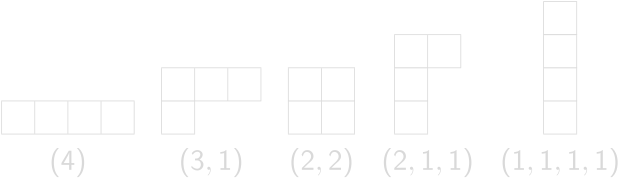

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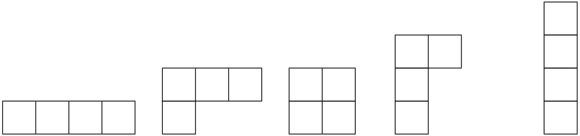

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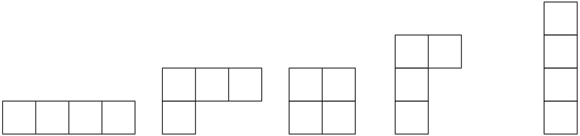

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New Results: Probabilistic Case

- Maximally entangled state as a resource state

$$p = \frac{1}{d^N} \sum_{\alpha \vdash N-1} m_\alpha \min_{\mu = \alpha + \Box} \frac{d_\mu}{m_\mu}$$

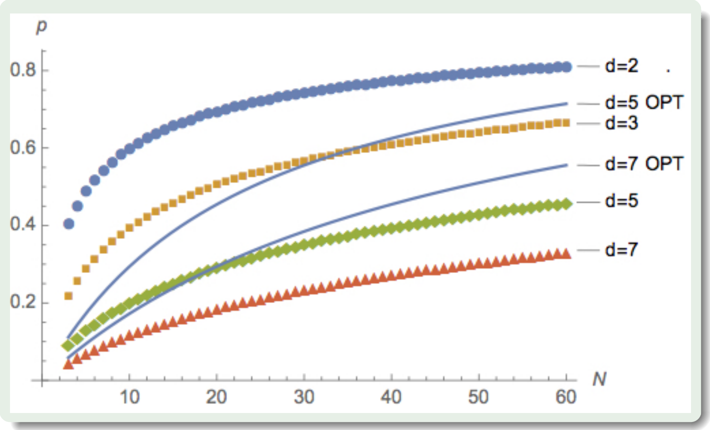
- Optimization over Alice's measurements and resource state

$$p = 1 - \frac{d^2 - 1}{N + d^2 - 1}$$

New Results: Probabilistic Case

- Max

- Opt



New Results: Deterministic Case

- Maximally entangled state as a resource state

$$F = \frac{1}{d^{N+2}} \sum_{\alpha \vdash N-1} \left(\sum_{\mu = \alpha + \Box} \sqrt{d_\mu m_\mu} \right)^2$$

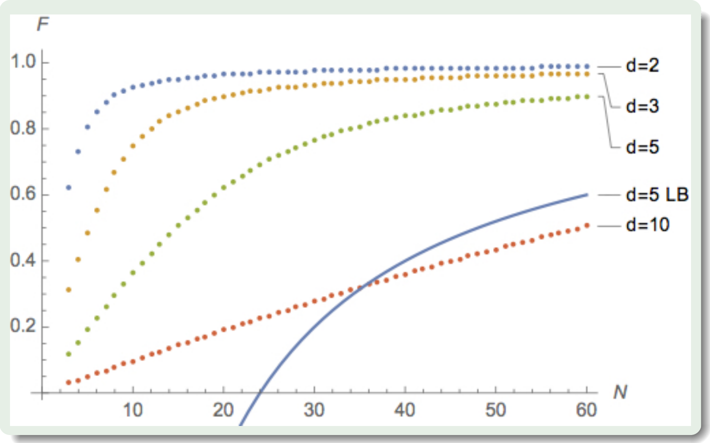
- Optimization over Alice's measurements and resource state

$$F = \frac{1}{d^2} ||M_F^d||_\infty ,$$

where M_F is the teleportation matrix.

New Results: Deterministic Case

- Maximally entangled state as a resource state



- Opt

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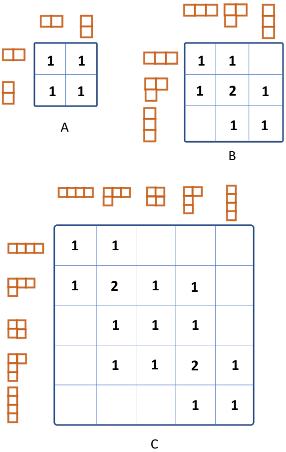
New Results: Teleportation Matrix



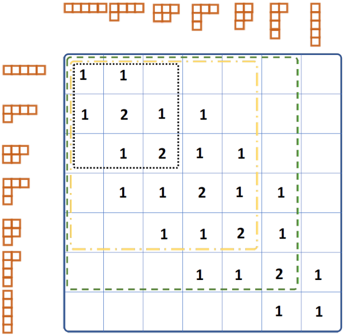
1	1
1	1

1	1	
1	2	1
	1	1

1	1			
1	2	1	1	
	1	1	1	
	1	1	2	1
			1	1

Figure: Teleportation Matrix for $N = 2, 3, 4$

$$M_F \equiv (n_\mu \delta_{\mu,\nu} + \Delta_{\mu,\nu})$$



1	1					
1	2	1	1			
	1	2	1	1		
	1	1	2	1	1	
		1	1	2	1	
			1	1	2	1
					1	1

Figure: Teleportation Matrix for $N = 5$ with principal submatrices.

Spectrum of Teleportation Matrix

We have two cases:

- $d \geq N$ - it is enough to compute maximal eigenvalue of M_F

$$\operatorname{spec} M_F = \{0, 1, 2, \ldots, N-2, N\} \qquad \text{so} \qquad F = \frac{N}{d^2}$$

- $d < N$ - we have to consider principal submatrices M_F^d of M_F. For $d = 2$ we know analytical expression.

Spectrum of Teleportation Matrix

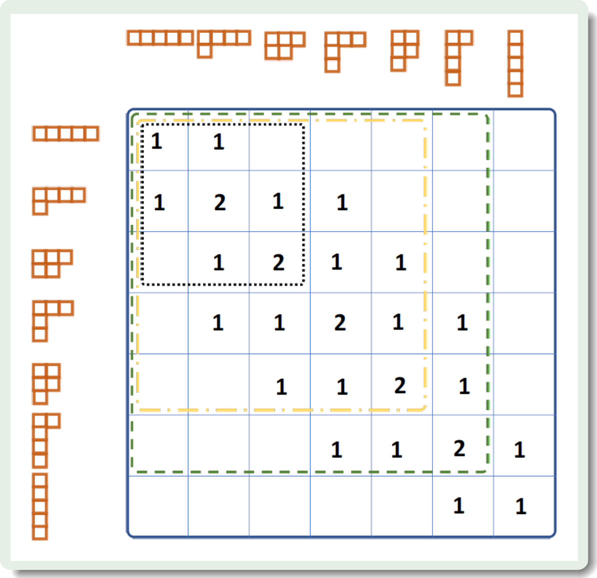
We have

- $d \geq$ of M_F

$= \frac{N}{d^2}$

- $d <$ $_F^d$ of M_F.

For



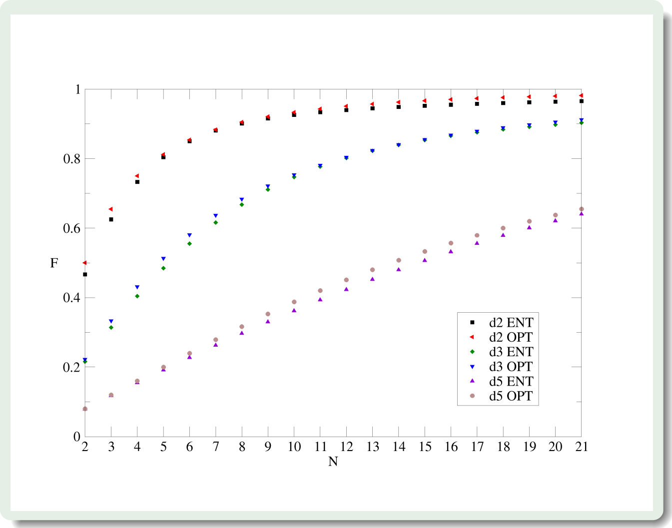
1	1					
1	2	1	1			
	1	2	1	1		
	1	1	2	1	1	
		1	1	2	1	
			1	1	2	1
					1	1

Spectrum of Teleportation Matrix

We have

- $d \geq$ F
- $d <$ $_F$.

  For

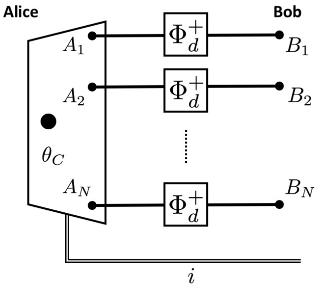


Path to Solution for $d \geq 2$

Results based on:

1) *M. Studziński, S. Strelchuk, M. Mozrzymas, M. Horodecki, Sci. Rep.* ***7****, 10871 (2017)*
2) *M. Mozrzymas, M. Studziński, S. Strelchuk, M. Horodecki, arXiv: 1707.08456*

Natural Symmetries in PBT



$$\underbrace{U^* \otimes \overbrace{U \otimes \cdots \otimes U}^{N}}_{n}$$

$n = N + 1$,

N – number of ports,

n – N + teleported particle

1. Projection onto maximally entangled state $|\psi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_k |k\rangle \otimes |k\rangle$
2. The state $|\psi_d^+\rangle$ is $U^* \otimes U$ invariant
3. Measurements on Alice's side have *natural $U^* \otimes U \otimes \cdots \otimes U$ symmetry*

Natural Symmetries in PBT

$$|\psi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_k |k\rangle \otimes |k\rangle$$

1. Description of the commutant of $U^* \otimes U \otimes \cdots \otimes U$ is needed
2. Complex conjugation translates into *partial transpose*
3. The commutant is spanned by the operators $V^{t_n}(\sigma) \;:\; \sigma \in S(n) \to \mathcal{A}_d^{t_n}(n)$ where $(n = N + 1)$

$$V(\sigma)|e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}\rangle = |e_{i_{\sigma^{-1}(1)}} \otimes e_{i_{\sigma^{-1}(2)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(n)}}\rangle$$

4. $V(1n)^{t_n} = dP_+ = d|\psi_d^+\rangle\langle\psi_d^+|_{1n} \otimes \mathbf{1}$

$$\rho = \frac{1}{d^N} \sum_{i=1}^{N} V(in)^{t_n} \in \mathcal{A}_d^{t_n}(n)$$

Rigorous Definition of Algebra $\mathcal{A}_d^{t_n}(n)$

Definition

For $\mathcal{A}_n(d) = \operatorname{Span}_{\mathbb{C}}\{V(\sigma) : \sigma \in S(n)\}$ we define a new complex algebra

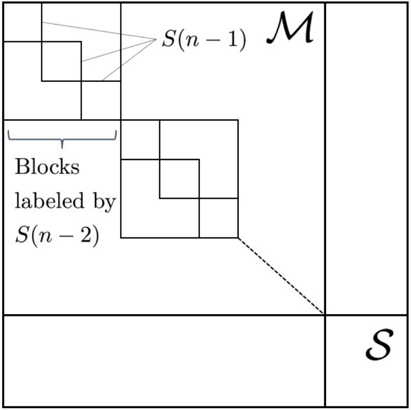
$$\mathcal{A}_d^{t_n}(n) := \operatorname{Span}_{\mathbb{C}}\{V(\sigma)^{t_n} : \sigma \in S(n)\} \subset \operatorname{Hom}((\mathbb{C}^d)^{\otimes n}),$$

where the symbol t_n describes the partial transpose in the last place in the space $\operatorname{Hom}((\mathbb{C}^d)^{\otimes n})$. The elements $V(\sigma)^{t_n} : \sigma \in S(n)$ will be called natural generators of the algebra $\mathcal{A}_d^{t_n}(n)$.

$$V(kn)V(kn) = \mathbf{1} \qquad V(kn)^{t_n}V(kn)^{t_n} = dV(kn)^{t_n}$$

Structure of Algebra $\mathcal{A}_d^{t_n}(n)$

$$\mathcal{A}_d^{t_n}(n) = \mathcal{M} \oplus \mathcal{N}, \qquad \text{support}(\rho) = \mathcal{M}$$



Spectral Decomposition of PBT Operator

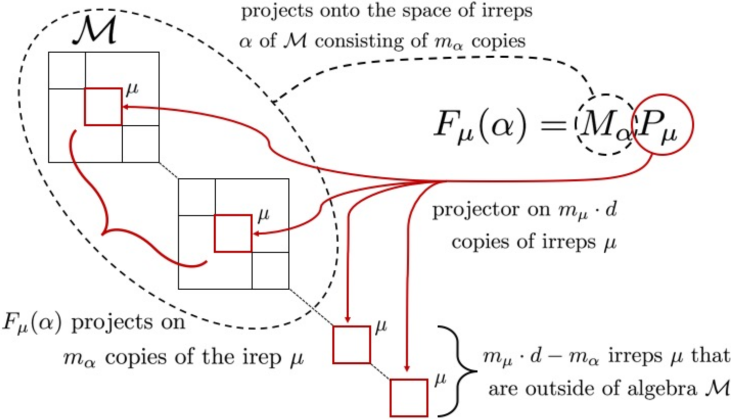
Spectral decomposition of PBT operator

$$\rho = \frac{1}{d^N} \sum_{i=1}^{N} V^{t_n}(in) \equiv \sum_{\alpha \vdash N} \sum_{\mu = \alpha + \Box} \lambda_\mu(\alpha) F_\mu(\alpha)$$

Eigenvalues of PBT operator

$$\lambda_\mu(\alpha) = \frac{N}{d^N} \frac{m_\mu d_\alpha}{m_\alpha d_\mu}$$

Irreducible Representations of Algebra $\mathcal{A}_d^{t_n}(n)$



projects onto the space of irreps α of $\mathcal{M}$ consisting of m_α copies

$$F_\mu(\alpha) = M_\alpha P_\mu$$

projector on $m_\mu \cdot d$ copies of irreps μ

$F_\mu(\alpha)$ projects on m_α copies of the irep μ

$m_\mu \cdot d - m_\alpha$ irreps μ that are outside of algebra $\mathcal{M}$

A Few Words More About Teleportation Matrix M_F

- Deeper connection with $S(N)$:

$$T^{\dagger}(C)M_F T(C) = \mathrm{diag}(0, 1, 2, \ldots, N-2, N)$$

$$M_F T(C) = kT(C) \Leftrightarrow \sum_{\mu}(M_F)_{\mu\nu}\chi^{\mu}(C) = k\chi^{\nu}(C)$$

- Optimal POVMs are connected with **eigenvectors** of Teleportation Matrix:

$$\Pi_i = \Pi\sigma_i\Pi, \qquad \Pi = \frac{d^N}{\sqrt{N}}\sum_{\alpha}\sum_{\mu\in\alpha}\sqrt{\frac{m_\alpha}{d_\alpha}\frac{v_\mu}{m_\mu}}F_\mu(\alpha),$$

for $i = 1, \ldots, N$.

References:

Port-based Teleportation protocols for qubits:

- *S. Ishizaka, T. Hiroshima, PRA* ***79****, 042306 (2009)*
- *S. Ishizaka, T. Hiroshima, PRL* ***101****, 240501 (2008)*

Solution by Temperely-Lieb algebra for $N = 2, 3, 4$

- *Z.-W. Wang, S.L. Braunstein, Sci. Reps* ***6****, 33004 (2016)*

Solution for arbitrary dimension and number of ports:

- *M. Mozrzymas et al. arXiv: 1707.08456*
- *M. Studziński et al. Sci. Rep.* ***7****, 10871 (2017)*

Papers about algebra $\mathcal{A}_n^{t_n}(d)$:

- *M. Mozrzymas et al. arXiv: 1708.02434*
- *M. Mozrzymas et al. JMP* ***55****, 032202 (2014)*
- *M. Studziński et al. JPA* ***46****, 395303 (2013)*