COMPRESSION OF IDENTICALLY PREPARED QUDITS

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INTRODUCTION

Why do we study identically prepared state compression

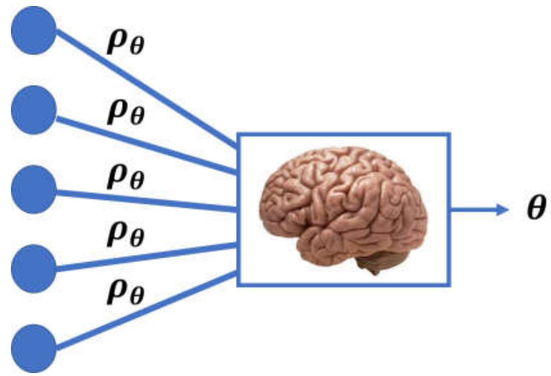
POPULATION CODING

- How a stimulus is encoded in the "states" of neurons.
- "States" of a neuron: probability distributions of reactions to different stimuli.

$$\rho_{bite} = 0.9|1\rangle\langle 1| + 0.1|0\rangle\langle 0| \quad \rho_{touch} = 0.2|1\rangle\langle 1| + 0.8|0\rangle\langle 0|$$

1/0: a spike/no spike.

- A group of $n \gg 1$ neurons reacting to the same stimulus $\rightarrow$ tensor-power form states $\rho_\theta^{\otimes n}$ ($\theta =$ bite/touch).
- Population coding: the state $\rho_\theta^{\otimes n}$ of a large group of neurons is a coding for the stimulus θ.



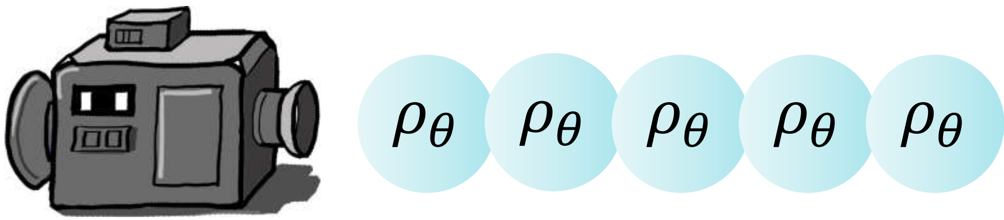
A QUANTUM EXTENSION OF POPULATION CODING

- Population coding → A population of quantum states $\rho_\theta^{\otimes n}$ carrying θ.
- Goal: reduce the cost of transmission of identically prepared state $\rho_\theta^{\otimes n}$ with **unknown** θ.

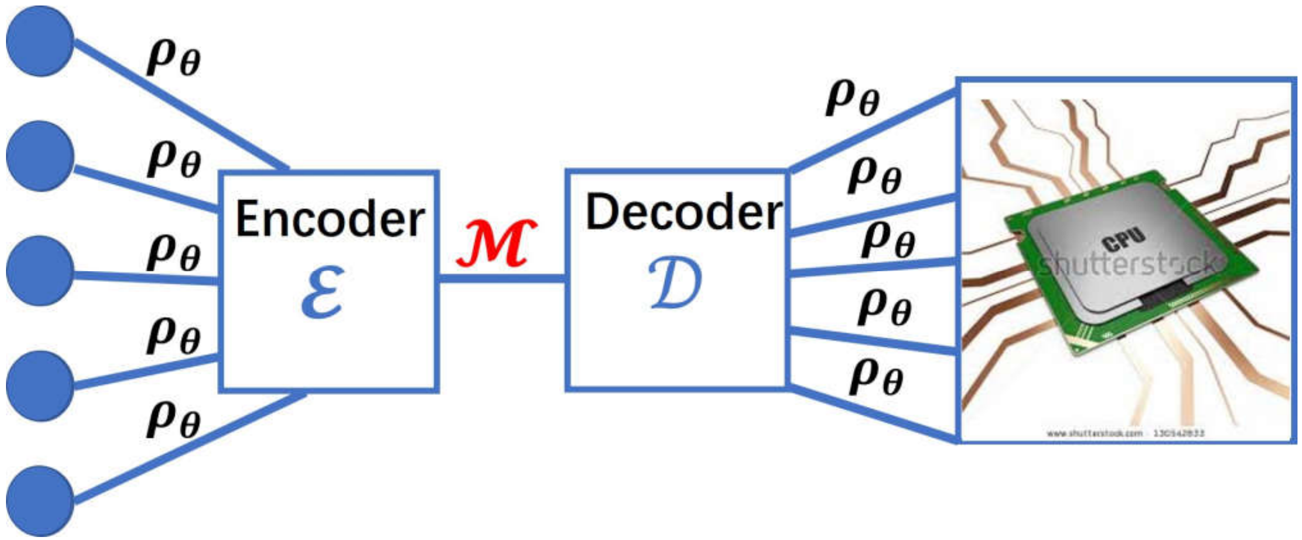
A TYPICAL SCENARIO IN QIP

A source emitting identical copies of a quantum state, unknown to the experimenter (e.g. cloning, metrology, learning …)

- How much information is in it?
- What is the shortest description length (and how to reach it)?



COMPRESS IDENTICAL COPIES OF AN UNKNOWN STATE



- Goal: minimize the total (classical + quantum) memory cost $\mathcal{M}$ (focusing on the leading order of n).
- Constraint: consider only faithful protocols:

$$\lim_{n\to\infty} \epsilon_n = 0 \qquad \epsilon_n := \frac{1}{2}\sup_\theta \left\| \rho_\theta^{\otimes n} - \mathcal{D}\circ\mathcal{E}\left(\rho_\theta^{\otimes n}\right)\right\|_1$$

RELATED WORKS

Families of identically prepared states	Qubits	Bits
Pure qubit states [Plesch and Buzek'10]	$\log n$	0
Mixed qubit states [YY, Chiribella, and Hayashi'16]	$\log n$	$1/2 \log n$
Mixed qubit clocks [YY, Chiribella, and Hayashi arXiv 1703.05876]	$1/2 \log n$	$1/2 \log n$
Classical populations with d parameters [Hayashi and Tan'17]	0	$d/2 \log n$
Qudit states [YY, Chiribella, and Ebler'16]	$O(\log n)$	0

Not general

Not optimal

- This work: a compression protocol for **qudits states,** requiring the **minimum total memory** and **less quantum memory**.

MEMORY COST OF COMPRESSION

How many bits and qubits do we need to encode $\rho_\theta^{\otimes n}$

CLASSICAL AND QUANTUM PARAMETERS

A non-degenerate state family $\{\rho_\theta^{\otimes n} : \theta = (\mu, \xi) \in \Theta\}$ is characterized by two kinds of **real** parameters:

$$\rho_\theta = U_\xi \rho_\mu U_\xi^\dagger$$

- **Classical** (independent) parameters μ: determining the spectrum
- **Quantum** (independent) parameters ξ: determining the eigenbasis

EXAMPLES

- Full qudit state family: $f_c = d - 1$ and $f_q = d^2 - d$.

- Phase-covariant state family: $f_c = 0$ and $f_q = d - 1$

$$\rho_\theta = U_\theta \rho_0 U_\theta^\dagger \quad U_\theta = \sum_k e^{i\theta_k} |k\rangle\langle k|.$$

- Classical distribution family: $f_c = d - 1$ and $f_q = 0$.

MEMORY COST OF THE COMPRESSION

- [**What we expect**] (by observation from previous results):

1. $1/2 \log n$ bits per classical parameter
2. $1/2 \log n$ qubits per quantum parameter

- **[Main result]** For each independent parameter t, it takes:

1. $(1/2 + \delta)\log n$ bits for t classical
2. $1/2 \log n$ bits + $\delta \log n$ qubits for t quantum

to encode faithfully the n-copy state.

- $\delta > 0$ is a parameter independent of n (proportional to the quantum/classical ratio), which can be arbitrarily close to zero.

COMPRESSION PROTOCOL

How to achieve the minimal memory cost

PROTOCOL FOR QUDIT STATES

- Localization.

- Local asymptotic equivalence of n-tensor power qudit states and Gaussian (displaced thermal $\otimes$ classical Gaussian) states.

- Compression of displaced thermal $\otimes$ classical Gaussian states.

LOCALIZATION

- Take out $n^{1-\delta/2}$ copies ($\delta > 0$ is a small constant) and use them for tomography. Left with $n - n^{1-\delta/2}$ copies (the lost copies can be retrieved later by amplification).

- Good enough tomography pins θ to a neighborhood
$$\Theta_L := \left\{\theta : ||\theta|| < n^{-\frac{1}{2}+\frac{\delta}{3}}\right\}$$
of size $O\left(n^{-\frac{1}{2}+\frac{\delta}{3}}\right)$ with exponentially vanishing error.

- Encode the tomography outcome [to $1/\sqrt{n}$ precision] into a classical memory; $1/2 \log n$ bits per independent parameter.

- Total memory = tomography outcome + compression in Θ_L (the overall quantum memory cost can be reduced).

QUANTUM LOCAL ASYMPTOTIC NORMALITY (Q-LAN)

- Q-LAN [Kahn and Guta '09]

In the neighborhood Θ_L, $\rho_\theta^{\otimes n}$ is asymptotically equivalent to a classical-quantum Gaussian state:

$$\rho_\theta^{\otimes n} \overset{Q-LAN}{\Longleftrightarrow} \gamma_{\sqrt{n}\theta} = \gamma_{\sqrt{n}\mu}^{class} \otimes \gamma_{\sqrt{n}\xi}^{quant}$$

- Classical mode γ^{class} : a Gaussian distribution with f_c variates;
- Quantum mode γ^{quant}: a multimode (number of modes depending on f_q) displaced thermal state ("noisy" laser).
- Problem reduced to compression of Gaussian states in a small neighborhood Θ_{L}.

COMPRESSION OF GAUSSIAN STATES

- Compress $\gamma_{\mu'}^{class} \otimes \gamma_{\xi'}^{quant}$ with $||\xi'||,||\mu'|| < n^{\frac{\delta}{3}}$
- To compress f_c–variate Gaussian $\gamma_{\mu'}^{class}$ with constant covariance matrix and unknown mean $||\mu'|| < n^{\frac{\delta}{3}}$: truncation in an-$O(n^{\delta f_c})$ hypercube is enough.
- To compress the multimode displaced thermal ("noisy laser") state $\gamma_{\xi'}^{quant}$ with f_q unknown parameters $||\mu'|| < n^{\frac{\delta}{3}}$ of intensity/phase :
  photon number truncation in an-$O(n^{\delta f_q})$ hypercube is enough

Memory cost	tomography	Gaussian compression
per classical parameter	½ $\log n$ bits	$\delta \log n$ bits
per quantum parameter	½ $\log n$ bits	$\delta \log n$ qubits

ERROR BOUND

- The compression error is upper bounded as

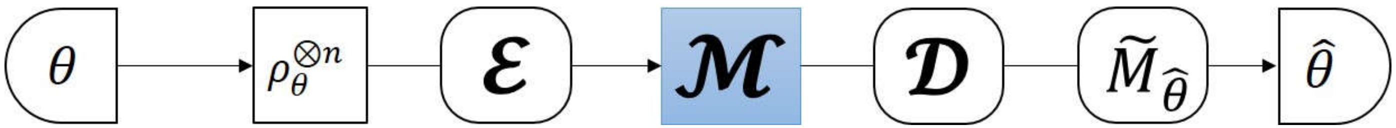
$$\epsilon_n = O\left(n^{-\delta/2}\right) + O\left(n^{-\kappa(\delta)}\right),$$

where the latter is the error of Q-LAN. Especially, $\kappa(\delta) > 0$ for $\delta \in (0, 2/9)$.

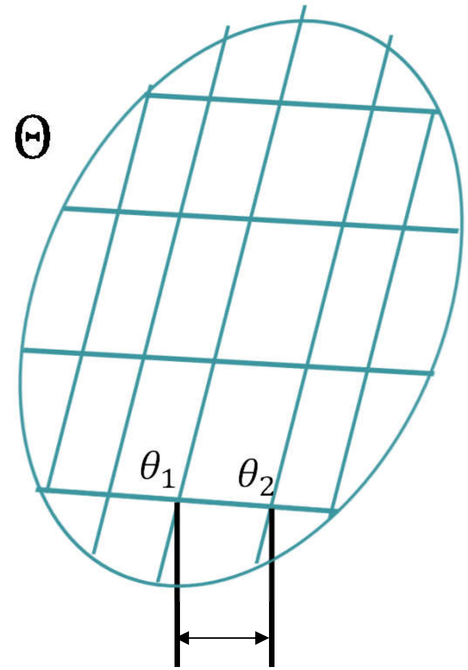
- Faithfulness $\lim_{n \to \infty} \epsilon_n = 0$ is guaranteed as long as $\delta > 0$.
- The error vanishes slower when less quantum memory is used.
- # qubits / # bits can be made arbitrarily close to 0.

OPTIMALITY

- Construct a mesh on Θ containing $\mathrm{n}^{f/2-\delta}$ mutually distinguishable states for any $\delta > 0$. $f = f_c + f_q$.
- Consider any faithful compression protocol $(\mathcal{E}\mathcal{D})$:



- Can faithfully communicate $(f/2-\delta)\log n$ bits of messages.
- The communication cost $\log|\mathcal{M}|$ cannot be smaller than the amount of messages.



$O(n^{-1/2+\alpha})\alpha > 0$

$\Rightarrow \rho_{\theta_1}^{\otimes n}$ distinguishable from $\rho_{\theta_2}^{\otimes n}$

QUANTUM MEMORY IS ESSENTIAL

Why fully classical memory doesn't work

DOES FULLY CLASSICAL MEMORY WORK?

- No constraint on the size of the memory: Is it possible to reversibly convert $\rho_\theta^{\otimes n}$ into classical bits with an error vanishing in n ?

$$\rho_\theta^{\otimes n} \xrightarrow{measurement} \theta \xrightarrow{re-preparation} \rho_\theta^{\otimes n}$$

- A positive answer might trivialize many quantum information tasks.
- Fact: a state family can be perfectly compressed into classical memory iff. it is classical, i.e. $[\rho_1 \rho_2] = 0$ for any $\rho_1 \rho_2$ from the family.
- Compression is only approximately perfect. Cannot directly apply the fact.

PROOF FOR ϵ-APPROXIMATE COMPRESSION

- Bures distance (see e.g. [Nielsen&Chuang]) and (quantum) Hellinger distance [Holevo'72]:

$$d_H(\rho_1, \rho_2) := \sqrt{2 - 2\,\mathrm{Tr}\left(\rho_1^{1/2}\rho_2^{1/2}\right)}$$
$$d_B(\rho_1, \rho_2) := \sqrt{2 - 2\,\mathrm{Tr}\left|\rho_1^{1/2}\rho_2^{1/2}\right|}.$$

- **Lemma** [Luo, Zhang'04] $d_H \geq d_B$; equality holds iff. $[\rho_1, \rho_2] = 0$.

- **Theorem** If $\mathcal{E}$ is a q.-c. channel and $\mathcal{D}$ is a c.-q. channel satisfying $||\mathcal{D}\mathcal{E}(\rho_i) - \rho_i||_1 \leq \epsilon$ $(i = 1, 2)$, then the following holds

$$d_H(\rho_1, \rho_2) \leq d_B(\rho_1, \rho_2) + 2\sqrt{\epsilon}.$$

- **Faithful compression using classical memory works only for states whose $d_H \to d_B$.**

PROOF FOR ϵ-APPROXIMATE COMPRESSION

- Consider $\rho_{\theta_0}^{\otimes n}$ and $\rho_{\theta_0+t/\sqrt{n}}^{\otimes n}$; $t > 0$ is a vector of quantum parameters.
  - Approximation by Gaussian states:

$$\rho_{\theta_0}^{\otimes n} \overset{Q-LAN}{\Longleftrightarrow} \gamma_0$$

$$\rho_{\theta_0+t/\sqrt{n}}^{\otimes n} \overset{Q-LAN}{\Longleftrightarrow} \gamma_t := D_t \gamma_0 D_t^{\dagger}$$

  - A compression protocol for $\rho_{\theta_0}^{\otimes n}$ $\rho_{\theta_0+t/\sqrt{n}}^{\otimes n}$ $\rightarrow$ a protocol for γ_0, γ_t.
  - By **Theorem** $d_H(\gamma_0, \gamma_t) \rightarrow d_B(\gamma_0, \gamma_t)$ if there is a faithful protocol with solely classical memory
  - $d_H(\gamma_0, \gamma_t) > d_B(\gamma_0, \gamma_t)$ (i ndependent of n). Contradiction
  - State families containing both $\rho_{\theta_0}^{\otimes n}$ and $\rho_{\theta_0+t/\sqrt{n}}^{\otimes n}$ cannot be faithfully encoded in a classical memory.

SUMMARY AND OPEN PROBLEMS

- Compression of $\rho_\theta^{\otimes n}$:
  I. minimal memory cost: approximately $1/2 \log n$ for each degree of freedom;
  II. the required memory is mainly classical; (classical profile)
  III. a fully classical memory is not OK (quantum signature required)

- Extension to non-product states and many-body systems.
- Minimal quantum memory?

Thanks for your attention!

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