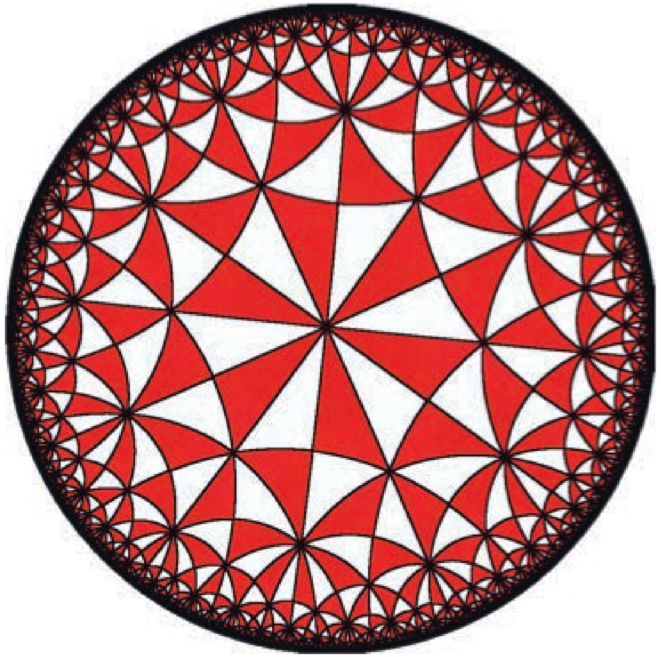
Approximate Operator Algebra Quantum Error Correction

(Decoding the Hologram in AdS/CFT)

arXiv:1704.05839



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QIP 2018 Delft
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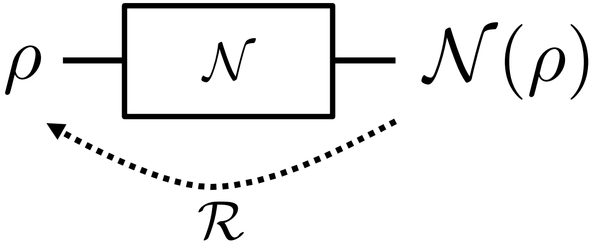
Outline

- **Part 1:** Universal recovery for algebras
- Review of recovery channels
- A theorem (universal recovery for algebras)

- **Part 2:** Application to AdS/CFT
- Review of AdS/CFT
- Entanglement wedge reconstruction

Recovery channels

Problem: Given a quantum channel $\mathcal{N}$, find another quantum channel $\mathcal{R}$ that reverses the action of $\mathcal{N}$ (i.e., a recovery map)



$$\mathcal{R} \circ \mathcal{N}(\rho) = \rho$$

Monotonicity of relative entropy: $$D(\rho\|\sigma) \geq D\left(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)\right)$$

Equality in the relative entropy condition means that we don’t lose any information

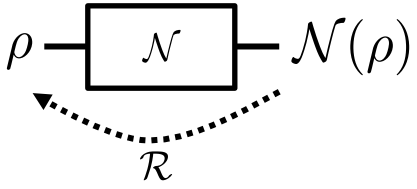
This suggests that we should be able to “undo” the channel and recover the initial state

Relative Entropy: $$D(\rho\|\sigma) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \sigma)$$

Equal to zero iff $\rho = \sigma$.

A measure of distinguishability.

Exact recovery: Petz map



$$D(\rho\|\sigma) \overset{!}{=} D\left(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)\right)$$

$$\Longleftrightarrow$$

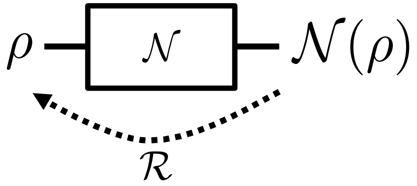
$\exists\ \mathcal{P}$ such that, $\forall\, \rho, \sigma,$ $\quad \mathcal{P}\circ\mathcal{N}(\rho) = \rho \qquad \mathcal{P}\circ\mathcal{N}(\sigma) = \sigma$

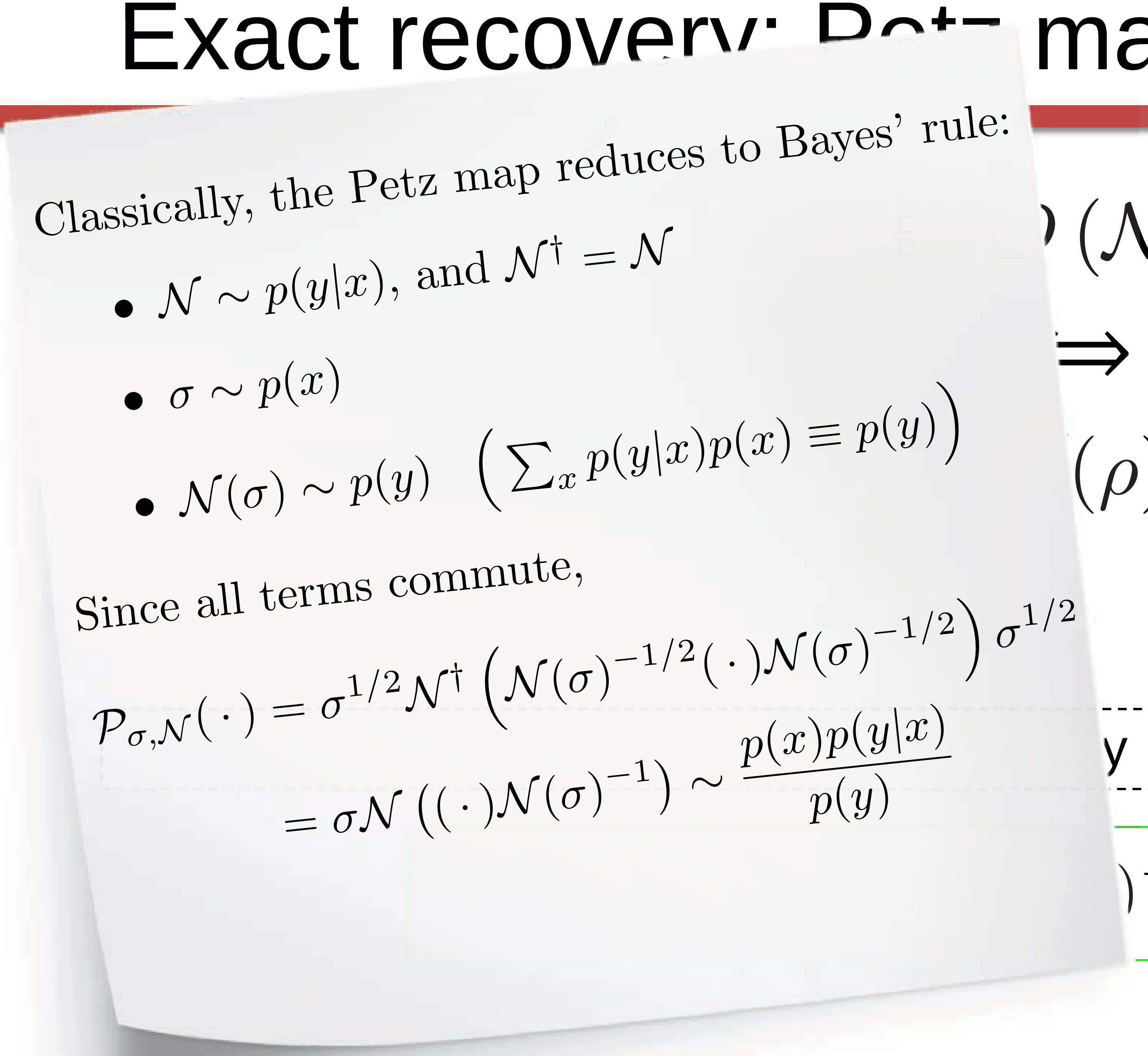
$\mathcal{P}$ is called the **Petz map**, and it is given by

$$\boxed{\mathcal{P}_{\sigma,\mathcal{N}}(\,\cdot\,) = \sigma^{1/2}\mathcal{N}^{\dagger}\left(\mathcal{N}(\sigma)^{-1/2}(\,\cdot\,)\mathcal{N}(\sigma)^{-1/2}\right)\sigma^{1/2}}$$

M. Ohya and D. Petz. Quantum Entropy and Its Use. Springer-Verlag, 1993.

Exact recovery: Petz map





Classically, the Petz map reduces to Bayes' rule:

- $\mathcal{N} \sim p(y|x)$, and $\mathcal{N}^\dagger = \mathcal{N}$
- $\sigma \sim p(x)$
- $\mathcal{N}(\sigma) \sim p(y) \quad \left(\sum_x p(y|x)p(x) \equiv p(y)\right)$

Since all terms commute,

$$\mathcal{P}_{\sigma,\mathcal{N}}(\,\cdot\,) = \sigma^{1/2}\mathcal{N}^\dagger\left(\mathcal{N}(\sigma)^{-1/2}(\,\cdot\,)\mathcal{N}(\sigma)^{-1/2}\right)\sigma^{1/2}$$

$$= \sigma\mathcal{N}\left((\,\cdot\,)\mathcal{N}(\sigma)^{-1}\right) \sim \frac{p(x)p(y|x)}{p(y)}$$

$$(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

$$\Rightarrow$$

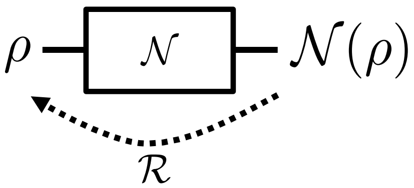
$$(\rho) = \rho \qquad \mathcal{P}\circ\mathcal{N}(\sigma) = \sigma$$

y

$$)^{-1/2}(\,\cdot\,)\mathcal{N}(\sigma)^{-1/2}\Big)\,\sigma^{1/2}$$

M. Ohya and D. Petz. Quantum Entropy and Its Use. Springer-Verlag, 1993.

Approximate recovery: universal channels



$$\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho) \approx \rho$$

For any channel $\mathcal{N}$ there exists a recovery channel $\mathcal{R}_{\sigma,\mathcal{N}}$ depending only on σ and $\mathcal{N}$ such that

$$\boxed{D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -2\log F(\rho, (\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho))}$$

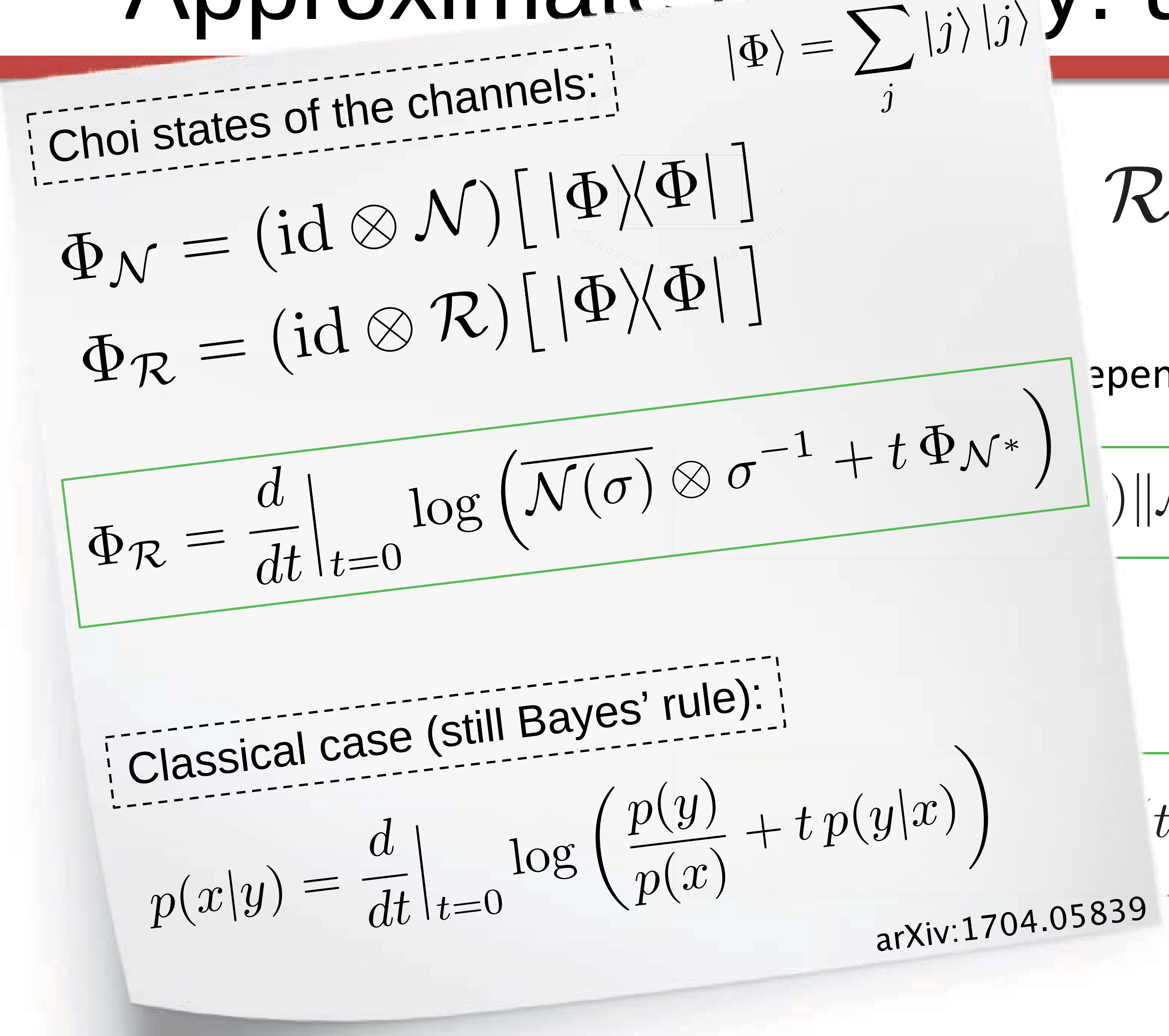
Fidelity $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$

Explicit form of recovery channel $\boxed{\mathcal{R}_{\sigma,\mathcal{N}}(\,\cdot\,) = \int_{\mathbb{R}} dt\,\beta_0(t)\sigma^{\frac{1-it}{2}}\mathcal{N}^\dagger\left(\mathcal{N}(\sigma)^{-\frac{1-it}{2}}(\,\cdot\,)\mathcal{N}(\sigma)^{-\frac{1+it}{2}}\right)\sigma^{\frac{1+it}{2}}}$

$$\beta_o(t) = \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}$$

M. Junge, R. Renner, D. Sutter, M. M. Wilde, and A. Winter, arXiv preprint arXiv:1509.07127 (2015).

Approximate recovery: universal channels



Choi states of the channels:

$$|\Phi\rangle = \sum_j |j\rangle|j\rangle$$

$$\Phi_{\mathcal{N}} = (\mathrm{id} \otimes \mathcal{N})\big[\,|\Phi\rangle\langle\Phi|\,\big]$$

$$\Phi_{\mathcal{R}} = (\mathrm{id} \otimes \mathcal{R})\big[\,|\Phi\rangle\langle\Phi|\,\big]$$

$$\boxed{\Phi_{\mathcal{R}} = \frac{d}{dt}\Big|_{t=0} \log\left(\overline{\mathcal{N}(\sigma)} \otimes \sigma^{-1} + t\,\Phi_{\mathcal{N}^*}\right)}$$

Classical case (still Bayes' rule):

$$p(x|y) = \frac{d}{dt}\Big|_{t=0} \log\left(\frac{p(y)}{p(x)} + t\,p(y|x)\right)$$

arXiv:1704.05839

$$\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho) \approx \rho$$

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Fidelity $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$

$$\boxed{t)\sigma^{\frac{1-it}{2}}\mathcal{N}^{\dagger}\left(\mathcal{N}(\sigma)^{-\frac{1-it}{2}}(\,\cdot\,)\mathcal{N}(\sigma)^{-\frac{1+it}{2}}\right)\sigma^{\frac{1+it}{2}}}$$

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Theorem: Approximate recovery for algebras

Let (finite dim) von Neumann Algebras

$\mathcal{M}_a \subseteq \mathcal{M}_A$

$\mathcal{M}_b \subseteq \mathcal{M}_B$

$$\begin{tikzcd}
\mathcal{M}_a \arrow[r, hook, "\text{incl}"] \arrow[d, dashed, "\mathcal{R}^*"'] & \mathcal{M}_A \\
\mathcal{M}_b \arrow[r, hook, "\text{incl}"'] & \mathcal{M}_B \arrow[u, "\mathcal{N}^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
S(\mathcal{M}_a) & S(\mathcal{M}_A) \arrow[l, "\text{res}"'] \arrow[d, "\mathcal{N}"] \\
S(\mathcal{M}_b) \arrow[u, dashed, "\mathcal{R}"] & S(\mathcal{M}_B) \arrow[l, "\text{res}"]
\end{tikzcd}$$

Quantum channel: $\mathcal{N}\colon S(\mathcal{M}_A) \to S(\mathcal{M}_B)$

$$|D(\rho_a \| \sigma_a) - D(\mathcal{N}[\rho]_b \| \mathcal{N}[\sigma]_b)| \leq \epsilon$$

(note the restriction to the subalgebras)

Then

(i) $\|\rho_a - \mathcal{R}[\mathcal{N}[\rho]_A]\|_1 \leq \delta,$

(ii) $\left|\langle \mathcal{R}^*[\phi_a]\rangle_{\mathcal{N}[\rho]} - \langle \phi_a \rangle_\rho\right| \leq \delta \|\phi_a\|,$

(iii) $\left|\langle \mathcal{R}^*[\phi'_a]\mathcal{R}^*[\phi_a]\rangle_{\mathcal{N}[\rho]} - \langle \phi'_a \phi_a \rangle_\rho\right| \leq \delta' \max\{\|\phi'_a\|^2, \|\phi_a\|^2\},$

$$\boxed{\mathcal{R}^*[\phi_a] = \int dt\, \beta_0(t)\, e^{\frac{1-it}{2} H_A} \mathcal{N}\left[\mathcal{E}_a[e^{-\frac{1-it}{2} H_a} \phi_a e^{-\frac{1+it}{2} H_a}]\right]_A e^{\frac{1+it}{2} H_A}}$$

$$H_a = -\log \sigma_a$$

$$H_A = -\log \mathcal{N}[\mathcal{E}_a[\sigma_a]]_A$$

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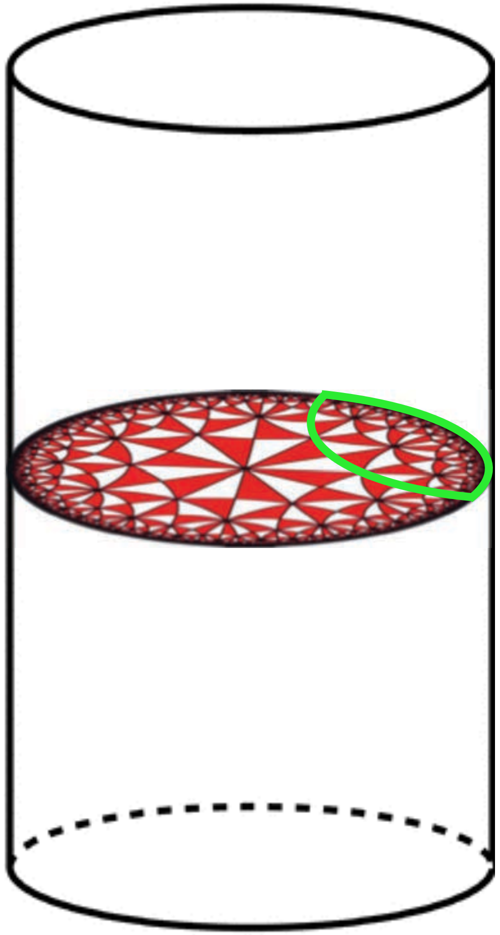
AdS/CFT

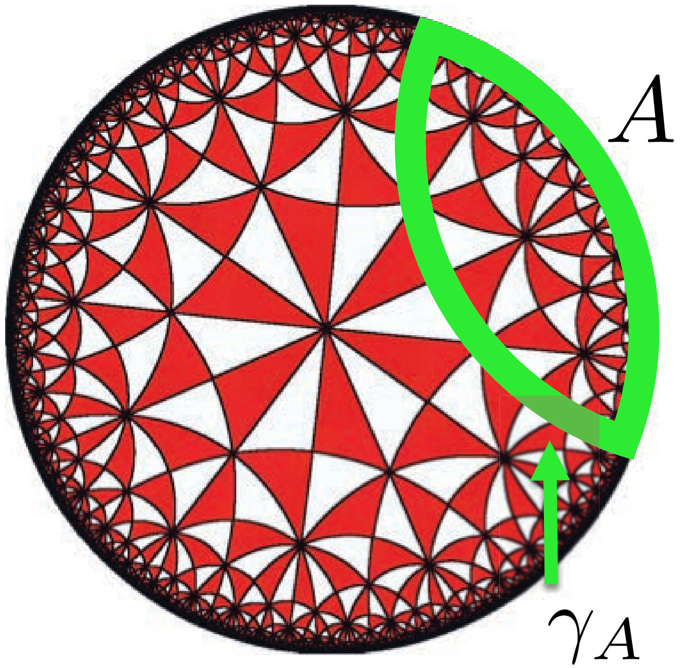
Gravity in anti-de Sitter space in d+1 dimensions

dual to

Conformal field theory in d dimensions

$$\mathcal{H}_{\mathrm{QG}} \longleftrightarrow \mathcal{H}_{\mathrm{CFT}}$$





We often think of the CFT as living on the boundary of AdS

The entropy of a boundary subregion is given by the area of the RT surface

$$S(A) = \frac{|\gamma_A|}{4G_N} + \ldots$$

Causal wedge reconstruction

Gravity in anti-de Sitter space in d+1 dimensions

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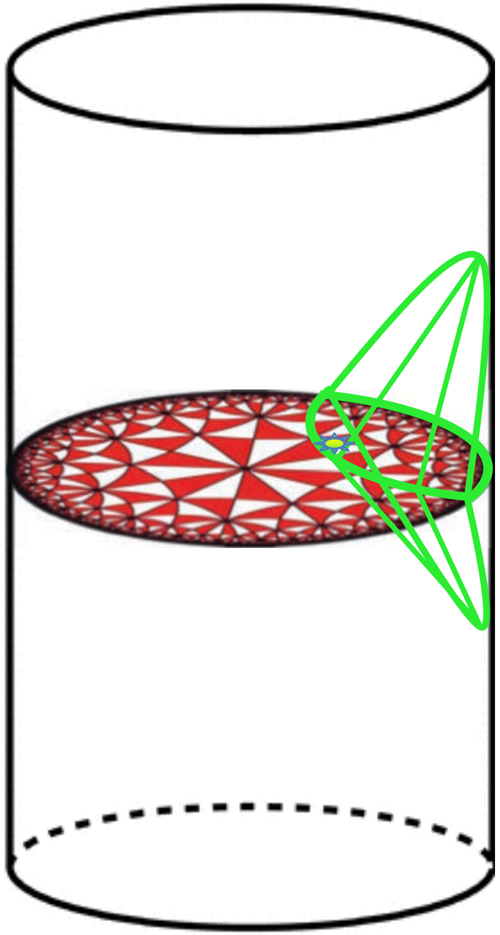
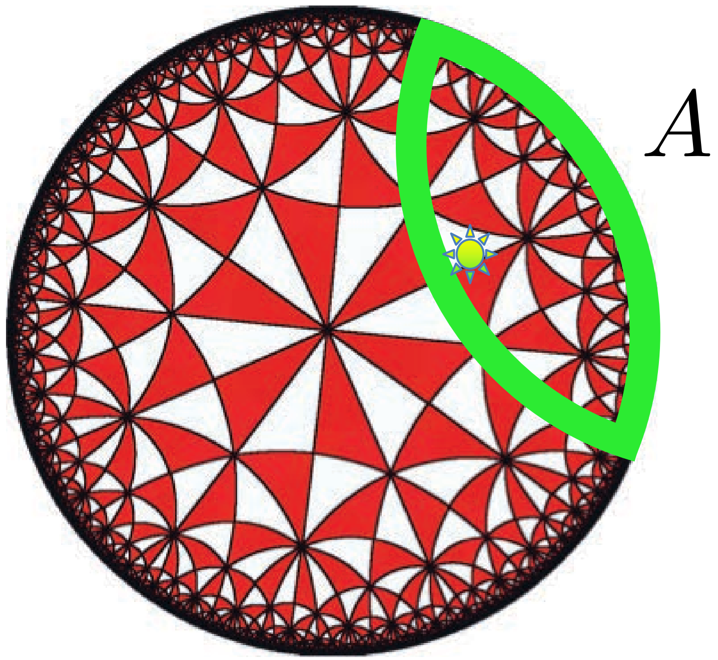




It's possible to reconstruct bulk operators in a "local" way

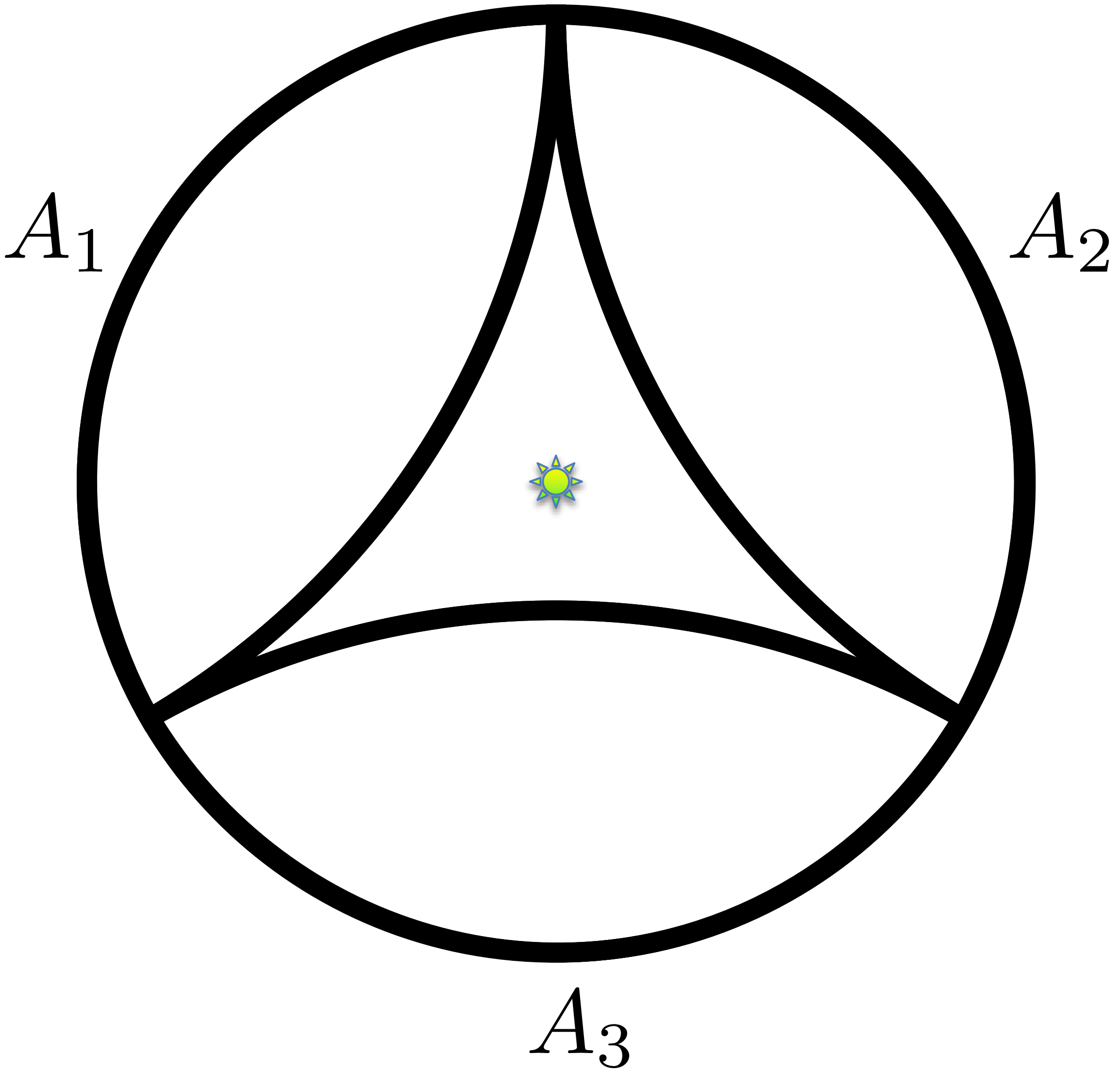
All operators in the "causal wedge" of A can be supported only on A

$$\phi(x,z) = \int_{x\in A} dx' K(x,z,x')\mathcal{O}(x')$$

$$\mathrm{Tr}\,\phi_{\mathrm{bulk}}\rho_{\mathrm{bulk}} = \mathrm{Tr}\,\mathcal{O}_{\mathrm{bdy}}\rho_{\mathrm{bdy}}$$

BUT! Causal wedge reconstruction is not the whole story…

Holographic quantum error correction



Use causal wedge reconstruction for each of

$$A_1 \cup A_2,\ A_1 \cup A_3,\ A_2 \cup A_3$$

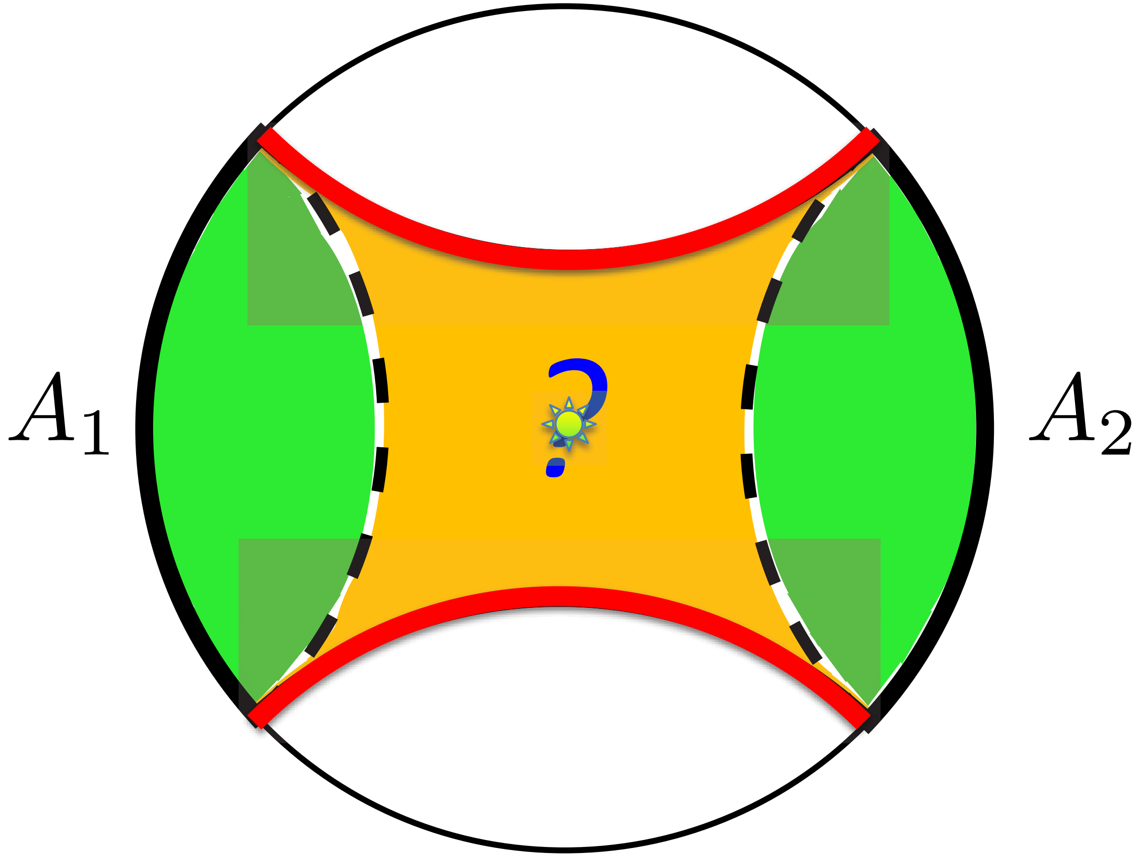
This is a (2,3)-threshold secret sharing quantum error correcting code!

Capable of correcting for loss of any 1 out of the three regions

$$\mathcal{H}_{\text{code}} \subseteq \mathcal{H}_{\text{CFT}}$$

A. Almheiri, X. Dong, and D. Harlow, JHEP 04, 163 (2015), arXiv:1411.7041

Causal wedge reconstruction is not enough



$$A = A_1 \cup A_2$$

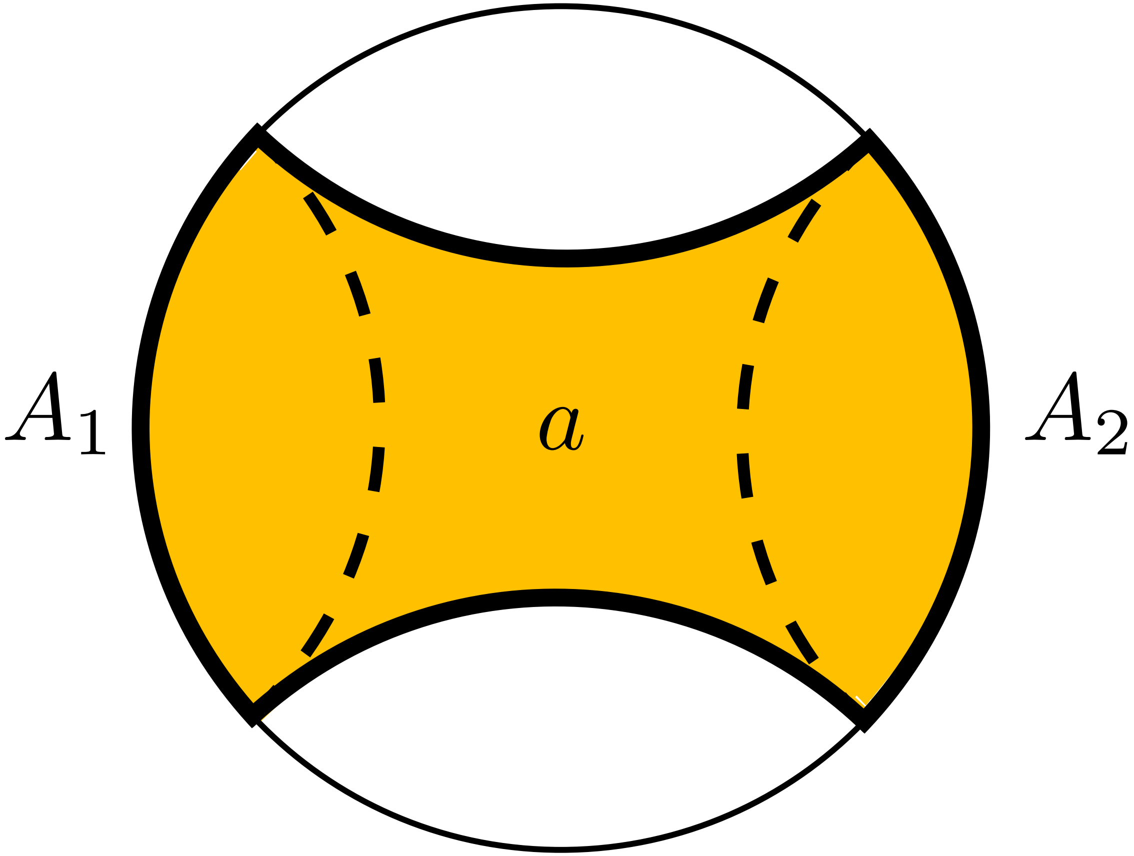
The entropy of the boundary reduced density matrix is given by

$$S(A) = \frac{|\gamma_A|}{4G_N} + S(a) + \ldots$$

Properties of the density matrix on A are sensitive to operators living outside of the causal wedges

This leads to the entanglement wedge hypothesis

Entanglement wedge hypothesis



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Hypothesis: any bulk operators in the entanglement wedge can be reconstructed on the associated boundary subregion

Bulk and boundary relative entropies

JLMS: boundary relative entropy ≈ bulk relative entropy

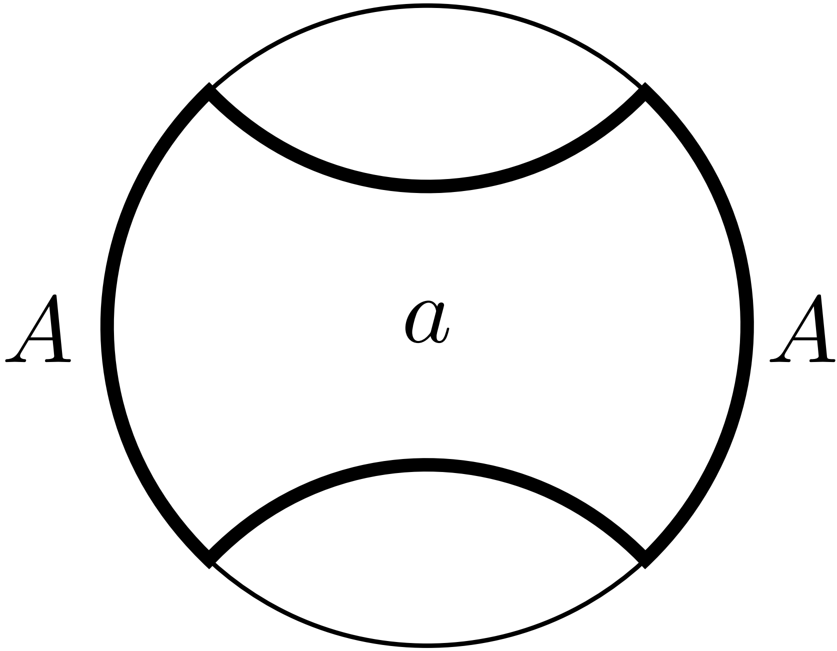
$$D(\rho_A\|\sigma_A) = D(\rho_a\|\sigma_a) + O(G_N)$$

When this condition holds exactly (e.g., $N \to \infty$), the entanglement wedge hypothesis has been argued to be true.

The proof relies on algebraic consequences of exact equality in the relative entropy condition

These algebraic consequences are known not to hold when the relative entropy condition is only approximately satisfied

No explicit expression for reconstruction



$$\left\{D(\rho_A\|\sigma_A) = D(\rho_a^{\{\sigma\}}\|\sigma_a) + \left[\mathrm{Tr}\left(\rho_a^{\{\sigma\}}\mathcal{A}_{\mathrm{loc}}^{\{\sigma\}}\right) - \mathrm{Tr}\left(\rho_a^{\{\rho\}}\mathcal{A}_{\mathrm{loc}}^{\{\rho\}}\right) + S\left(\rho_a^{\{\sigma\}}\right) - S\left(\rho_a^{\{\rho\}}\right)\right]\right\}$$

D. L. Jafferis, A. Lewkowycz, J. Maldacena, and S. J. Suh, JHEP 2016, 1 (2016)

X. Dong, D. Harlow, and A. C. Wall, Phys. Rev. Lett. 117, 021601 (2016)

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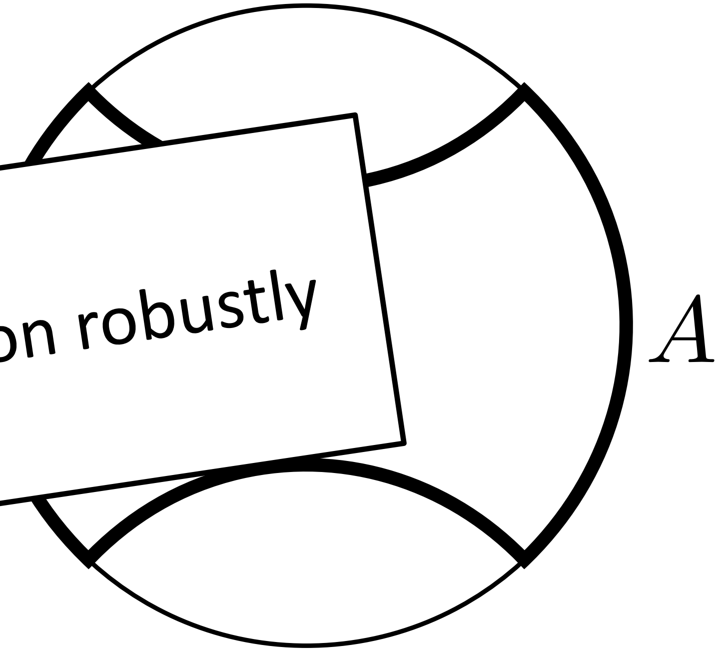
No explicit expression for reconstruction

Our goal:

- Prove entanglement wedge reconstruction robustly
- Give an explicit formula



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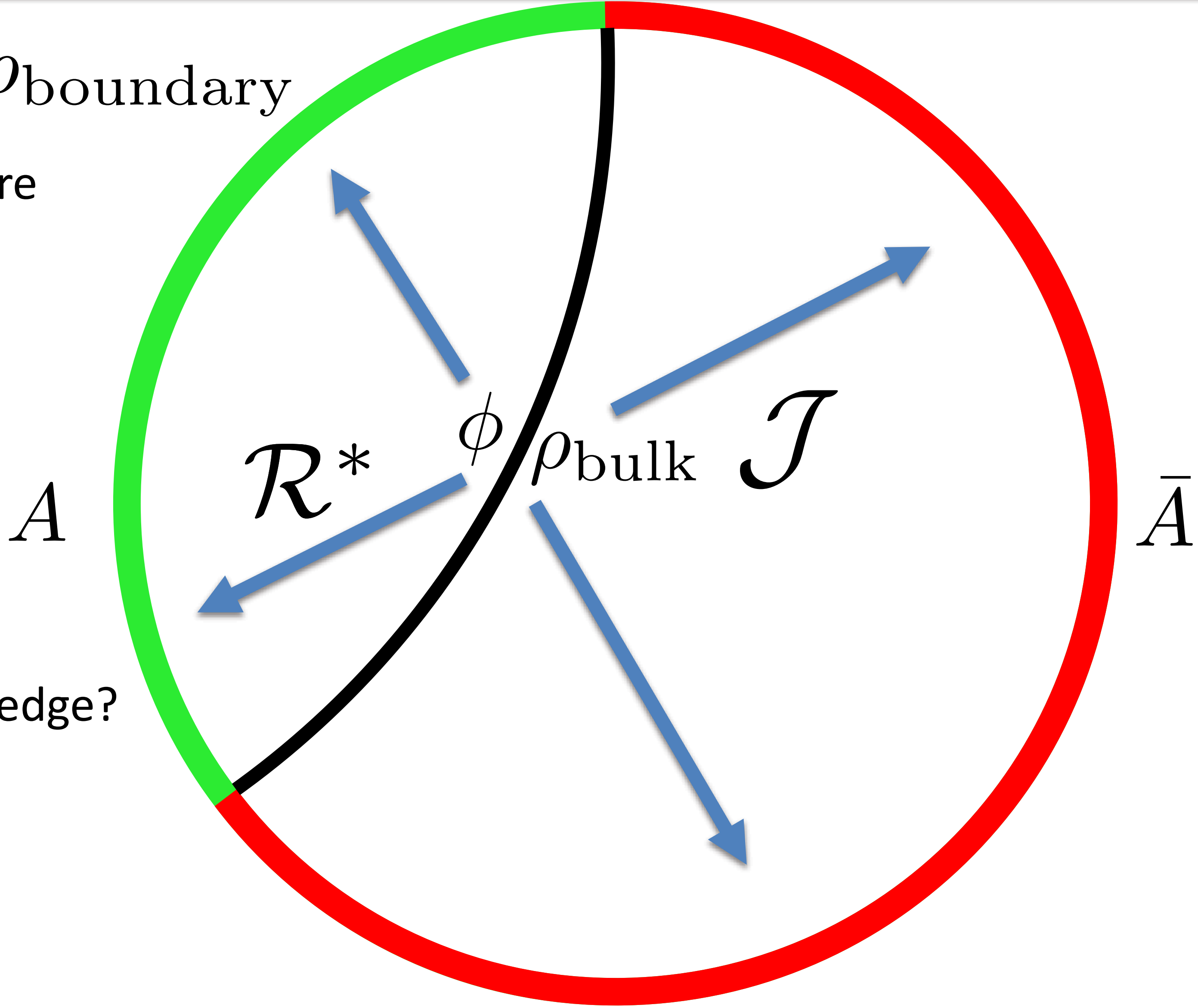
Entanglement wedge reconstruction

$$\rho_A = \mathrm{Tr}_{\bar{A}}\, \rho_{\mathrm{boundary}}$$

- Use recovery channels to correct the erasure
- Switch to the Heisenberg picture (adjoint)
  - Bulk operators mapped to boundary

Apply this technique to the entanglement wedge?



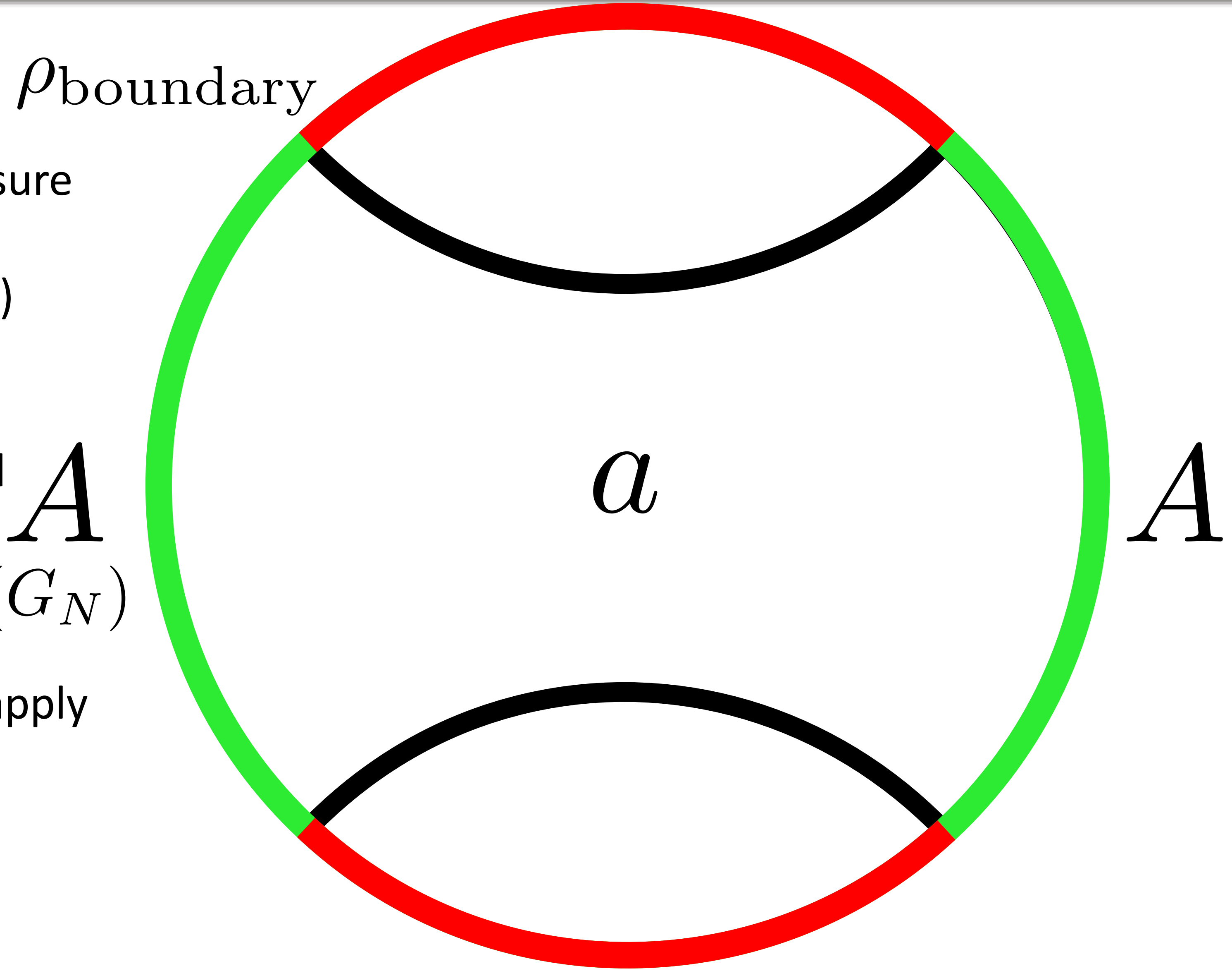
Entanglement wedge reconstruction

$$\rho_A = \mathrm{Tr}_{\bar{A}}\ \rho_{\mathrm{boundary}}$$

- Use recovery channels to correct the erasure
- Switch to the Heisenberg picture (adjoint)
    - Bulk operators mapped to boundary
- JLMS say relative entropies are preserved

$$D(\rho_A \| \sigma_A) = D(\rho_a \| \sigma_a) + O(G_N)$$

- This is not the right condition to naively apply recovery results
    - Extra trace over $\bar{a}$
- Fortunately we have a stronger theorem!



Entanglement wedge reconstruction

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$$D(\rho_A \| \sigma_A) =$$

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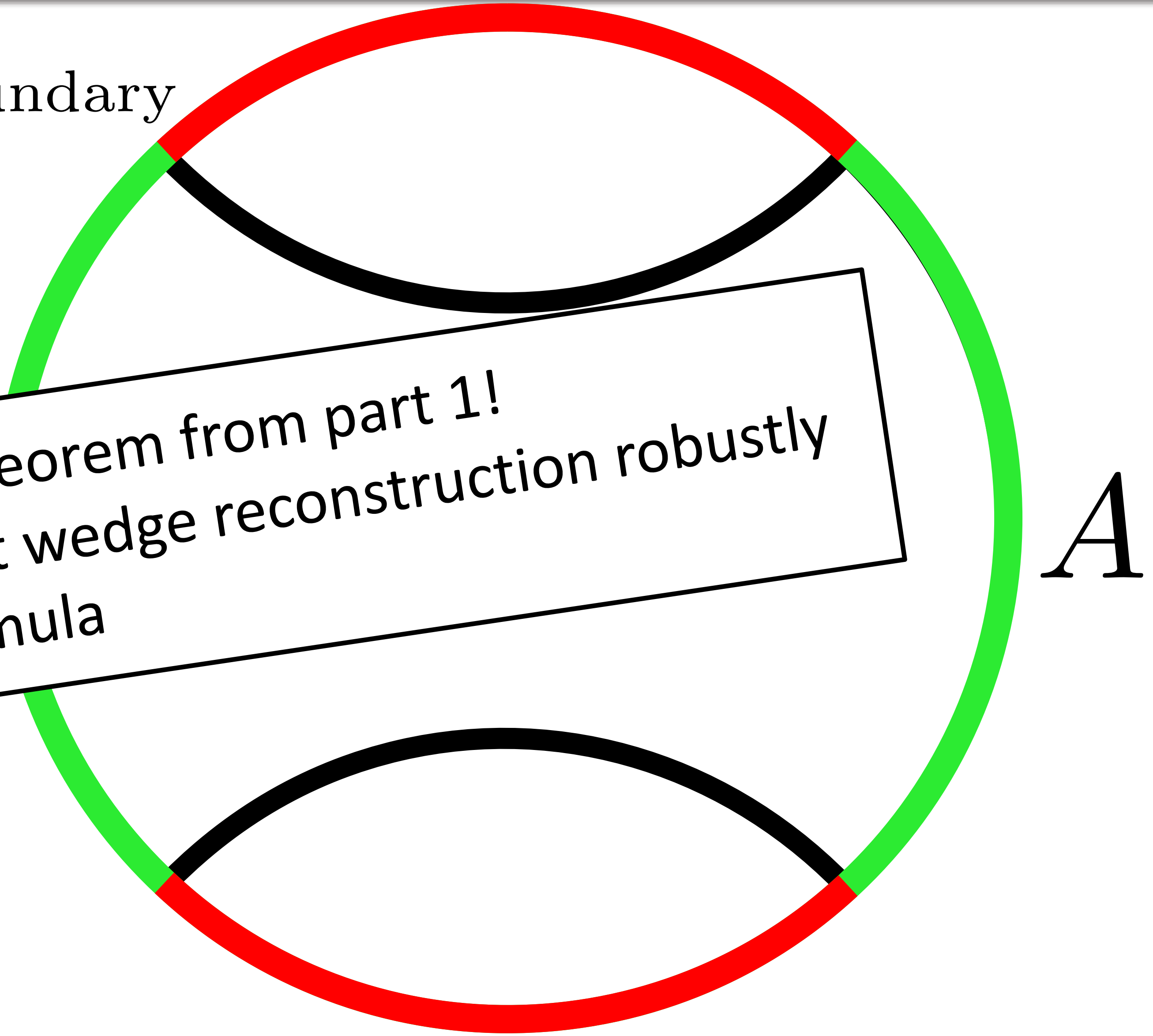
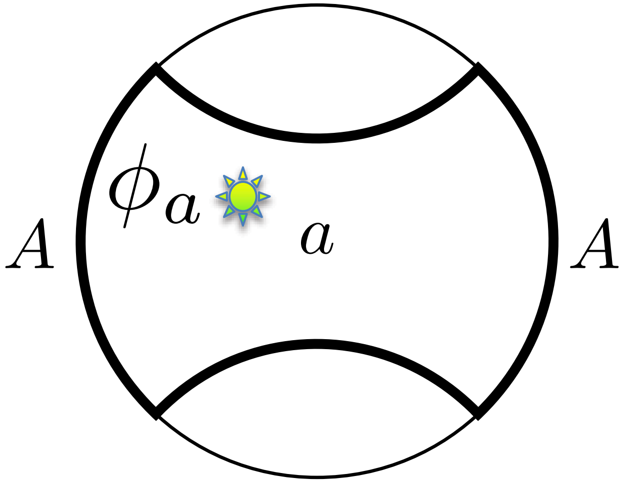


Direct application of Theorem from part 1!

- ✓ Prove entanglement wedge reconstruction robustly
- Give an explicit formula

Algebraic formalism



For all $\phi_a \in \mathcal{M}_a$

$$\mathcal{R}^*(\phi_a) = \frac{1}{d_{\text{code}}} \int dt\, \beta_0(t) e^{\frac{1}{2}(1-it)H_A} \mathcal{J}[\phi_a]_A e^{\frac{1}{2}(1+it)H_A}$$

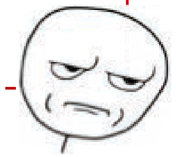
$$\mathcal{R}^*(\phi_a) = -\frac{1}{d_{\text{code}}} \frac{d}{dt}\bigg|_{t=0} H_A(\tau_{\text{code}} + t\,\phi_a) \qquad H_A[\rho] := -\log \mathcal{J}[\rho]_A$$

The boundary operator corresponding to ϕ_a can be computed as a response in the boundary modular Hamiltonian H_A to a perturbation of the maximally mixed code state in the direction of ϕ_a

Two-point functions:

$$\boxed{\langle \phi_a \phi'_a \rangle_\rho \approx \langle \mathcal{O}_A \mathcal{O}'_A \rangle_{J\rho J^\dagger}}$$

In the approximate case, our proof does not generalize to higher-point functions…



Summary

- Generalized universal recovery channels finite dimensional von Neumann algebras

$$\text{(i)}\quad \left\|\rho_a - \mathcal{R}[\mathcal{N}[\rho]_A]\right\|_1 \leq \delta,$$

$$\text{(ii)}\quad \left|\langle \mathcal{R}^*[\phi_a]\rangle_{\mathcal{N}[\rho]} - \langle \phi_a \rangle_\rho\right| \leq \delta \|\phi_a\|,$$

$$\text{(iii)}\quad \left|\langle \mathcal{R}^*[\phi_a']\mathcal{R}^*[\phi_a]\rangle_{\mathcal{N}[\rho]} - \langle \phi_a'\phi_a \rangle_\rho\right| \leq \delta' \max\{\|\phi_a'\|^2, \|\phi_a\|^2\},$$

$$\boxed{\mathcal{R}^*[\phi_a] = \int dt\, \beta_0(t)\, e^{\frac{1-it}{2}H_A} \mathcal{N}\big[\mathcal{E}_a[e^{-\frac{1-it}{2}H_a}\phi_a e^{-\frac{1+it}{2}H_a}]\big]_A\, e^{\frac{1+it}{2}H_A}}$$

- Choi state of universal recovery channel: quantum Bayes’ rule

$$\boxed{\Phi_{\mathcal{R}} = \frac{d}{dt}\bigg|_{t=0} \log\left(\overline{\mathcal{N}(\sigma)} \otimes \sigma^{-1} + t\, \Phi_{\mathcal{N}^*}\right)}$$

- Proved entanglement wedge reconstruction robustly and gave explicit formula

$$\boxed{\mathcal{R}^*(\phi_a) = -\frac{1}{d_{\text{code}}}\frac{d}{dt}\bigg|_{t=0} H_A(\tau_{\text{code}} + t\,\phi_a)} \qquad \boxed{\langle \phi_a \phi_a' \rangle_\rho \approx \langle \mathcal{O}_A \mathcal{O}_A' \rangle_{J\rho J^\dagger}}$$

- Provided an interpretation of reconstructed operators