Quantum walk speedup of backtracking algorithms

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- We might want to find one assignment to the variables that satisfies all the constraints, or list all such assignments.
- For many CSPs, the best algorithms known for either task have exponential runtime in n.
- A fundamental example: boolean satisfiability with at most 3 variables per clause (3-SAT).

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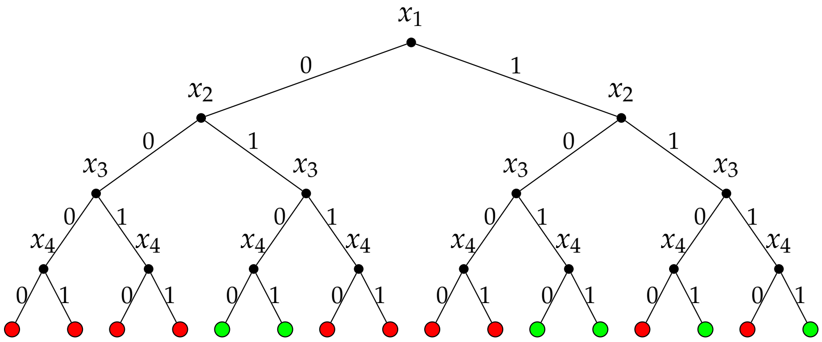
Imagine we want to find all satisfying assignments. One naïve way of doing this is exhaustive search:



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- Exploring the tree corresponds to substituting variable values into the formula.
- At each vertex, we determine which variable to choose next using a heuristic.

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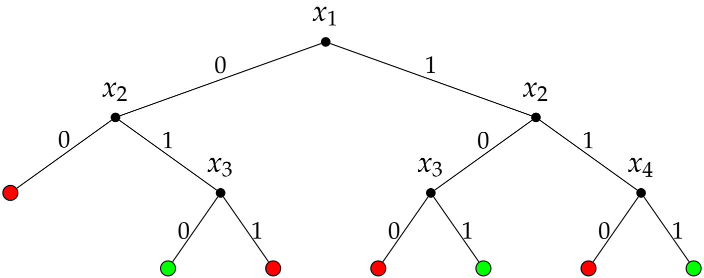

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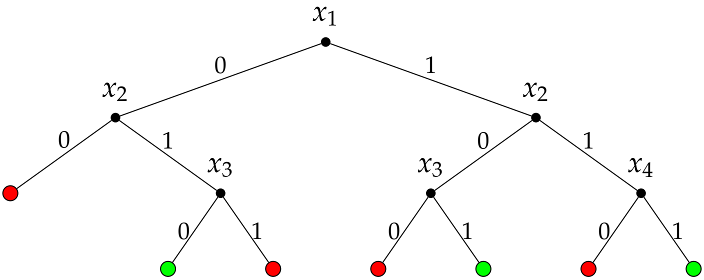

This algorithm is a simple variant of the DPLL algorithm, which forms the basis of many of the most efficient SAT solvers used in practice.

General backtracking framework

Suppose we want to solve a constraint satisfaction problem on n variables, each picked from $[d] := \{0, \ldots, d-1\}$.

- Write $\mathcal{D} := ([d] \cup \{*\})^n$, where $*$ means "not assigned yet".

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  which returns the next index to branch on from a given partial assignment.
- Also allows randomised heuristics, as distributions over deterministic functions h.

Main result

Theorem

Let T be the number of vertices in the backtracking tree. Then there is a bounded-error quantum algorithm which evaluates P and h $O(\sqrt{T}n^{3/2}\log n)$ times each, and outputs x such that $P(x)$ is true, or "not found" if no such x exists.

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If we are promised that there exists a unique x_0 such that $P(x_0)$ is true, this is improved to $O(\sqrt{Tn}\log^3 n)$.

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- Note that the algorithm does not need to know T.

Previous work

Some previous works have developed quantum algorithms related to backtracking:

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By contrast, the algorithm presented here achieves a (nearly) quadratic separation for all trees.

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These can be overcome using work of [Belovs '13] relating quantum walks to effective resistance in an electrical network.

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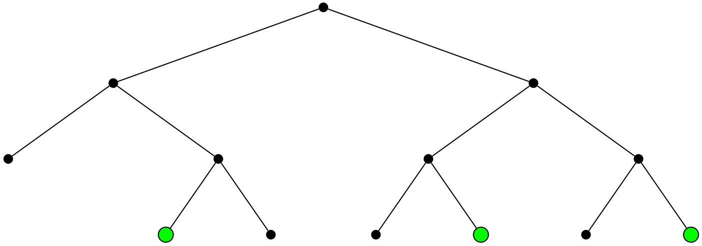

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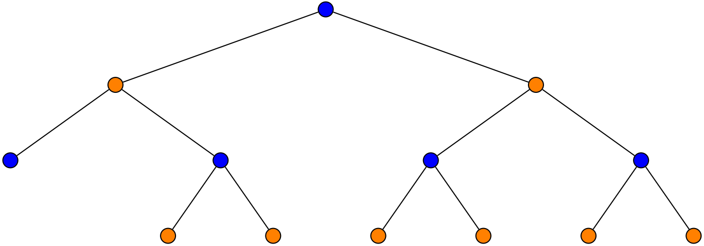

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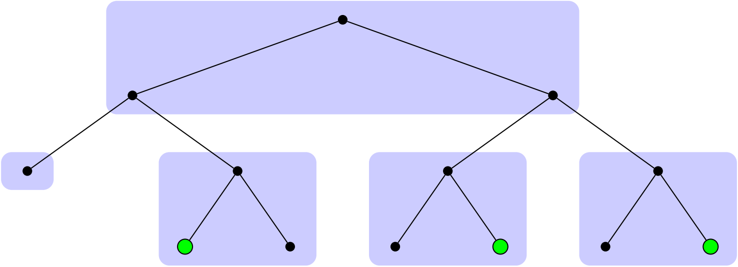

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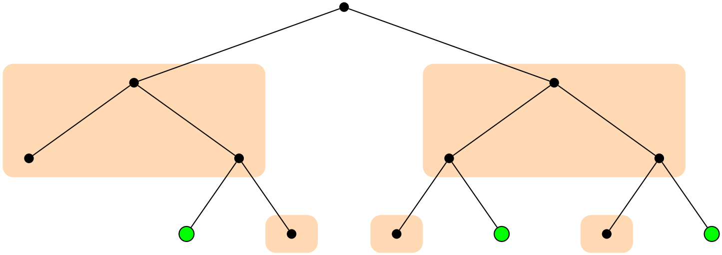

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We apply phase estimation to $R_B R_A$ on state $|r\rangle$ with precision $O(1/\sqrt{Tn})$, where n is an upper bound on the depth of the tree, and accept if the eigenvalue is 1.

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It follows that we can use the above subroutine to detect a marked vertex with $O(\sqrt{Tn})$ uses of $R_B R_A$.

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- We first apply the procedure to the whole tree. If it outputs "marked vertex exists" we apply it to the subtree rooted at each of the children of the root in turn and repeat.

- There is a more efficient algorithm if there is exactly one marked vertex, using the fact that the eigenvector with eigenvalue 1 encodes the entire path from the root to the marked vertex.

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- Recall that we have access to P and h.
- Represent each vertex in the tree by a string $(i_1, v_1), \ldots, (i_\ell, v_\ell)$ giving the indices and values of the variables set so far.
- Then we can use P and h to determine the neighbours of each vertex. This allows us to implement the D_x operations (efficiently).

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- If there are k solutions, can we find them all in time $O(\sqrt{Tnk})$?
- What else can we do using the electrical circuit framework of [Belovs '13]?

Thanks!

General backtracking framework

Backtracking algorithm

Return $\texttt{bt}(*^n)$, where $\texttt{bt}$ is the following recursive procedure:

$\texttt{bt}(x)$:

1. If $P(x)$ is true, output x and return.
2. If $P(x)$ is false, return.
3. Set $j = h(x)$.
4. For each $w \in [d]$:
    1. Set y to x with the j'th entry replaced with w.
    2. Call $\texttt{bt}(y)$.

This algorithm runs in time at most $O(d^n)$, but on some instances its runtime can be substantially lower.

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Claim

Pick a random 3-SAT instance on n variables by choosing m random clauses, where $\Pr[m = m'] \propto 2^{-Cn^{3/2}/\sqrt{m'}}$.

Then there exists a constant C such that the expected quantum runtime is $\mathrm{poly}(n)$, but a simple backtracking algorithm has expected runtime exponential in n.

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- So for $\beta > -2$ the average classical complexity is large.
- But, if $-2 < \beta < -3/2$, the average number of steps used by the quantum backtracking algorithm is

$$\mathbb{E}_X[O(\sqrt{T(X)}\,\mathrm{poly}(n))] \leqslant \sum_{t \geqslant 1} O(\sqrt{t} \cdot t^{\beta}\,\mathrm{poly}(n)) = \mathrm{poly}(n).$$

Proof: marked element case

Claim

Let x_0 be a marked element. Then

$$|\phi\rangle = \sqrt{n}|r\rangle + \sum_{x \neq r, x \rightsquigarrow x_0} (-1)^{\ell(x)} |x\rangle$$

is an eigenvector of $R_B R_A$ with eigenvalue 1, where $\ell(x)$ is the distance of x from the root.

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- Also,

$$\frac{\langle r|\phi\rangle}{\||\phi\rangle\|} \geqslant \frac{1}{\sqrt{2}}.$$

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Effective spectral gap lemma [Lee et al. '11]

Set $R_A = 2\Pi_A - I$, $R_B = 2\Pi_B - I$. Let P_χ be the projector onto the span of the eigenvectors of $R_B R_A$ with eigenvalues $e^{2i\theta}$ such that $|\theta| \leqslant \chi$. Then, for any $|\psi\rangle$ such that $\Pi_A|\psi\rangle = 0$, we have

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- These spaces are spanned by vectors of the form $|\psi_x^\perp\rangle$ for $x \in A$, $x \in B$ respectively.

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- Here $|\psi_x^\perp\rangle$ is orthogonal to $|\psi_x\rangle$ and has support only on $\{|x\rangle\} \cup \{|y\rangle : x \to y\}$; in addition to $|r\rangle$ in the case of R_B.

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- On each subspace $\mathcal{H}_x$, $x \in A$, $|\eta\rangle \propto |\psi_x\rangle$, so $\Pi_A|\eta\rangle = 0$. Similarly $\Pi_B|\eta\rangle = |r\rangle$.
- By the effective spectral gap lemma,

$$\|P_\chi |r\rangle\| = \|P_\chi \Pi_B |\eta\rangle\| \leqslant \chi \| |\eta\rangle \| \leqslant \chi \sqrt{Tn}.$$