Universal recoverability in quantum information theory

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David Sutter

Institute for Theoretical Physics, ETH Zurich

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ETH zürich

Joint work with Omar Fawzi, Marius Junge, Renato Renner, Mark Wilde, and Andreas Winter

The Fawzi-Renner bound (see previous talk)

- Von Neuman entropy $H(\rho_A) = H(A)_\rho =: -\mathrm{tr}(\rho_A \log \rho_A)$
- Quantum conditional mutual information
  $I(A : C|B)_\rho := H(AB)_\rho + H(BC)_\rho - H(ABC)_\rho - H(B)_\rho$
- Fidelity $F(\rho, \sigma) := \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1 \in [0, 1]$

Theorem: For any ρ_{ABC} there exists $\mathcal{R}_{B\to BC}$ such that

$$I(A : C|B)_\rho \geq -2 \log F\big(\rho_{ABC}, \mathcal{R}_{B\to BC}(\rho_{AB})\big) \geq 0$$

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Remarks:

- Strong subadditivity $I(A:C|B)_\rho \geq 0$ [Lieb-Ruskai-73]
- Quantum Markov chains (QMC) ρ_{ABC} s.t. $\exists \mathcal{R}_{B\to BC}$ s.t. $\rho_{ABC} = \mathcal{R}_{B\to BC}(\rho_{AB})$
- ρ_{ABC} is QMC $\iff$ $I(A:C|B)_\rho = 0$ [Petz-88]
- $I(A:C|B)_\rho \leq \epsilon$ then $\exists \mathcal{R}_{B\to BC}$ s.t. $\rho_{ABC} \approx_\epsilon \mathcal{R}_{B\to BC}(\rho_{AB})$

The Fawzi-Renner bound (con't)

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$$I(A:C|B)_\rho \geq -2\log F\big(\rho_{ABC}, \mathcal{R}_{B\to BC}(\rho_{AB})\big) \geq 0 \qquad (1)$$

$$\mathcal{R}_{B\to BC} : X_B \mapsto V_{BC}\rho_{BC}^{\frac{1}{2}}(\rho_B^{-\frac{1}{2}} U_B X_B U_B^\dagger \rho_B^{-\frac{1}{2}} \otimes \mathrm{id}_C)\rho_{BC}^{\frac{1}{2}} V_{BC}^\dagger$$

- V_{BC} and U_B unknown unitaries that could depend on ρ_{ABC}

Question 1 Can we prove (1) for an explicit recovery map $\mathcal{R}_{B\to BC}$ that only depends on ρ_{BC}? (Universality property)

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$$D(\rho\|\sigma) := \begin{cases} \mathrm{tr}(\rho\log\rho) - \mathrm{tr}(\rho\log\sigma) & \text{if } \mathrm{supp}(\rho)\subseteq\mathrm{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases}$$

Monotonicity of relative entropy (data processing inequality)

$$D(\rho\|\sigma) - D\big(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)\big) \geq 0$$

Let $\rho = \rho_{ABC}$, $\sigma = \rho_{BC}$, and $\mathcal{N}(\cdot) = \mathrm{tr}_C(\cdot)$

$$D(\rho\|\sigma) - D\big(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)\big) = I(A:C|B)_\rho$$

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Theorem: For any ρ_{ABC} there exists $\mathcal{R}_{B\to BC}$ such that

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Question 1 Can we prove (1) for an explicit recovery map $\mathcal{R}_{B\to BC}$ that only depends on ρ_{BC}? (Universality property)

Question 2 Can we generalize (1) to

$$D(\rho\|\sigma) - D\big(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)\big) \geq -2\log F(\rho, (\mathcal{R}\circ\mathcal{N})(\rho))$$

Main result

Theorem: For any ρ, σ, $\mathcal{N}$ we have

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -2\log F(\rho, (\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho))$$

where

$$\mathcal{R}_{\sigma,\mathcal{N}}(\cdot) := \int_{\mathbb{R}} \mathrm{d}t\, \beta_0(t)\, \mathcal{R}_{\sigma,\mathcal{N}}^{\frac{t}{2}}(\cdot)$$

with

$$\mathcal{R}_{\sigma,\mathcal{N}}^{t} : X_B \mapsto \sigma^{\frac{1}{2}-\mathrm{i}t}\mathcal{N}^{\dagger}\big(\mathcal{N}(\sigma)^{-\frac{1}{2}+\mathrm{i}t} X_B\, \mathcal{N}(\sigma)^{-\frac{1}{2}-\mathrm{i}t}\big)\sigma^{\frac{1}{2}+\mathrm{i}t}$$

and a probability density

$$\beta_0(t) := \frac{\pi}{2}\big(\cosh(\pi t) + 1\big)^{-1}$$

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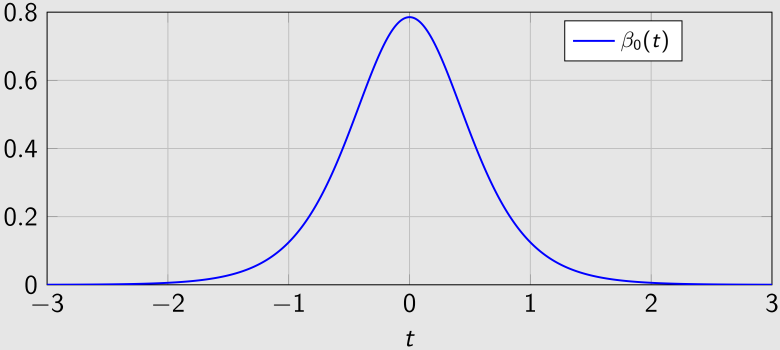

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and a probability density

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To prove the theorem we need interpolation theory

Stein-Hirschman operator interpolation theorem

Strengthening of the Hadamard three lines theorem

- $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$
- $L(\mathcal{H})$ is the space of bounded linear operators acting on $\mathcal{H}$
- Let $G : \overline{S} \to L(\mathcal{H})$ be
  - bounded on $\overline{S}$
  - holomorphic on S
  - continuous on the boundary $\partial\overline{S}$
- Let $\theta \in (0,1)$ and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ where $p_0, p_1 \in [1, \infty]$

$$\log \|G(\theta)\|_{p_\theta} \leq \int_{\mathbb{R}} \mathrm{d}t \left(\alpha_\theta(t) \log \|G(\mathrm{i}t)\|_{p_0}^{1-\theta} + \beta_\theta(t) \log \|G(1+\mathrm{i}t)\|_{p_1}^{\theta} \right)$$

with

$$\alpha_\theta(t) := \frac{\sin(\pi\theta)}{2(1-\theta)\left[\cosh(\pi t) - \cos(\pi\theta)\right]}$$

$$\beta_\theta(t) := \frac{\sin(\pi\theta)}{2\theta\left[\cosh(\pi t) + \cos(\pi\theta)\right]}$$

Proof sketch of Recoverability Theorem

1. Tune parameters
   - Let U be the Stinespring isometry corresponding to $\mathcal{N}$
   - Pick $G(z) := \left(\mathcal{N}(\rho)^{\frac{z}{2}}\mathcal{N}(\sigma)^{-\frac{z}{2}} \otimes I_E\right) U \sigma^{\frac{z}{2}} \rho^{\frac{1}{2}}$
   - Pick $p_0 = 2$, $p_1 = 1$, $\theta \in (0,1) \Rightarrow p_\theta = \frac{2}{1+\theta}$

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2. Evaluate norms
   - $\|G(\mathrm{i}t)\|_2 \leq \left\|\rho^{\frac{1}{2}}\right\|_2 = 1$
   - $\|G(1+\mathrm{i}t)\|_1 = F\left(\rho, (\mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho)\right)^{\frac{1}{2}}$

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3. Apply the Stein-Hirschman theorem

$$\log \left\| \left(\mathcal{N}(\rho)^{\frac{\theta}{2}}\mathcal{N}(\sigma)^{-\frac{\theta}{2}} \otimes I_E\right) U \sigma^{\frac{\theta}{2}} \rho^{\frac{1}{2}} \right\|_{2/(1+\theta)} \leq \int_{\mathbb{R}} \mathrm{d}t\ \beta_\theta(t) \log F\left(\rho, (\mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho)\right)^{\frac{\theta}{2}}$$

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4. Final step
   - Multiply both sides by $-\frac{2}{\theta}$ and consider $\theta \downarrow 0$

Remarks about the Recoverability Theorem

Theorem: For any ρ, σ, $\mathcal{N}$ we have

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -2\log F(\rho, (\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho))$$

where

$$\mathcal{R}_{\sigma,\mathcal{N}}(\cdot) := \int_{\mathbb{R}} \mathrm{d}t\, \beta_0(t)\, \mathcal{R}^{\frac{t}{2}}_{\sigma,\mathcal{N}}(\cdot)$$

- ρ density operator on a *separable* Hilbert space A
- σ non-negative operator on A
- $\mathcal{N}$ TPCP map from A to a separable Hilbert space B
- $\mathcal{R}_{\sigma,\mathcal{N}}$ does not depend on $\rho \Rightarrow$ universality property
- For $\rho = \rho_{ABC}$, $\sigma = \rho_{BC}$ and $\mathcal{N}(\cdot) = \mathrm{tr}_C(\cdot)$ we have
  $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = I(A:C|B)_\rho$
  $\Rightarrow$ FR-bound with explicit and universal recovery map.

Tighten the bound?

Theorem: For any ρ, σ, $\mathcal{N}$ we have

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -2\log F(\rho, (\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho))$$

- $D(\rho\|\sigma) \geq -2\log F(\rho, \sigma)$

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Dream: For any ρ, σ, $\mathcal{N}$ there exits $\mathcal{R}_{\sigma,\mathcal{N}}$ such that

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- Not clear how to prove it ☹

Tighten the bound?

Theorem: For any ρ, σ, $\mathcal{N}$ we have

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- $D(\rho\|\sigma) \geq -2\log F(\rho,\sigma)$

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- Not clear how to prove it ☹
- Measured relative entropy

$$D_{\mathbb{M}}(\rho\|\sigma) := \sup\Big\{ D\big(\mathcal{M}(\rho)\big\|\mathcal{M}(\sigma)\big) : \mathcal{M}(\rho) = \sum_x \mathrm{tr}(\rho M_x)|x\rangle\langle x| \text{ with } \sum_x M_x = \mathrm{id}\Big\}$$

- $D(\rho\|\sigma) \geq D_{\mathbb{M}}(\rho\|\sigma) \geq -2\log F(\rho,\sigma)$

Second result (see also [S-Tomamichel-Harrow-15])

Theorem: For any σ, $\mathcal{N}$ there exists $\mathcal{R}_{\sigma,\mathcal{N}}$ such that for any ρ

$$\begin{aligned} D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) &\geq D_{\mathbb{M}}(\rho\|(\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho)) \\ &\geq -2\log F(\rho, (\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho)) \end{aligned}$$

Second result (see also [S-Tomamichel-Harrow-15])

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D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) &\geq D_{\mathbb{M}}(\rho\|(\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho)) \\
&\geq -2\log F(\rho, (\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho))
\end{aligned}
$$

- $\mathcal{R}_{\sigma,\mathcal{N}}$ does not depend on ρ ($\rightarrow$ it is universal)
- $\mathcal{R}_{\sigma,\mathcal{N}}$ is not known explicitly
- Totally different proof technique
  - Pinching maps
  - Variational formula of (measured) relative entropy
  - Sion's minimax theorem
- see also [Brandão-Harrow-Oppenheim-Strelchuk-14]

Schematic overview of different proofs

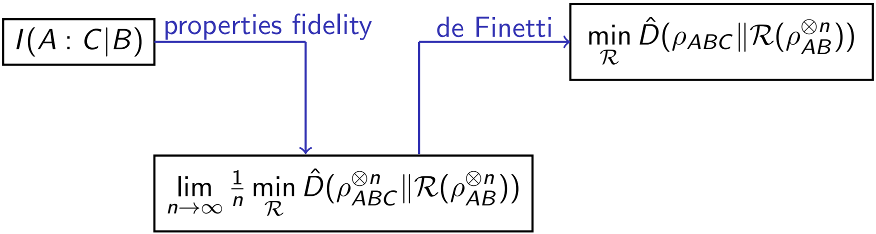
$$I(A:C|B)$$

$$\min_{\mathcal{R}} \hat{D}(\rho_{ABC} \| \mathcal{R}(\rho_{AB}^{\otimes n}))$$

$$\lim_{n\to\infty} \frac{1}{n} \min_{\mathcal{R}} \hat{D}(\rho_{ABC}^{\otimes n} \| \mathcal{R}(\rho_{AB}^{\otimes n}))$$

Schematic overview of different proofs

- Fawzi-Renner (October 2014)



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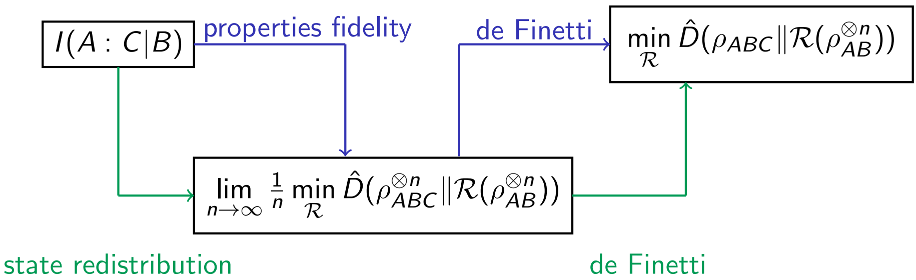

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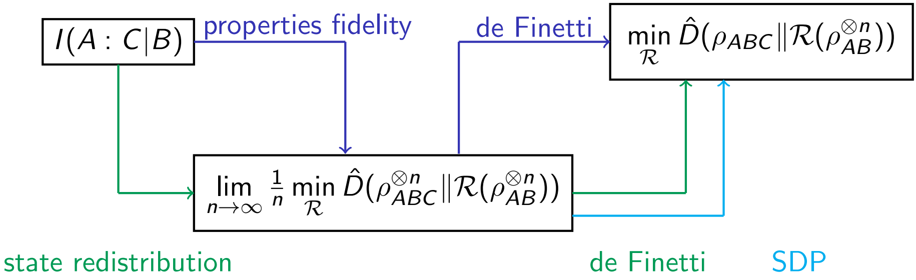

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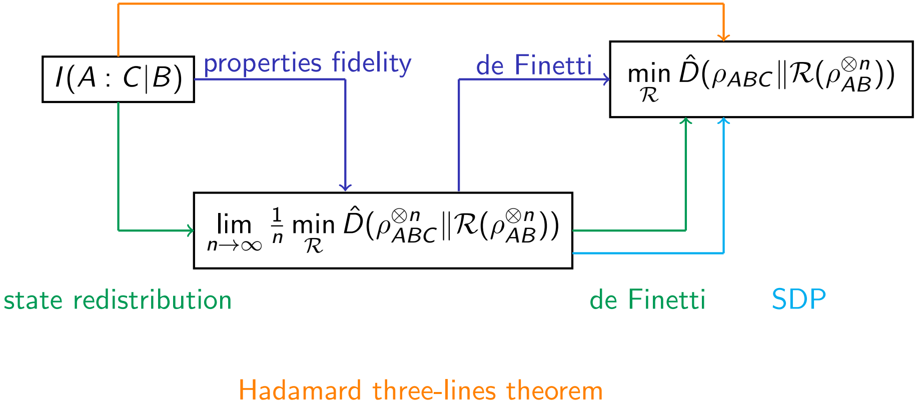

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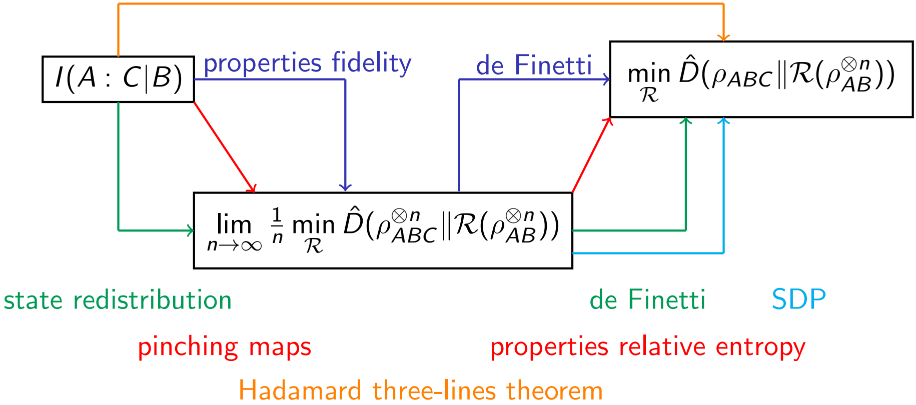
- Wilde (May 2015)
- S-Tomamichel-Harrow (July 2015)



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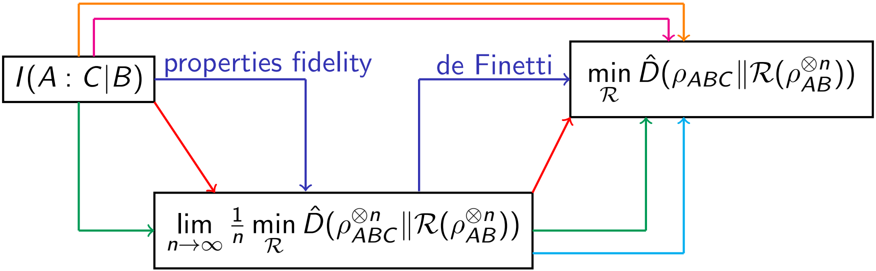
- S-Tomamichel-Harrow (July 2015)
- Junge-Renner-S-Wilde-Winter (September 2015)



state redistribution

de Finetti

SDP

pinching maps

properties relative entropy

Hadamard three-lines theorem

Stein-Hirschman theorem

Summary

- Two different measures of correlations
  - Conditional mutual information (information theoretic, easy to compute)
  - Recoverability (operational, more difficult to evaluate)
- FR-bound (and its follow up versions) provides a link
- Can we prove a remainder term in terms of a relative entropy?
- Does the Petz recovery map satisfy all the bounds seen in this talk?
- Can we prove an upper bound for the conditional mutual information with a similar form as the FR-lower-bound? (see [Wilde-15] for partial progress)

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Proof sketch [S-Tomamichel-Harrow-15]

Thm : $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq D_{\mathbb{M}}(\rho\|(\mathcal{R}\circ\mathcal{N})(\rho))$

- For $H = \sum_x \lambda_x |x\rangle\langle x|$ let $P_\lambda := \sum_{x:\lambda_x=\lambda} |x\rangle\langle x|$ and define the pinching map

$$\mathcal{P}_H \; : \; X \mapsto \sum_{\lambda\in\mathrm{spec}(H)} P_\lambda \, X \, P_\lambda$$

- Pinching recovery map $\mathcal{R}^n_{\sigma,\mathcal{N}}$

$$X \mapsto (\sigma^{\frac{1}{2}})^{\otimes n}\mathcal{P}_{\sigma^{\otimes n}}\left((\mathcal{N}^\dagger)^{\otimes n}\left[(\mathcal{N}(\sigma)^{-\frac{1}{2}})^{\otimes n}\mathcal{P}_{\mathcal{N}(\sigma)^{\otimes n}}(X)(\mathcal{N}(\sigma)^{-\frac{1}{2}})^{\otimes n}\right]\right)(\sigma^{\frac{1}{2}})^{\otimes n}$$

Lem : $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq \liminf_{n\to\infty} \frac{1}{n} D(\rho^{\otimes n}\|(\mathcal{R}^n_{\sigma,\mathcal{N}} \circ \mathcal{N}^{\otimes n}(\rho^{\otimes n}))$

Proof sketch [S-Tomamichel-Harrow-15] (con't)

Lem : $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq \liminf_{n\to\infty} \frac{1}{n} D(\rho^{\otimes n}\|(\mathcal{R}^n_{\sigma,\mathcal{N}} \circ \mathcal{N}^{\otimes n}(\rho^{\otimes n}))$

- $\mathcal{P}_H(X)$ commutes with H
- Pinching inequality: $\mathcal{P}_H(X) \geq \frac{1}{|\mathrm{spec}(H)|} X$ for all $X \in \mathrm{Pos}(A)$
- For any $\rho \in \mathrm{Pos}(A)$ we have $|\mathrm{spec}(\rho^{\otimes n})| = O(\mathrm{poly}(n))$
- Operator logarithm is concave and monotone
- $\mathrm{tr}\big(\mathcal{N}(\rho) \log \sigma\big) \leq \mathrm{tr}\big(\rho \log \mathcal{N}^\dagger(\sigma)\big)$

Lem : $\mathcal{R}^n_{\sigma,\mathcal{N}}(\,\cdot\,) = \frac{1}{(2\pi)^{d_1}} \int_{[0,2\pi]^{\times d_1}} \mathrm{d}\vartheta \;\; \frac{1}{(2\pi)^{d_2}} \int_{[0,2\pi]^{\times d_2}} \mathrm{d}\varphi \;\; (\mathcal{T}^{\varphi,\vartheta}_{\sigma,\mathcal{N}})^{\otimes n}(\,\cdot\,)$

with

$$\mathcal{T}^{\varphi,\vartheta}_{\sigma,\mathcal{N}} \;:\; X_B \mapsto U^{\vartheta}_{\sigma} \sigma^{\frac{1}{2}} \mathcal{N}^\dagger\big(\mathcal{N}(\sigma)^{-\frac{1}{2}} U^{\varphi}_{\mathcal{N}(\sigma)} X_B U^{\varphi\dagger}_{\mathcal{N}(\sigma)} \mathcal{N}(\sigma)^{-\frac{1}{2}}\big) \sigma^{\frac{1}{2}} U^{\vartheta\dagger}_{\sigma}$$

Proof sketch [S-Tomamichel-Harrow-15] (con't)

Lem : $\frac{1}{n} D_{\mathbb{M}} \left(\rho^{\otimes n} \middle\| \int \mu(\mathrm{d}\sigma) \sigma^{\otimes n} \right) \geq \min_{\nu} D_{\mathbb{M}} \left(\rho \middle\| \int \nu(\mathrm{d}\sigma) \sigma \right)$

Combining these three lemmas gives

$$
\begin{aligned}
& D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \\
& \geq \liminf_{n\to\infty} \frac{1}{n} D\left(\rho^{\otimes n}\|(\mathcal{R}^n_{\sigma,\mathcal{N}} \circ \mathcal{N}^{\otimes n}(\rho^{\otimes n})\right) \\
& = \liminf_{n\to\infty} \frac{1}{n} D\left(\rho^{\otimes n}\| \frac{1}{(2\pi)^{d_1}} \int_{[0,2\pi]^{\times d_1}} \mathrm{d}\vartheta \frac{1}{(2\pi)^{d_2}} \int_{[0,2\pi]^{\times d_2}} \mathrm{d}\varphi (\mathcal{T}^{\varphi,\vartheta}_{\sigma,\mathcal{N}})^{\otimes n} (\mathcal{N}(\rho)^{\otimes n})\right) \\
& \geq \liminf_{n\to\infty} \frac{1}{n} D_{\mathbb{M}}\left(\rho^{\otimes n}\| \frac{1}{(2\pi)^{d_1}} \int_{[0,2\pi]^{\times d_1}} \mathrm{d}\vartheta \frac{1}{(2\pi)^{d_2}} \int_{[0,2\pi]^{\times d_2}} \mathrm{d}\varphi (\mathcal{T}^{\varphi,\vartheta}_{\sigma,\mathcal{N}})^{\otimes n} (\mathcal{N}(\rho)^{\otimes n})\right) \\
& \geq \min_{\vartheta,\varphi} D_{\mathbb{M}}\left(\rho\|(\mathcal{T}^{\varphi,\vartheta}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho)\right)
\end{aligned}
$$