Dvoretzky's theorem and the complexity of entanglement detection

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Foreword

We present a lower bound for complexity of entanglement detection which (ultimately) relies on the 1961 Dvoretzky's theorem, a fundamental result from Asymptotic Geometric Analysis asserting that high-dimensional convex sets typically look round when we observe only their section with a randomly chosen subspaces of smaller dimension.

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We present a lower bound for complexity of entanglement detection which (ultimately) relies on the 1961 Dvoretzky's theorem, a fundamental result from Asymptotic Geometric Analysis asserting that high-dimensional convex sets typically look round when we observe only their section with a randomly chosen subspaces of smaller dimension.

More on the interface between Asymptotic Geometric Analysis and Quantum Information Theory can be found in the forthcoming book

G. Aubrun and S. Szarek, *Alice and Bob meet Banach*

of which a preliminary version is available via Aubrun's web page.

Outline

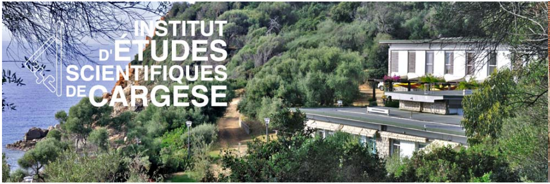
- Notation
- Background: Gurvits, Horodeckis, Størmer, Skowronek
- The main result
- Strategy behind the proof:

  Tangible version of Dvoretzky's theorem (Milman 1971)

  Face/vertex counting (Figiel–Lindenstrauss–Milman 1977)

  Bounds on verticial and facial complexity of sets of quantum states
- Sketch of the proof
- Conclusions

Before we proceed. . . an announcement



Fall 2017 (Sep. 4 - Dec. 15): Trimester on

ANALYSIS IN QUANTUM INFORMATION THEORY

at the Institut Henri Poincaré in Paris

Pre-school, September 4-8, Cargèse, Corsica

http://www.ihp.fr/en/activities/trimester-thematic/calendar

Organizers: G. Aubrun, B. Collins, I. Nechita, S. Szarek

Don't be shy and let one of us know if you are interested!

Entanglement

A quantum state on a finite-dimensional complex Hilbert space $\mathcal{H}$ is a positive operator of trace 1. Denote by $\mathrm{D} = \mathrm{D}(\mathcal{H})$ the set of states on $\mathcal{H}$. Note that

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When $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a bipartite Hilbert space, the set of separable states is the subset $\mathrm{Sep}(\mathcal{H}) \subset \mathrm{D}(\mathcal{H})$ defined as

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The dichotomy between entanglement vs. separability is fundamental in quantum theory.
Entanglement is indispensable for most protocols of quantum information theory/cryptography/computing/teleportation...

Certifying/witnessing entanglement

Since entanglement is defined as non-membership in a (closed) convex set, it follows from the Hahn–Banach separation theorem that for every entangled state ρ, there is a linear form f such that $f \leqslant a$ on Sep and $f(\rho) > a$. Such f certifies, or witnesses, the entanglement of ρ.

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More generally, it is known (Gurvits 2003) that deciding whether a state is entangled or separable is, in general, NP-hard. Later refinements have been due to Ioannou (2007), Gharibian (2010) and others; in particular some upper bounds were supplied by Brandão et al (2011).

The Horodecki criterion

A more structured scheme of witnessing entanglement is given by the following (M_d stands for the space of $n \times n$ complex matrices).

Theorem (The Horodecki criterion 1996)

A state $\rho \in \mathrm{Sep}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is **entangled** *if and only if there is a* **positive map** $\Phi : \mathrm{M}_d \to \mathrm{M}_d$ *such that the operator* $(\mathrm{Id} \otimes \Phi)\rho$ **is not** *positive semi-definite (one says that Φ witnesses the entanglement of ρ).*

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A map $\mathrm{M}_d \to \mathrm{M}_d$ is completely positive (CP) if it is a positive linear combination of maps of the form $\Phi_A : X \mapsto AXA^\dagger$. CP maps cannot be witnesses since $\mathrm{Id} \otimes \Phi_A = \Phi_{\mathrm{I} \otimes A}$ is also positive.

Størmer's theorem

In dimension 2 the cone of positive maps has a rather simple structure.

Theorem (Størmer 1963)

Any positive map $\Phi : \mathrm{M}_2 \to \mathrm{M}_2$ *can be written as* $\Phi = \Phi_1 + \Phi_2 \circ T$*, where* Φ_1, Φ_2 *are CP maps and* T *is the transposition on* M_2*.*

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This is specific to dimension 2.

Theorem (Skowronek, unpublished)

Let $d \geqslant 3$*. Let* $\mathcal{F}$ *be a family of positive maps* $\mathrm{M}_d \to \mathrm{M}_d$ *such that any entangled state* ρ *on* $\mathbb{C}^d \otimes \mathbb{C}^d$ *is witnessed by an element of* $\mathcal{F}$*. Then* $\mathcal{F}$ *is infinite.*

In fact, any closed universal family of witnesses must be uncountable.

Our main result

Denote by $\bullet$ homotheties with respect to $\rho_* := \frac{\mathrm{I}_{\mathcal{H}}}{\dim \mathcal{H}}$:

$$t \bullet \rho := t\rho + (1-t)\rho_*.$$

Say that a state ρ is robustly entangled if $\frac{1}{2} \bullet \rho$ is entangled. Robustly entangled states remain entangled in the presence of randomizing noise.

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Suppose that $\Phi_1, \dots, \Phi_N$ are positive maps on M_d such that, for any robustly entangled state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$, there is an index i such that $(\mathrm{Id} \otimes \Phi_i)(\rho)$ is not positive semi-definite. Then $N \geqslant \exp(cd^3/\log d)$ for some universal constant $c > 0$.

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This shows that the set of separable states is complex, and not because of some fine features of its boundary. Results about NP-hardness of entanglement detection have usually focused on boundary effects.

Verticial and facial dimensions of convex bodies

Let $K \subset \mathbb{R}^n$ be a convex compact set with 0 in the interior. Define the verticial and facial dimensions of K as

$$\dim_V(K) := \log \inf\{\#\text{vertices}(P) \;:\; K \subset P \subset 4K\}$$

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These are dual concepts since $\dim_F(K) = \dim_V(K^\circ)$, where $K^\circ := \{x \in \mathbb{R}^n \ : \ \langle x, y\rangle \leqslant 1, \forall y \in K\}$ is the polar of K.

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One has $\dim_F(K) = O(n)$ and $\dim_V(K) = O(n)$ if (say) the origin is the center of mass of K.

If $E \subset \mathbb{R}^n$ is a linear subspace, then $\dim_F(K \cap E) \leqslant \dim_F(K)$.

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Another parameter is the asphericity of K defined as

$$a(K) := \inf\left\{R/r \;:\; rB_2^n \subset K \subset RB_2^n\right\}.$$

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This result is a consequence of the tangible version of Dvoretzky's theorem due to Milman, which gives a sharp formula for the dimension of almost Euclidean sections of convex bodies.

If $K \subset \mathbb{R}^n$ is a convex body such that $rB_2^n \subset K$ and $M = M(K)$ denotes the average of the "norm" $\| \cdot \|_K$ over the sphere, then K has ***lots of*** *almost Euclidean sections of dimension $k = \Omega(nr^2 M^2)$.*

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Thus the facial dimension of K exceeds ck. Applying the same argument to K° and using the inequality $M(K)M(K^\circ) \geqslant 1$ yields the FLM bound.

The FLM bound – examples

We illustrate the FLM bound on some examples where it is sharp up to polylog factors

$$\dim_V(K) \cdot \dim_F(K) \geqslant c \left(\frac{n}{a(K)} \right)^2$$

K	dimension	$a(K)$	$\dim_V(K)$	$\dim_F(K)$
B_2^n	n	1	$\Theta(n)$	$\Theta(n)$
$[-1,1]^n$	n	$\sqrt{n}$	$\Theta(n)$	$\Theta(\log n)$
Δ_n	n	n	$\Theta(\log n)$	$\Theta(\log n)$

Recall that B_2^n is the n-dimensional Euclidean ball, while Δ_n is the n-dimensional simplex.

Quantum-related examples

And here are some more examples related to entanglement detection.

K	dimension	$a(K)$	$\dim_V(K)$	$\dim_F(K)$
$\mathrm{D}(\mathbb{C}^m)$	$m^2 - 1$	$m - 1$		
$\mathrm{Sep}(\mathbb{C}^d \otimes \mathbb{C}^d)$	$d^4 - 1$	$d^2 - 1$		

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- For a well chosen (e.g., random) $\frac{1}{10}$-net $\mathscr{N}$ in the unit sphere of $\mathbb{C}^m$ and $P = \mathrm{conv}\{|\psi\rangle\langle\psi| \ : \ \psi \in \mathscr{N}\}$, we have $\frac{1}{4} \bullet \mathrm{D} \subset P \subset \mathrm{D}$.

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- For a well chosen (e.g., random) $\frac{1}{10}$-net $\mathcal{N}$ in the unit sphere of $\mathbb{C}^m$ and $P = \mathrm{conv}\{|\psi\rangle\langle\psi| \ : \ \psi \in \mathcal{N}\}$, we have $\frac{1}{4} \bullet \mathrm{D} \subset P \subset \mathrm{D}$.
- Similarly, for any $\frac{1}{10d}$-net $\mathcal{N}'$ in the unit sphere of $\mathbb{C}^d$ and $P' = \mathrm{conv}\{|\psi \otimes \varphi\rangle\langle\psi \otimes \varphi| \ : \ \psi, \varphi \in \mathcal{N}'\}$, we have $\frac{1}{4} \bullet \mathrm{Sep} \subset P' \subset \mathrm{Sep}$.

Quantum-related examples, II

We now complete the table.

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$\mathrm{Sep}(\mathbb{C}^d \otimes \mathbb{C}^d)$	d^4-1	d^2-1	$\Theta(d \log d)$	$\Omega(d^3/\log d)$

The set D is self-dual (or, more precisely, $\mathrm{D}^\circ = (-m) \bullet \mathrm{D}$), so its facial dimension equals its verticial dimension.

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The lower bound on the facial dimension of Sep follows from the Figiel–Lindenstrauss–Milman inequality

$$\dim_V(\mathrm{Sep}) \cdot \dim_F(\mathrm{Sep}) \geqslant c \left(\frac{d^4 - 1}{d^2 - 1} \right)^2 > cd^4$$

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We now complete the table.

K	dimension	$a(K)$	$\dim_V(K)$	$\dim_F(K)$
$\mathrm{D}(\mathbb{C}^m)$	$m^2 - 1$	$m - 1$	$\Theta(m)$	$\Theta(m)$
$\mathrm{Sep}(\mathbb{C}^d \otimes \mathbb{C}^d)$	$d^4 - 1$	$d^2 - 1$	$\Theta(d \log d)$	$\Omega(d^3 / \log d)$

The set D is self-dual (or, more precisely, $\mathrm{D}^\circ = (-m) \bullet \mathrm{D}$), so its facial dimension equals its verticial dimension.

The lower bound on the facial dimension of Sep follows from the Figiel–Lindenstrauss–Milman inequality

$$\dim_V(\mathrm{Sep}) \cdot \dim_F(\mathrm{Sep}) \geqslant c \left(\frac{d^4 - 1}{d^2 - 1} \right)^2 > cd^4$$

However, it is conceivable that we actually have $\dim_F(\mathrm{Sep}) = \Theta(d^4)$.

Sketch of the proof of the theorem

Let $\Phi_1, \ldots, \Phi_N$ be N positive maps on M_d with the property that for every robustly entangled state ρ, there exists an index i such that $(\Phi_i \otimes \mathrm{Id})(\rho)$ is not positive. This hypothesis is equivalent to the following inclusion

$$\bigcap_{i=1}^{N} \left\{ \rho \in \mathrm{D}(\mathbb{C}^d \otimes \mathbb{C}^d) \;:\; (\mathrm{Id} \otimes \Phi_i)(\rho) \text{ is PSD} \right\} \subset 2 \bullet \mathrm{Sep}.$$

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which satisfies $\mathrm{Sep} \subset K \subset 2 \bullet \mathrm{Sep}$. Note the trace-preserving condition assures that for all i's we are $\bullet$-dilating with respect to the same point.

Sketch of the proof of the theorem, II

Since the facial dimension of $\mathrm{D}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is of order d^2, there exists a polytope P with at most $\exp(Cd^2)$ facets such that $\frac{1}{2} \bullet \mathrm{D} \subset P \subset \mathrm{D}$. Then the polytope

$$Q = P \cap \bigcap_{i=1}^{N} (\mathrm{Id} \otimes \Phi_i)^{-1}(P)$$

satisfies $\frac{1}{2} \bullet \mathrm{Sep} \subset \frac{1}{2} \bullet K \subset Q \subset K \subset 2 \bullet \mathrm{Sep}$. Since

$$\#\text{facets}(P_1 \cap P_2) \leqslant \#\text{facets}(P_1) + \#\text{facets}(P_2),$$

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The general situation (without the trace-preserving restriction) is handled similarly starting with the assumption that $(1 - \frac{1}{2d}) \bullet \mathrm{D} \subset P \subset \mathrm{D}$.

Conclusion

We illustrated the complexity of robust entanglement by showing that super-exponentially many positive maps are needed to detect it – at least if used non-adaptively/without reflection.

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Some other directions in which this work can be continued are:

- Upper bounds; in particular, what is the order of $d_F(\mathrm{Sep})$?
- Less/more robust entanglement, i.e., replacing $\frac{1}{2}$ with $\varepsilon \in (0, 1)$
- What if we use witnesses $\Phi : \mathrm{M}_d \to \mathrm{M}_m$, where $m = \mathit{poly}(d)$?
- The multipartite or "unbalanced" ($\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^m$) setting