An Energy Barrier is Necessary for the Thermal Stability of Stabilizer Quantum Memories

by
Anna Kómár (IQIM Caltech)
Olivier Landon-Cardinal (Caltech, McGill U)
Kristan Temme (Caltech, IBM)

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arXiv:1412.2858
arXiv: 1601.01324

THERMALIZATION TIME BOUND

- For a variety of models (commuting Pauli stabilizers and Abelian quantum doubles):

$$t_{Qmem} < t_{mix} \leq \mathcal{O}(N^2 e^{2\beta\bar{\epsilon}})$$

K. Temme. *arXiv:1412.2858*
A. Kómár, O. Landon-Cardinal, K. Temme. *arXiv: 1601.01324*

Outline

- Motivation
- Framework
- Thermalization time bounds
- Evaluation of the energy barrier
- Sketch of the proof

QUANTUM MEMORIES

Q-

- Quantum memory: physical system that encodes quantum information
  - Degenerate ground space
  - Robust to perturbations
  - Long memory time (even in a thermal environment)
- Active error correction:

encode → (syndrome) measurements & operations → decode

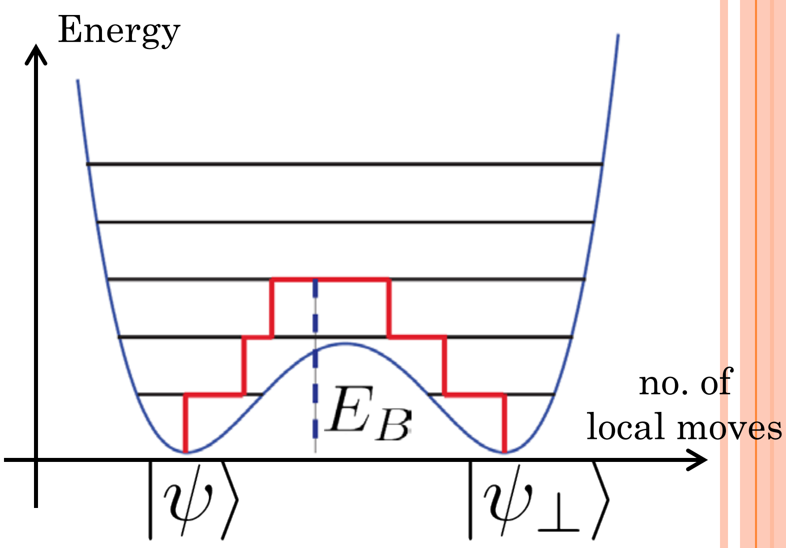
- Self-correction:

encode $\xrightarrow{\text{wait}}$ decode

Phenomenological Picture of Memory Time

- Arrhenius law:

$$t_{mem} \sim e^{\beta E_B}$$



	Constant Energy barrier	Scaling Energy barrier
non-SC	2D Toric code	welded code
SC	entropic codes* (?)	4D Toric code

Arrhenius bound

Arrhenius law

* Brown et al., *PRL (2014)*

Thermalization in 2D Toric Code

R. Alicki, M. Fannes, M. Horodecki J. Phys. A: Math. Theor. 42 (2009) 065303

- Thermalization represented by Liouvillian; with Gibbs state as a fixed point
- Liouvillian: detailed balanced, with gap λ
- Mixing time:
  - State is close to Gibbs state
  - Bounded by N/λ
- Spectral gap bound: $\lambda \geq \frac{1}{3} e^{-8\beta J}$
- Implies mixing time bound:

$$t_{mix} \leq \mathcal{O}(N e^{8\beta J})$$

Questions

- Can we have a more rigorous connection between the energy barrier and thermalization?

- Is entropic protection possible?

Framework

Stabilizer Codes

- Hamiltonian: $H = -J\sum_k g_k$ $\qquad \mathcal{H} = C^{2^N}$
  - With commuting Paulis:

    $\mathcal{G} = \{g_1, \ldots, g_M\} \quad [g_i, g_j] = 0$

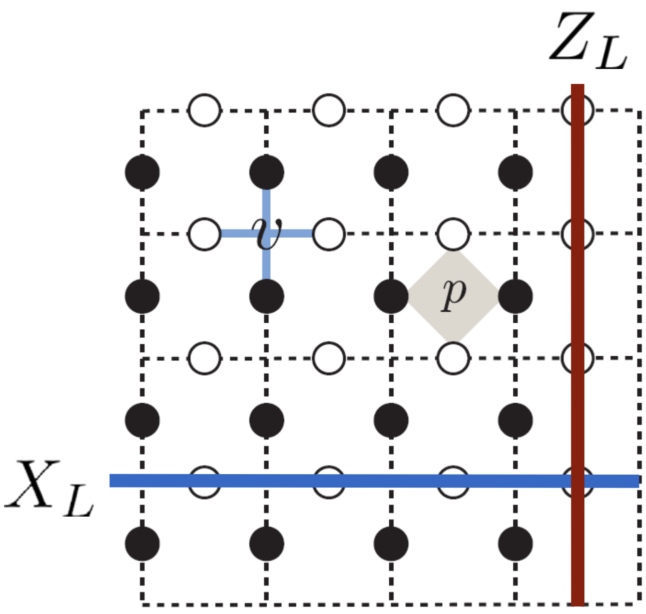
- Stabilizer group: $\mathcal{S} = \langle \mathcal{G} \rangle$

with codespace $\mathcal{C}(\mathcal{S})$ such that:

$s\,|\psi\rangle = |\psi\rangle$ for $\substack{s \in \mathcal{S} \\ |\psi\rangle \in \mathcal{C}(\mathcal{S})}$

- Logical operators:

$$Comm(\mathcal{S}) \backslash \mathcal{S}$$



$A(v) = Z^{\otimes 4}$

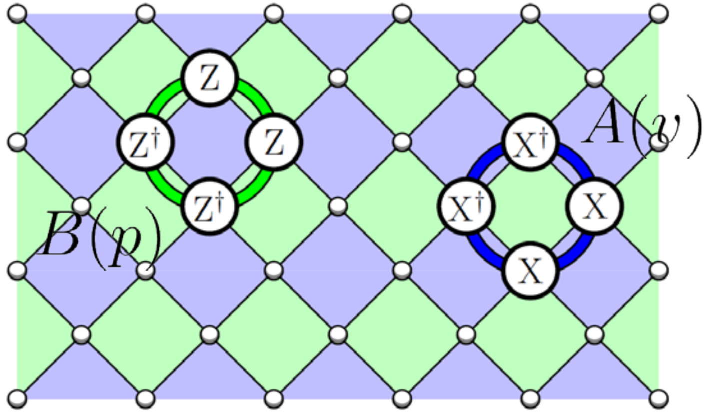
$B(p) = X^{\otimes 4}$

Abelian Quantum Doubles

Kitaev "Fault-tolerant quantum computation by anyons." *Annals of Physics* (2003)

- Generalization of Toric Code to $\mathbb{Z}_d$ qudits
- Generalized Paulis:

Source:
B.J. Brown, 2013



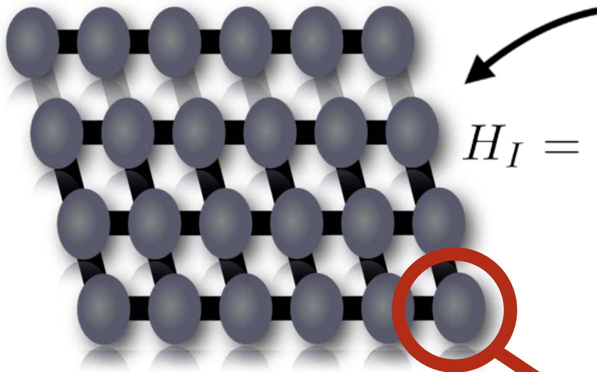
$$X^m |k\rangle = |k \oplus_d m\rangle$$

$$Z^l |k\rangle = e^{\frac{2\pi i}{d} lk} |k\rangle$$

- Hamiltonian:

$$H = \sum_v \sum_{a=0}^{d-1} J_v^a P_v^a + \sum_p \sum_{b=0}^{d-1} J_p^b Q_p^b$$

Noise Model



$$H_I = \lambda \sum_\alpha S_\alpha \otimes B_\alpha$$

H_S S_α

$$\rho_R \propto e^{-\beta H_B}$$

- Evolution: $\rho_S(t+\Delta t) = \mathrm{tr}[e^{-iH\Delta t}(\rho_S(t)\otimes\rho_R)e^{iH\Delta t}]$

- Markovian & Weak-Coupling limit:

$$\partial_t \rho_S = \mathcal{L}(\rho_S)$$

Davies Generator

E.B. Davies "Generators of dynamical semigroups." *Journal of Functional Analysis* (1979)

$$\mathcal{L}(\rho) = \sum_{\alpha,\omega} h^{\alpha}(\omega) \left(S_{\alpha}(\omega)\rho S_{\alpha}^{\dagger}(\omega) - \frac{1}{2}\{S_{\alpha}^{\dagger}(\omega)S_{\alpha}(\omega), \rho\}_{+} \right)$$

- Jump operators: $e^{iH_S t} S_{\alpha} e^{-iH_S t} = \sum_{\omega} S_{\alpha}(\omega) e^{i\omega t}$
- KMS condition: $h^{\alpha}(-\omega) = e^{-\beta\omega} h^{\alpha}(\omega)$
- Fixed point: Gibbs state: $\rho_G \propto e^{-\beta H_S}$

Thermalization Bounds and Energy Barrier

Mixing Time Bound

- Mixing time: state is close to Gibbs state (trace-norm)

Theorem *For any commuting N-qubit Pauli* or *N-qudit Abelian quantum double Hamiltonian, the mixing time is bounded by*

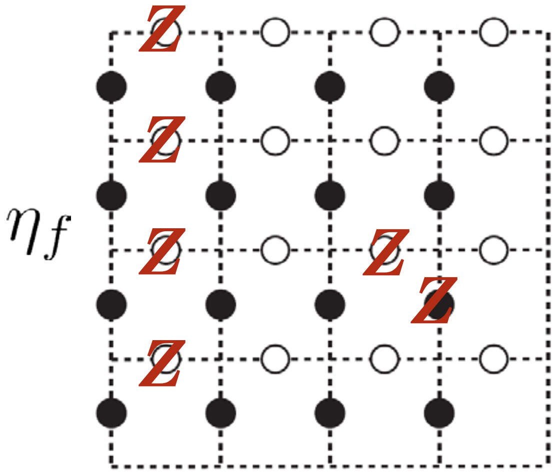
$$t_{mix} < \mathcal{O}\left(N \frac{4\mu^*}{h^*} e^{2\beta\bar{\epsilon}}\right) \tag{1}$$

- Largest (generalized) Pauli path: $\mu^* \leq \mathcal{O}(N)$
- Smallest transition rate: $h^* \geq c\, e^{-\beta\Delta}$
- Generalized energy barrier: $\bar{\epsilon}$

Pauli stabilizers: *Temme. arXiv:1412.2858*
Abelian Quantum Doubles: *AK, Landon-Cardinal, Temme. arXiv: 1601.01324*

Generalized Energy Barrier – Example on Toric Code

- Choose any error η_f (doesn't have to be a logical operator)



Generalized Energy Barrier – Example on Toric Code

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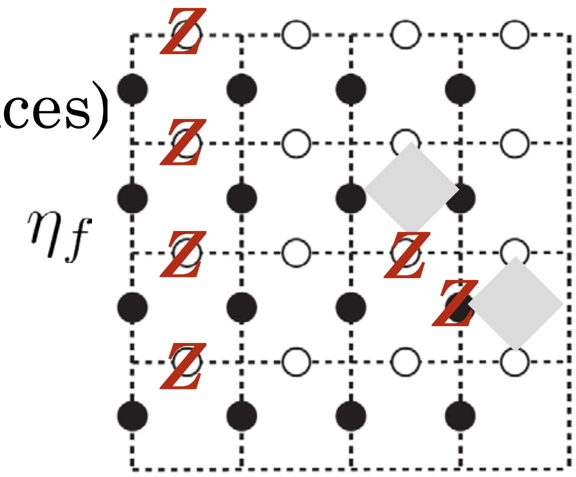
- Throw out violated plaquettes (vertices)



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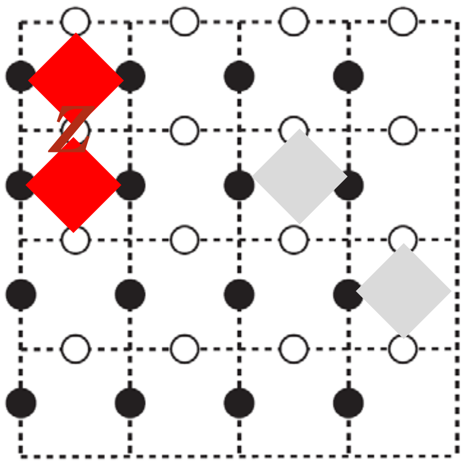

- Construct η_f by applying local errors
- Count number of violated plaquettes **at each step**

- Take maximum of these for given ordering;

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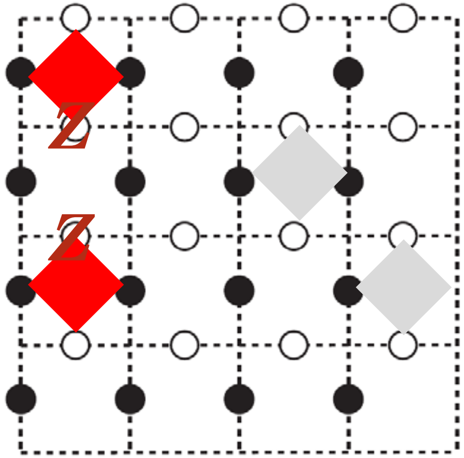

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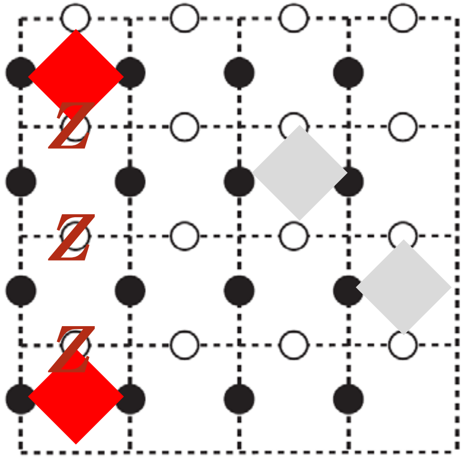

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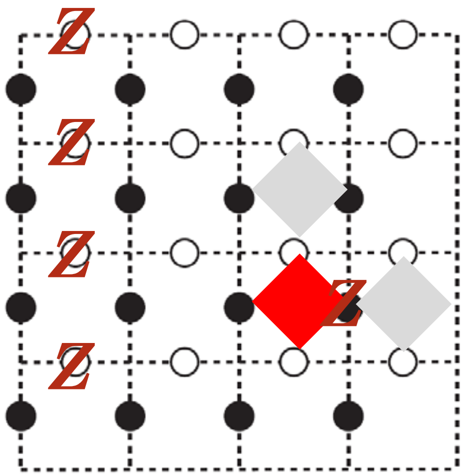

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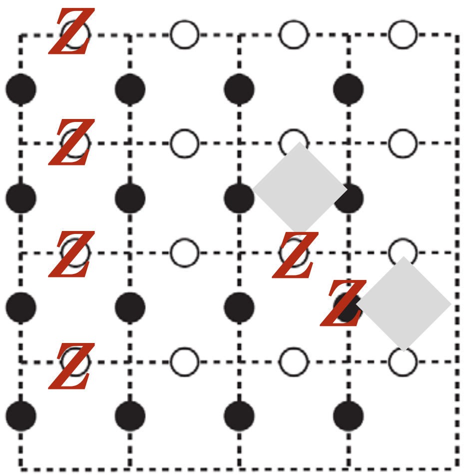


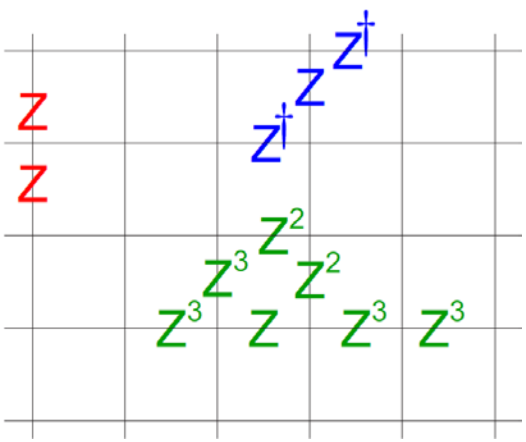
- Construct η_f by applying local errors
- Count number of violated plaquettes **at each step**

- Take maximum of these for given ordering;
- But: choose an optimal ordering: $\bar{\epsilon}(\eta_f)$

$$\Rightarrow \quad \bar{\epsilon} = \max_{\eta_f} \bar{\epsilon}(\eta_f)$$

Evaluating the Energy Barrier for Quantum Doubles

- (d-1) electric (magnetic) excitations + vacuum
- Break errors down to simple structures
- Generalized Pauli η_f = string-net of excitations



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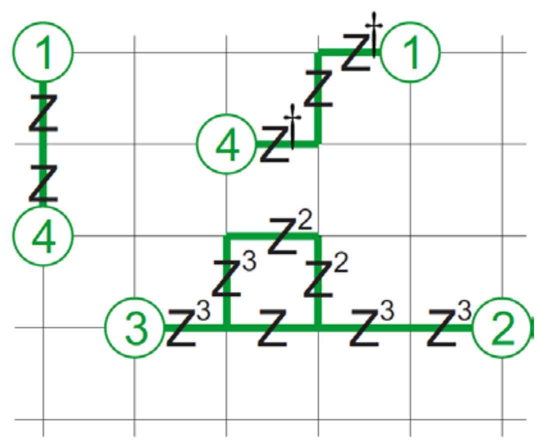
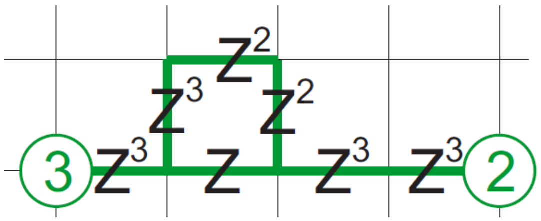


(d = 5)

- String-net = (charge) flow of excitations



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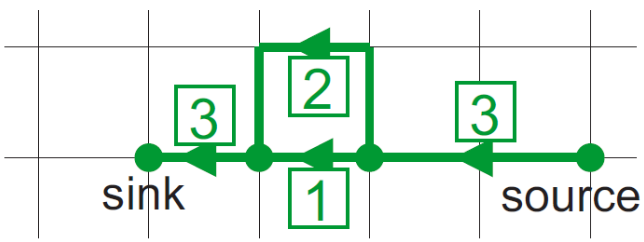
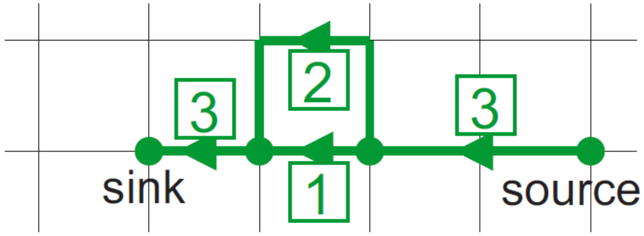


Evaluating the Energy Barrier for Quantum Doubles

- Helmholtz-Hodge decomposition:



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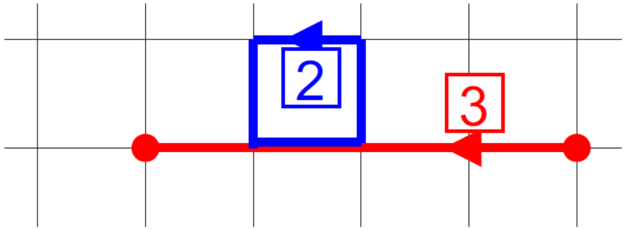


- In general: η_f configuration = loops + trees

- Energy barrier:
  - Of loops: 2
  - Of trees (simple union of strings): 2

Sketch of Proof

Spectral Gap and Thermalization Time

- Convergence to unique fixed point σ :

$$||e^{t\mathcal{L}}(\rho_0) - \sigma||_{tr} \leq \sqrt{||\sigma^{-1}||}e^{-\lambda t}$$

- For thermal σ :

$$||\sigma^{-1}|| \sim e^{c\beta N} \quad \Longrightarrow \quad t_{mix} \sim \mathcal{O}(\beta N \lambda^{-1})$$

Poincaré Inequality

$$\lambda = \min_{f \in \mathcal{M}_{dN}} \frac{\mathcal{E}(f,f)}{\mathrm{Var}_\sigma(f,f)}$$

Poincaré Inequality

$$\lambda = \min_{f \in \mathcal{M}_{d^N}} \frac{-\operatorname{tr}[\sigma f^\dagger \mathcal{L}^*(f)]}{\operatorname{tr}[\sigma f^\dagger f] - \operatorname{tr}[\sigma f]^2}$$

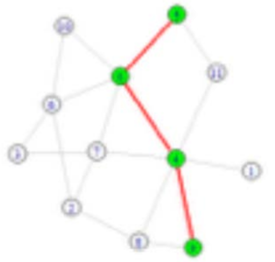
- Any $\mu > 0$ such that $\mu \leq \frac{\mathcal{E}(f,f)}{\mathrm{Var}_\sigma(f,f)}$ for all f is a lower bound to λ ⇨ maximize μ

- Equivalently: minimize τ subject to $\tau\hat{\mathcal{E}} - \hat{\mathcal{V}} \geq 0$

  $\mathcal{E}(f,f) = (f|\hat{\mathcal{E}}|f)$ $\mathrm{Var}_\sigma(f,f) = (f|\hat{\mathcal{V}}|f)$

(Quantum) Canonical Paths

- With defining a quantum canonical path we can factorize: $\hat{\mathcal{E}} = AA^\dagger$ and $\hat{\mathcal{V}} = BB^\dagger$ and define $AW = B$
- Then the spectral gap bound: $\tau = \min_W ||W||^2$
- For classical Markov processes:
  - M. Jerrum, A. Sinclair. "Approximating the permanent." *SIAM journal on computing 18.6 (1989): 1149-1178.*
  - Geometric picture: graph of states



- Generalized to Quantum setting:
  - K. Temme. "Thermalization time bounds for Pauli stabilizer Hamiltonians." *arXiv preprint arXiv:1412.2858* (2014).

Conclusion & Open Questions

- Commuting Pauli stabilizer Hamiltonians and Abelian quantum doubles follow the Arrhenius law
  - Even with defect lines of *Brown et al., PRL (2014)*

$$t_{mix} \leq \mathcal{O}(\beta N^2 e^{2\beta\bar{\epsilon}})$$

- No self-correction without a scaling energy barrier

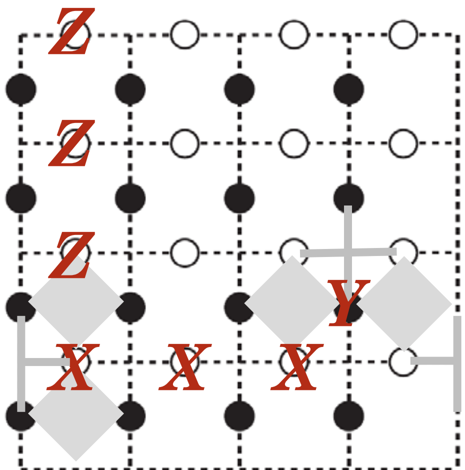
- Not excluded of entropic protection:
  - Different type of defects (*Bombin, PRL (2010)*)
  - Different noise model
  - Non-Abelian quantum doubles

K. Temme. *arXiv:1412.2858*
A. Kómár, O. Landon-Cardinal, K. Temme. *arXiv: 1601.01324*

Generalized Energy Barrier – Commuting Pauli Stabilizers

Defined for **any** operator, not only logical ops.

- Path on the Pauli group: $\eta_0, \eta_1, \ldots \eta_f \in \mathcal{P}_N$



- Reduced set $\mathcal{G}_\eta$:
  - Removing stabilizer generators violated in the final error configuration, we only consider the other stabilizer generators
- Energy barrier of a Pauli:

$$\bar{\epsilon}(\eta_f) = \min_{\eta_t} \max_t 2J^* \#\{g_k \in \mathcal{G}_\eta \,|\, [g_k, \eta_t] \neq 0\}$$

- Generalized energy barrier:

$$\bar{\epsilon} = \max_{\eta_f} \bar{\epsilon}(\eta_f)$$

For 2D Toric Code:

$$t_{mix} < \beta 8 N^2 e^{\beta 6 J}$$

Generalized Energy Barrier – Abelian Quantum Doubles

Defined for **any** operator, not only logical ops.

- Path on the *generalized* Pauli group: $\eta_0, \eta_1, \ldots \eta_f \in \mathcal{P}_N$
- Set of projectors: $\mathcal{A} = \{P_v^l, Q_p^l \,|\, l = 0, 1, \ldots d-1, \; all \; v, \; p\}$
- Reduced set $\mathcal{A}_\eta$:
  - Removing operators violated in the final error configuration, we only consider the other elements of $\mathcal{A}$
- Energy barrier of a Pauli:

$$\bar{\epsilon}(\eta_f) = \min_{\eta_t} \max_t 2J^* \#\{a_k^l \in \mathcal{A}_\eta \,|\, [a_k^l, \eta_t] \neq 0\}$$

- Generalized energy barrier:

$$\bar{\epsilon} = \max_{\eta_f} \bar{\epsilon}(\eta_f)$$