Approximate degradable quantum channels

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Joint work with Volkher Scholz, Andreas Winter and Renato Renner
Quantum capacity of a channel

- Quantum channel (TPCPM) \( \Phi : S(A) \rightarrow S(B) \)
  - By Stinespring \( \Phi : \rho_A \mapsto \text{tr}_E(V_{BE}\rho_A V_{BE}^\dagger) \)
  - Complementary channel \( \Phi^c : \rho_A \mapsto \text{tr}_B(V_{BE}\rho_A V_{BE}^\dagger) \)

How much (quantum) information can we reliably send over such a channel?

Quantum capacity \([\text{Lloyd-Shor-Devetak-97}]\)

\[ Q(\Phi) = \lim_{k \to \infty} \frac{1}{k} Q(1)(\Phi^\otimes k) \]

Coherent information

\[ Q(1)(\Phi) := \max_{\rho \in S(A)} H(\Phi(\rho)) - H(\Phi^c(\rho)) \]

with

\[ H(\rho) := -\text{tr}(\rho \log \rho) \]

"Problems" with the LSD-formula

- Regularization makes it difficult to compute
- \( Q(1)(\Phi) \leq Q(\Phi) \) however \( Q(1)(\Phi) < Q(\Phi) \) possible \([\text{DiVincenzo-Shor-Smolin-98}]\)

Single letter upper bounds are difficult to find

Would like to have UBs that are efficiently computable
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  - Quantum capacity [Lloyd-Shor-Devetak-97]
    
    $Q(\Phi) = \lim_{k \rightarrow \infty} \frac{1}{k} Q^{(1)}(\Phi \otimes k)$
  
  - Coherent information
    
    $Q^{(1)}(\Phi) := \max_{\rho \in S(A)} H(\Phi(\rho)) - H(\Phi^c(\rho))$ with
    
    $H(\rho) := -\operatorname{tr}(\rho \log \rho)$

$A \xrightarrow{\Phi} B$

$E$

$\Phi^c$
Quantum capacity of a channel

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- “Problems” with the LSD-formula
  - Regularization makes it difficult to compute
  - $Q^{(1)}(\Phi) \leq Q(\Phi)$ however $Q^{(1)}(\Phi) < Q(\Phi)$ possible [DiVincenzo-Shor-Smolin-98]
  - Single letter upper bounds are difficult to find
  - Would like to have UBs that are efficiently computable
Degradable channels

- A channel $\Phi : S(A) \rightarrow S(B)$ is *degradable* if $\exists$ a channel $\Theta : S(B) \rightarrow S(E)$ such that $\Phi^c = \Theta \circ \Phi$.
- If $\Phi$ is degradable then $Q^{(1)}(\Phi) = Q(\Phi)$ [Devetak-Shor-05]

Diagram:

```
A ───Φ─── B
 |   \   |
 Φ^c \     / Θ
   \   /     E
     /     /
```

Examples of degradable channels:
- Dephasing channels, e.g. $\rho \mapsto (1-p)\rho + pX\rho X$
- Amplitude damping channels

Not all channels are degradable:
- Depolarizing channel, i.e., $\rho \mapsto (1-p)\rho + p\pi$.
- BB84 channel (independent bit and phase flip error)

Concept of degradable channels is not robust
Degradable channels

- A channel $\Phi : S(A) \to S(B)$ is **degradable** if $\exists$ a channel $\Theta : S(B) \to S(E)$ such that $\Phi^c = \Theta \circ \Phi$.
- If $\Phi$ is degradable then $Q^{(1)}(\Phi) = Q(\Phi)$ [Devetak-Shor-05]
- Examples of degradable channels
  - Dephasing channels, e.g. $\rho \mapsto (1 - p)\rho + pX\rho X$
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- Not all channels are degradable 😞
  - Depolarizing channel, i.e., $\rho \mapsto (1 - p)\rho + p\pi$
  - BB84 channel (independent bit and phase flip error)
- Concept of degradable channels is not robust
Approximate degradable channels

TPCPM $\Xi : S(A) \rightarrow S(B)$

$$\|\Xi\|_\diamond := \max_{\rho \in S(A \otimes A')} \| (\Xi \otimes I_{A'})(\rho) \|_1$$

- A channel $\Phi : S(A) \rightarrow S(B)$ is degradable if $\exists$ a channel $\Theta : S(B) \rightarrow S(E)$ such that $\Phi^c = \Theta \circ \Phi$
- A channel $\Phi : S(A) \rightarrow S(B)$ is $\varepsilon$-degradable if $\exists$ a channel $\Theta : S(B) \rightarrow S(E)$ such that $\| \Phi^c - \Theta \circ \Phi \|_\diamond \leq \varepsilon$
- Every channel is $\varepsilon$-degradable with some $\varepsilon \in [0, 2]$
Approximate degradable channels

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$\|\Xi\| : = \max_{\rho \in S(A \otimes A')} \|(\Xi \otimes I_{A'}) (\rho)\|_1$

- A channel $\Phi : S(A) \rightarrow S(B)$ is **degradable** if $\exists$ a channel $\Theta : S(B) \rightarrow S(E)$ such that $\Phi^c = \Theta \circ \Phi$

- A channel $\Phi : S(A) \rightarrow S(B)$ is $\varepsilon$-**degradable** if $\exists$ a channel $\Theta : S(B) \rightarrow S(E)$ such that $\|\Phi^c - \Theta \circ \Phi\|_\diamond \leq \varepsilon$

- Every channel is $\varepsilon$-degradable with some $\varepsilon \in [0, 2]$

Theorem. Let $\Phi$ be $\varepsilon$-degradable, then

$$Q^{(1)}(\Phi) \leq Q(\Phi) \leq Q^{(1)}(\Phi) + \frac{\varepsilon}{2} \log(|E| - 1) + h\left(\frac{\varepsilon}{2}\right) + \varepsilon \log |E|$$

$$+ \left(1 + \frac{\varepsilon}{2}\right) h\left(\frac{\varepsilon}{2 + \varepsilon}\right)$$

with $|E| := \text{dim } E$ and $h(x) := -x \log x - (1 - x) \log(1 - x)$
A few remarks about how to prove the theorem

- Strengthened Alicki-Fannes inequality [Winter-1507.07775]:
  If $\|\rho_{AB} - \sigma_{AB}\|_1 \leq \varepsilon \leq 2$ then
  $|H(A|B)_\rho - H(A|B)_\sigma| \leq \varepsilon \log |A| + (1 + \frac{\varepsilon}{2})h\left(\frac{\varepsilon}{2+\varepsilon}\right)$
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  \[
  |H(A|B)_\rho - H(A|B)_\sigma| \leq \varepsilon \log |A| + (1 + \frac{\varepsilon}{2})h(\frac{\varepsilon}{2+\varepsilon})
  \]

  strictly better than Alicki-Fannes

  \[
  |H(A|B)_\rho - H(A|B)_\sigma| \leq 4\varepsilon \log |A| + 2h(\varepsilon)
  \]
A few remarks about how to prove the theorem

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  If $\|\rho_{AB} - \sigma_{AB}\|_1 \leq \varepsilon \leq 2$ then
  $|H(A|B)_{\rho} - H(A|B)_{\sigma}| \leq \varepsilon \log |A| + (1 + \frac{\varepsilon}{2})h(\frac{\varepsilon}{2+\varepsilon})$

- Following the Devetak-Shor proof and applying Alicki-Fannes a few times (similar technique as in [Leung-Smith-0810.4931])

- Degradability is used via the data processing inequality, i.e.,
  $I(A : B) \geq I(A : E)$
A few remarks about how to prove the theorem

- Strengthened Alicki-Fannes inequality [Winter-1507.07775]:
  
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  \[
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  \]

An important comment

Unclear if \( \varepsilon \)-degradable channels are close to a degradable channel. Channels that are close to degradable ones are \( \varepsilon \)-degradable.

\[
\|\Phi^c - \Theta \circ \Phi\|_\diamond \leq \varepsilon \quad \|\Phi - \Xi\|_\diamond \leq \xi
\]

\[
\varepsilon = \xi + 2\sqrt{\xi}
\]
Approximate degradable channels (con’t)

- A channel \( \Phi : S(A) \rightarrow S(B) \) is \( \varepsilon \)-degradable if \( \exists \) a channel \( \Theta : S(B) \rightarrow S(E) \) such that \( \| \Phi^c - \Theta \circ \Phi \|_\diamond \leq \varepsilon \)

- How to find the smallest \( \varepsilon \) such that \( \Phi \) is \( \varepsilon \)-degradable?

\[
\varepsilon_\Phi := \left\{ \begin{array}{l}
\min_{\Theta} \| \Phi^c - \Theta \circ \Phi \|_\diamond \\
\text{s.t. } \Theta : S(B) \rightarrow S(E) \text{ is tpcp}
\end{array} \right.
\] (1)
Approximate degradable channels (con’t)

- A channel $\Phi : S(A) \to S(B)$ is $\varepsilon$-degradable if $\exists$ a channel $\Theta : S(B) \to S(E)$ such that $\|\Phi^c - \Theta \circ \Phi\|_\diamond \leq \varepsilon$

- How to find the smallest $\varepsilon$ such that $\Phi$ is $\varepsilon$-degradable?

\[
\varepsilon_{\Phi} := \left\{ \begin{array}{c}
\min_{\Theta} \|\Phi^c - \Theta \circ \Phi\|_\diamond \\
\text{s.t. } \Theta : S(B) \to S(E) \text{ is tpcp}
\end{array} \right. \tag{1}
\]

**Proposition.** (1) can be expressed as a semidefinite program

\[
Q^{(1)}(\Phi) \leq Q(\Phi) \leq Q^{(1)}(\Phi) + \frac{\varepsilon_{\Phi}}{2} \log(|E| - 1) + h\left(\frac{\varepsilon_{\Phi}}{2}\right) + \varepsilon_{\Phi} \log |E| + (1 + \frac{\varepsilon_{\Phi}}{2}) h\left(\frac{\varepsilon_{\Phi}}{2 + \varepsilon_{\Phi}}\right)
\]

is efficiently computable if we know $Q^{(1)}(\Phi)$
Proof sketch of the proposition

- The diamond norm of a difference of two channels can be phrased as an SDP [Watrous-09]

\[
\|\Xi_1 - \Xi_2\|_\diamond = \begin{cases} 
\inf \| \text{tr}_B(Z) \|_\infty \\
s.\ t. \ Z \geq J(\Xi_1 - \Xi_2) \\
Z \geq 0
\end{cases}
\]

- The mapping \( J(\Theta) \mapsto J(\Theta \circ \Phi) \) is linear, thus

\[
\varepsilon_{\Phi} = \begin{cases} 
\inf \| \Phi^c - \Theta \circ \Phi \|_\diamond \\
s.\ t. \ \Theta : S(\mathcal{H}_B) \to S(\mathcal{H}_E) \text{ is tpcp}
\end{cases}
\]

\[
= \begin{cases} 
\inf \| \text{tr}_E(Z) \|_\infty \\
s.\ t. \ Z \geq J(\Phi^c) - J(\Theta \circ \Phi) \\
Z \geq 0 \\
J(\Theta) \geq 0 \\
\text{tr}_E(J(\Theta)) = 1_B
\end{cases}
\]

Choi state of \( \Xi_1 - \Xi_2 \)
UB as a convex optimization problem

Recall

\[ Q^{(1)}(\Phi) \leq Q(\Phi) \leq Q^{(1)}(\Phi) + \frac{\varepsilon}{2} \log(|E| - 1) + h\left(\frac{\varepsilon}{2}\right) \]

\[ + \varepsilon \Phi \log |E| + \left(1 + \frac{\varepsilon}{2}\right) h\left(\frac{\varepsilon}{2 + \varepsilon}\right) \]

is efficiently computable if we know \( Q^{(1)}(\Phi) \).

\( Q^{(1)}(\Phi) := \max_{\rho \in S(A)} H(\Phi(\rho)) - H(\Phi^c(\rho)) \)

- Single letter formula 😊
- Sometimes closed form solution (e.g. depolarizing channel) 😊
- In general difficult — non-convex optimization problem 😞

**Question:** How to efficiently compute \( Q^{(1)}(\Phi) \)?
UB as a convex optimization problem (con’t)

Channel $\Phi$ from $A$ to $B$ and a degrading channel $\Xi$ from $B$ to $\tilde{E} \simeq E$. Choose Stinespring isometric dilations $V : A \leftrightarrow B \otimes E$ and $W : B \leftrightarrow \tilde{E} \otimes F$. Define

$$U_\Xi(\Phi) := \max_{\rho \in S(A)} \left\{ H(F|\tilde{E})_\omega : \omega^{E\tilde{E}F} = (W \otimes 1)V\rho V^\dagger(W \otimes 1)^\dagger \right\}$$

**Proposition.** If $\Phi : S(A) \rightarrow S(B)$ is an $\varepsilon$-degradable channel with a degrading map $\Xi : S(B) \rightarrow S(E)$, then

$$|Q^{(1)}(\Phi) - U_\Xi(\Phi)| \leq \frac{\varepsilon}{2} \log(|E| - 1) + h\left(\frac{\varepsilon}{2}\right)$$

- $U_\Xi(\Phi)$ is given via a convex optimization problem
- $Q(\Phi) \leq U_\Xi(\Phi) + \varepsilon \log |E| + \left(1 + \frac{\varepsilon}{2}\right) h\left(\frac{\varepsilon}{2+\varepsilon}\right)$
First application: depolarizing channel

\[ \mathcal{D}_p : \rho \mapsto (1 - p)\rho + p \mathbb{1}, \text{ for } p \in [0, 1] \]

Universal hashing bound

\[ Q^{(1)}(\mathcal{D}_p) = 1 + (1 - p) \log(1 - p) + p \log \left( \frac{p}{3} \right) \]
Second application: BB84 channel

Independent bit and phase error $\mathcal{B}_{p_X,p_Z} : \rho \mapsto (1 - p_X - p_Z + p_X p_Z) \rho + (p_X - p_X p_Z) X \rho X + (p_Z - p_Z p_X) Z \rho Z + p_X p_Z Y \rho Y$

$$Q^{(1)}(\mathcal{B}_{p_X,p_Z}) = 1 - h(p_X) - h(p_Z)$$
Comments to existing upper bounds

- Convex decomposition into degradable channels [Smith-Smolin-Winter-08]
  - $\Phi = \sum_i p_i \Theta_i$, where $\{\Theta_i\}_i$ are degradable
  - $Q(\sum_i p_i \Theta_i) \leq \sum_i p_i Q(\Theta_i) = \sum_i p_i Q^{(1)}(\Theta_i)$
  - Channel specific 😞
  - Decomposition into degradable channels may not exist!

- The quantum capacity with symmetric side channels [Smith-Smolin-Winter-08]

- No cloning argument [Cerf & Bruss et al.-98]
  - Only good at very high noise levels

- New approach offers
  - universal upper bound (method works for any channel)
  - UB is efficiently computable (via an SDP)
  - UB is good at low noise levels (ideal channel is degradable)
What about high noise levels?

- A channel $\Phi : S(A) \to S(B)$ is \textit{anti-degradable} if $\exists$ a channel $\Theta : S(E) \to S(B)$ such that $\Phi = \Theta \circ \Phi^c$

- Anti-degradable channels cannot have positive quantum capacity (no-cloning)

- A channel $\Phi : S(A) \to S(B)$ is $\varepsilon$-\textit{anti-degradable} if $\exists$ a channel $\Theta : S(E) \to S(B)$ such that $\|\Phi - \Theta \circ \Phi^c\|_\diamond \leq \varepsilon$
What about high noise levels?

- A channel $\Phi : S(A) \rightarrow S(B)$ is **anti-degradable** if $\exists$ a channel $\Theta : S(E) \rightarrow S(B)$ such that $\Phi = \Theta \circ \Phi^c$

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- A channel $\Phi : S(A) \rightarrow S(B)$ is $\varepsilon$-anti-degradable if $\exists$ a channel $\Theta : S(E) \rightarrow S(B)$ such that $\|\Phi - \Theta \circ \Phi^c\|_{\diamond} \leq \varepsilon$

**Proposition.** If $\Phi$ is $\varepsilon$-anti-degradable, then

$$Q(\Phi) \leq \frac{\varepsilon}{2} \log(|B| - 1) + \varepsilon \log |B| + h\left(\frac{\varepsilon}{2}\right) + \left(1 + \frac{\varepsilon}{2}\right) h\left(\frac{\varepsilon}{2 + \varepsilon}\right)$$

- Proof works similar as for the $\varepsilon$-degradable case
Upper bound via convex decompositions of channels

Symmetric side-channel assisted quantum capacity [Smith-Smolin-Winter-08]

\[ Q_{ss}(\Phi) := \sup_{\Theta} Q(\Phi \otimes \Theta) = \sup_{\Theta} Q^{(1)}(\Phi \otimes \Theta) \]

- Single letter formula
- Clearly \( Q(\Phi) \leq Q_{ss}(\Phi) \)
- \( \Phi \mapsto Q_{ss}(\Phi) \) is convex \( \Rightarrow \) we can combine different UBs

If \( \Phi \) is an \( \varepsilon \)-degradable channel, with a degrading map \( \Xi \), then

\[ Q_{ss}(\Phi) \leq U_{\Xi}(\Phi) + \varepsilon \log |E| + \left(1 + \frac{\varepsilon}{2}\right) h\left(\frac{\varepsilon}{2 + \varepsilon}\right) \]
Private classical capacity of a quantum channel

Private classical capacity of $\Phi$

$$P(\Phi) = \lim_{k \to \infty} \frac{1}{k} P^{(1)}(\Phi \otimes^k),$$

with

$$P^{(1)}(\Phi) := \max_{\{\rho_i, p_i\}} H \left( \sum_i p_i \Phi(\rho_i) \right) - \sum_i p_i H(\Phi(\rho_i))$$

$$- H \left( \sum_i p_i \Phi^c(\rho_i) \right) + \sum_i p_i H(\Phi^c(\rho_i))$$

- $P^{(1)}(\Phi) \leq P(\Phi)$ and $P^{(1)}(\Phi) < P(\Phi)$ possible [Smith-Renes-Smolin-08]
- For degradable channels $P^{(1)}(\Phi) = P(\Phi) = Q^{(1)}(\Phi) = Q(\Phi)$ [Smith-08]
Private classical capacity of a quantum channel (con’t)

For degradable channels $P^{(1)}(\Phi) = P(\Phi) = Q^{(1)}(\Phi) = Q(\Phi)$

**Theorem.** Let $\Phi$ be $\varepsilon$-degradable, then

$$P^{(1)}(\Phi) \leq P(\Phi) \leq P^{(1)}(\Phi) + \frac{\varepsilon}{2} \log(|E| - 1) + h\left(\frac{\varepsilon}{2}\right) + 3\varepsilon \log |E|$$

$$+ 3\left(1 + \frac{\varepsilon}{2}\right) h\left(\frac{\varepsilon}{2 + \varepsilon}\right)$$

$$Q^{(1)}(\Phi) \leq P^{(1)}(\Phi) \leq Q^{(1)}(\Phi) + \frac{\varepsilon}{2} \log(|E| - 1) + h\left(\frac{\varepsilon}{2}\right) + \varepsilon \log |E|$$

$$+ \left(1 + \frac{\varepsilon}{2}\right) h\left(\frac{\varepsilon}{2 + \varepsilon}\right)$$

Efficient computable upper bounds for $P(\Phi)$
Summary & outlook

- Robust definition of degradable channels
- Approximately preserve properties of degradable channels
  - additivity of coherent information
- Useful for upper bounds to the quantum capacity
  - computable via SDP
- Same for private classical capacity of a quantum channel
- Useful to prove upper bounds for the quantum capacity of bosonic channels?