Approximate degradable quantum channels

arXiv: 1412.0980

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QIP 2016, Banff

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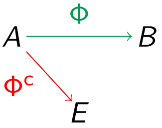
Joint work with Volkher Scholz, Andreas Winter and Renato Renner

Quantum capacity of a channel



- Quantum channel (TPCPM) $\Phi : \mathcal{S}(A) \to \mathcal{S}(B)$
  - By Stinespring $\Phi : \rho_A \mapsto \mathrm{tr}_E(V_{BE}\rho_A V_{BE}^\dagger)$
  - Complementary channel $\Phi^c : \rho_A \mapsto \mathrm{tr}_B(V_{BE}\rho_A V_{BE}^\dagger)$

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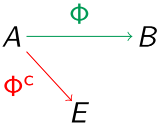

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- How much (quantum) information can we reliably send over such a channel?
  - Quantum capacity [Lloyd-Shor-Devetak-97]
    $Q(\Phi) = \lim_{k\to\infty} \frac{1}{k} Q^{(1)}(\Phi^{\otimes k})$
  - Coherent information
    $Q^{(1)}(\Phi) := \max_{\rho\in\mathcal{S}(A)} H\big(\Phi(\rho)\big) - H\big(\Phi^c(\rho)\big)$ with
    $H(\rho) := -\mathrm{tr}(\rho \log \rho)$

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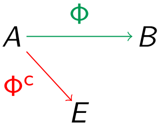

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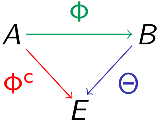
    $H(\rho) := -\mathrm{tr}(\rho \log \rho)$
- "Problems" with the LSD-formula
  - Regularization makes it difficult to compute
  - $Q^{(1)}(\Phi) \leq Q(\Phi)$ however $Q^{(1)}(\Phi) < Q(\Phi)$ possible [DiVincenzo-Shor-Smolin-98]
  - Single letter upper bounds are difficult to find
  - Would like to have UBs that are efficiently computable

Degradable channels



- A channel $\Phi : \mathcal{S}(A) \to \mathcal{S}(B)$ is *degradable* if $\exists$ a channel $\Theta : \mathcal{S}(B) \to \mathcal{S}(E)$ such that $\Phi^c = \Theta \circ \Phi$.
- If Φ is degradable then $Q^{(1)}(\Phi) = Q(\Phi)$ [Devetak-Shor-05]

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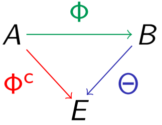

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- If Φ is degradable then $Q^{(1)}(\Phi) = Q(\Phi)$ [Devetak-Shor-05]
- Examples of degradable channels
  - Dephasing channels, e.g. $\rho \mapsto (1-p)\rho + pX\rho X$
  - Amplitude damping channels
- Not all channels are degradable ☹
  - Depolarizing channel, i.e., $\rho \mapsto (1-p)\rho + p\,\pi$.
  - BB84 channel (independent bit and phase flip error)
- Concept of degradable channels is not robust

Approximate degradable channels

TPCPM $\Xi : \mathcal{S}(A) \to \mathcal{S}(B)$

$\|\Xi\|_\diamond := \max_{\rho \in \mathcal{S}(A \otimes A')} \|(\Xi \otimes \mathcal{I}_{A'})(\rho)\|_1$

- A channel $\Phi : \mathcal{S}(A) \to \mathcal{S}(B)$ is *degradable* if $\exists$ a channel $\Theta : \mathcal{S}(B) \to \mathcal{S}(E)$ such that $\Phi^c = \Theta \circ \Phi$
- A channel $\Phi : \mathcal{S}(A) \to \mathcal{S}(B)$ is ε-*degradable* if $\exists$ a channel $\Theta : \mathcal{S}(B) \to \mathcal{S}(E)$ such that $\|\Phi^c - \Theta \circ \Phi\|_\diamond \leq \varepsilon$
- Every channel is ε-degradable with some $\varepsilon \in [0, 2]$

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Theorem. Let Φ be ε-degradable, then

$$Q^{(1)}(\Phi) \leq Q(\Phi) \leq Q^{(1)}(\Phi) + \frac{\varepsilon}{2} \log(|E| - 1) + h\left(\frac{\varepsilon}{2}\right) + \varepsilon \log |E| + \left(1 + \frac{\varepsilon}{2}\right) h\left(\frac{\varepsilon}{2 + \varepsilon}\right)$$

with $|E| := \dim E$ and $h(x) := -x \log x - (1 - x) \log(1 - x)$

A few remarks about how to prove the theorem

- Strengthened Alicki-Fannes inequality [Winter-1507.07775]:
  If $\|\rho_{AB} - \sigma_{AB}\|_1 \leq \varepsilon \leq 2$ then
  $|H(A|B)_\rho - H(A|B)_\sigma| \leq \varepsilon \log|A| + (1 + \frac{\varepsilon}{2}) h(\frac{\varepsilon}{2+\varepsilon})$

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  strictly better than Alicki-Fannes

  $$|H(A|B)_\rho - H(A|B)_\sigma| \leq 4\varepsilon \log |A| + 2h(\varepsilon)$$

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- Following the Devetak-Shor proof and applying Alicki-Fannes a few times (similar technique as in [Leung-Smith-0810.4931])
- Degradability is used via the data processing inequality, i.e., $I(A:B) \geq I(A:E)$

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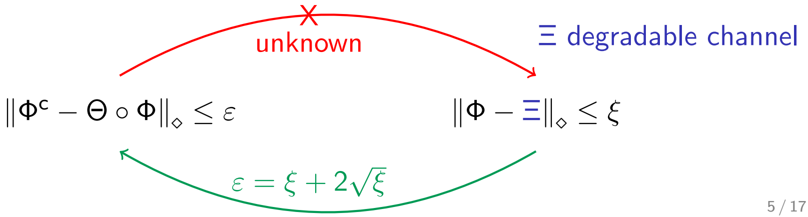
An important comment

Unclear if ε-degradable channels are close to a degradable channel.
Channels that are close to degradable ones are ε-degradable.



Approximate degradable channels (con't)

- A channel $\Phi : \mathcal{S}(A) \to \mathcal{S}(B)$ is ε-*degradable* if $\exists$ a channel $\Theta : \mathcal{S}(B) \to \mathcal{S}(E)$ such that $\|\Phi^{\mathsf{c}} - \Theta \circ \Phi\|_\diamond \leq \varepsilon$
- How to find the smallest ε such that Φ is ε-degradable?

$$\varepsilon_\Phi := \begin{cases} \min\limits_{\Theta} & \|\Phi^{\mathsf{c}} - \Theta \circ \Phi\|_\diamond \\ \text{s.t.} & \Theta : \mathcal{S}(B) \to \mathcal{S}(E) \text{ is tpcp} \end{cases} \tag{1}$$

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Proposition. (1) can be expressed as a semidefinite program

$$Q^{(1)}(\Phi) \leq Q(\Phi) \leq Q^{(1)}(\Phi) + \frac{\varepsilon_\Phi}{2} \log(|E| - 1) + h(\frac{\varepsilon_\Phi}{2}) + \varepsilon_\Phi \log |E| + (1 + \frac{\varepsilon_\Phi}{2}) h(\frac{\varepsilon_\Phi}{2 + \varepsilon_\Phi})$$

is efficiently computable if we know $Q^{(1)}(\Phi)$

Proof sketch of the proposition

- The diamond norm of a difference of two channels can be phrased as an SDP [Watrous-09]

$$\|\Xi_1 - \Xi_2\|_\diamond = \begin{cases} \inf_Z & \|\mathrm{tr}_B(Z)\|_\infty \\ \text{s.t.} & Z \geq J(\Xi_1 - \Xi_2) \\ & Z \geq 0 \end{cases}$$

Choi state of $\Xi_1 - \Xi_2$

- The mapping $J(\Theta) \mapsto J(\Theta \circ \Phi)$ is linear, thus

$$\varepsilon_\Phi = \begin{cases} \inf_\Theta & \|\Phi^c - \Theta \circ \Phi\|_\diamond \\ \text{s.t.} & \Theta : \mathcal{S}(\mathcal{H}_B) \to \mathcal{S}(\mathcal{H}_E) \text{ is tpcp} \end{cases}$$

$$= \begin{cases} \inf_{Z, J(\Theta)} & \|\mathrm{tr}_E(Z)\|_\infty \\ \text{s.t.} & Z \geq J(\Phi^c) - J(\Theta \circ \Phi) \\ & Z \geq 0 \\ & J(\Theta) \geq 0 \\ & \mathrm{tr}_E(J(\Theta)) = \mathbb{1}_B \end{cases}$$

UB as a convex optimization problem

- Recall

$$Q^{(1)}(\Phi) \le Q(\Phi) \le Q^{(1)}(\Phi) + \frac{\varepsilon_\Phi}{2}\log(|E|-1) + h\Big(\frac{\varepsilon_\Phi}{2}\Big) + \varepsilon_\Phi \log|E| + \Big(1+\frac{\varepsilon_\Phi}{2}\Big) h\Big(\frac{\varepsilon_\Phi}{2+\varepsilon_\Phi}\Big)$$

  is efficiently computable if we know $Q^{(1)}(\Phi)$.

- $Q^{(1)}(\Phi) := \max_{\rho\in\mathcal{S}(A)} H\big(\Phi(\rho)\big) - H\big(\Phi^c(\rho)\big)$
  - Single letter formula ☺
  - Sometimes closed form solution (e.g. depolarizing channel) ☺
  - In general difficult — non-convex optimization problem ☹

- **Question:** How to efficiently compute $Q^{(1)}(\Phi)$?

UB as a convex optimization problem (con't)

Channel Φ from A to B and a degrading channel Ξ from B to $\widetilde{E} \simeq E$. Choose Stinespring isometric dilations $V : A \hookrightarrow B \otimes E$ and $W : B \hookrightarrow \widetilde{E} \otimes F$. Define

$$U_{\Xi}(\Phi) := \max_{\rho \in \mathcal{S}(A)} \{ H(F|\widetilde{E})_{\omega} \; : \; \omega^{E\widetilde{E}F} = (W \otimes \mathbb{1}) V \rho V^{\dagger} (W \otimes \mathbb{1})^{\dagger} \}$$

Proposition. If $\Phi : \mathcal{S}(A) \to \mathcal{S}(B)$ is an ε-degradable channel with a degrading map $\Xi : \mathcal{S}(B) \to \mathcal{S}(E)$, then

$$\left| Q^{(1)}(\Phi) - U_{\Xi}(\Phi) \right| \leq \frac{\varepsilon}{2} \log(|E| - 1) + h\left(\frac{\varepsilon}{2}\right)$$

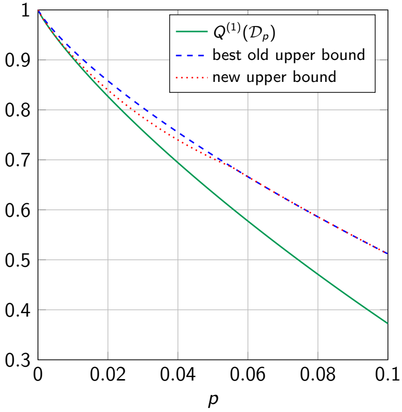
- $U_{\Xi}(\Phi)$ is given via a convex optimization problem
- $Q(\Phi) \leq U_{\Xi}(\Phi) + \varepsilon \log |E| + \left(1 + \frac{\varepsilon}{2}\right) h\left(\frac{\varepsilon}{2+\varepsilon}\right)$

First application: depolarizing channel

$\mathcal{D}_p : \rho \mapsto (1-p)\rho + p\,\mathbb{1}$, for $p \in [0,1]$

Universal hashing bound

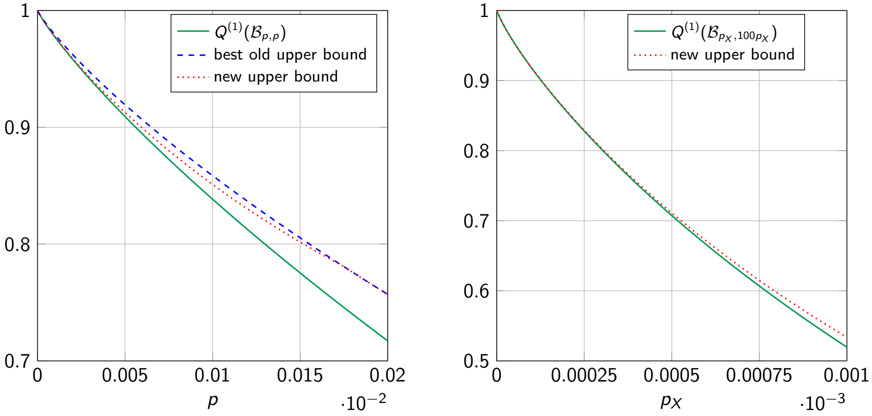
$$Q^{(1)}(\mathcal{D}_p) = 1 + (1-p)\log(1-p) + p\log\left(\frac{p}{3}\right)$$



Second application: BB84 channel

Independent bit and phase error $\mathcal{B}_{p_X,p_Z} : \rho \mapsto (1 - p_X - p_Z + p_X p_Z)\rho + (p_X - p_X p_Z) X\rho X + (p_Z - p_Z p_X) Z\rho Z + p_X p_Z Y\rho Y$

$$Q^{(1)}(\mathcal{B}_{p_X,p_Z}) = 1 - h(p_X) - h(p_Z)$$



Comments to existing upper bounds

- Convex decomposition into degradable channels [Smith-Smolin-Winter-08]
  - $\Phi = \sum_i p_i \Theta_i$, where $\{\Theta_i\}_i$ are degradable
  - $Q(\sum_i p_i \Theta_i) \leq \sum_i p_i Q(\Theta_i) = \sum_i p_i Q^{(1)}(\Theta_i)$
  - Channel specific ☹
  - Decomposition into degradable channels may not exist!
- The quantum capacity with symmetric side channels [Smith-Smolin-Winter-08]
- No cloning argument [Cerf & Bruss *et al.*-98]
  - Only good at very high noise levels
- New approach offers
  - universal upper bound (method works for *any* channel)
  - UB is efficiently computable (via an SDP)
  - UB is good at low noise levels (ideal channel is degradable)

What about high noise levels?

- A channel $\Phi : \mathcal{S}(A) \to \mathcal{S}(B)$ is *anti-degradable* if $\exists$ a channel $\Theta : \mathcal{S}(E) \to \mathcal{S}(B)$ such that $\Phi = \Theta \circ \Phi^c$
- Anti-degradable channels cannot have positive quantum capacity (no-cloning)
- A channel $\Phi : \mathcal{S}(A) \to \mathcal{S}(B)$ is *ε-anti-degradable* if $\exists$ a channel $\Theta : \mathcal{S}(E) \to \mathcal{S}(B)$ such that $\|\Phi - \Theta \circ \Phi^c\|_\diamond \leq \varepsilon$

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Proposition. If Φ is ε-anti-degradable, then

$$Q(\Phi) \leq \frac{\varepsilon}{2} \log(|B| - 1) + \varepsilon \log |B| + h\Big(\frac{\varepsilon}{2}\Big) + \Big(1 + \frac{\varepsilon}{2}\Big) h\Big(\frac{\varepsilon}{2+\varepsilon}\Big)$$

- Proof works similar as for the ε-degradable case

Upper bound via convex decompositions of channels

Symmetric side-channel assisted quantum capacity [Smith-Smolin-Winter-08]

$$Q_{ss}(\Phi) := \sup_{\Theta} Q(\Phi \otimes \Theta) = \sup_{\Theta} Q^{(1)}(\Phi \otimes \Theta)$$

- Single letter formula
- Clearly $Q(\Phi) \leq Q_{ss}(\Phi)$
- $\Phi \mapsto Q_{ss}(\Phi)$ is convex $\Rightarrow$ we can combine different UBs

If Φ is an ε-degradable channel, with a degrading map Ξ, then

$$Q_{ss}(\Phi) \leq U_{\Xi}(\Phi) + \varepsilon \log |E| + \left(1 + \frac{\varepsilon}{2}\right) h\left(\frac{\varepsilon}{2+\varepsilon}\right)$$

Private classical capacity of a quantum channel

Private classical capacity of Φ

$$P(\Phi) = \lim_{k\to\infty} \frac{1}{k} P^{(1)}(\Phi^{\otimes k}),$$

with

$$\begin{aligned} P^{(1)}(\Phi) := \max_{\{\rho_i, p_i\}} H\left(\sum_i p_i \Phi(\rho_i)\right) - \sum_i p_i H\big(\Phi(\rho_i)\big) \\ - H\left(\sum_i p_i \Phi^c(\rho_i)\right) + \sum_i p_i H\big(\Phi^c(\rho_i)\big) \end{aligned}$$

- $P^{(1)}(\Phi) \leq P(\Phi)$ and $P^{(1)}(\Phi) < P(\Phi)$ possible [Smith-Renes-Smolin-08]
- For degradable channels $P^{(1)}(\Phi) = P(\Phi) = Q^{(1)}(\Phi) = Q(\Phi)$ [Smith-08]

Private classical capacity of a quantum channel (con't)

For degradable channels $P^{(1)}(\Phi) = P(\Phi) = Q^{(1)}(\Phi) = Q(\Phi)$

Theorem. Let Φ be ε-degradable, then

$$P^{(1)}(\Phi) \leq P(\Phi) \leq P^{(1)}(\Phi) + \frac{\varepsilon}{2}\log(|E|-1) + h\Big(\frac{\varepsilon}{2}\Big) + 3\varepsilon\log|E| + 3\Big(1+\frac{\varepsilon}{2}\Big)h\Big(\frac{\varepsilon}{2+\varepsilon}\Big)$$

$$Q^{(1)}(\Phi) \leq P^{(1)}(\Phi) \leq Q^{(1)}(\Phi) + \frac{\varepsilon}{2}\log(|E|-1) + h(\frac{\varepsilon}{2}) + \varepsilon\log|E| + (1+\frac{\varepsilon}{2})h(\frac{\varepsilon}{2+\varepsilon})$$

Efficient computable upper bounds for $P(\Phi)$

Summary & outlook

arXiv:1412.0980

- Robust definition of degradable channels
- Approximately preserve properties of degradable channels
  - additivity of coherent information
- Useful for upper bounds to the quantum capacity
  - computable via SDP
- Same for private classical capacity of a quantum channel
- Useful to prove upper bounds for the quantum capacity of bosonic channels?