

Quantum Latin Squares and Unitary Error Bases

Ben Musto Jamie Vicary

Department of Computer Science
University of Oxford

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Latin squares

Definition

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By sending $k \in \{0, \dots, n-1\}$ to $|k\rangle \in \mathbb{C}^n$, we can turn a Latin square into an array of Hilbert space elements:

| | | | |
|---|---|---|---|
| 3 | 1 | 0 | 2 |
| 1 | 0 | 2 | 3 |
| 2 | 3 | 1 | 0 |
| 0 | 2 | 3 | 1 |

\rightsquigarrow

| | | | |
|-------------|-------------|-------------|-------------|
| $ 3\rangle$ | $ 1\rangle$ | $ 0\rangle$ | $ 2\rangle$ |
| $ 1\rangle$ | $ 0\rangle$ | $ 2\rangle$ | $ 3\rangle$ |
| $ 2\rangle$ | $ 3\rangle$ | $ 1\rangle$ | $ 0\rangle$ |
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For example:

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Quantum Latin squares

as linear maps in Hilbert space

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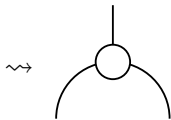
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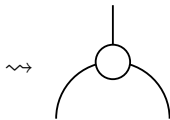
We can encode this data as a linear map.

Quantum Latin squares

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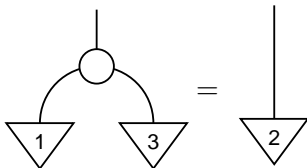
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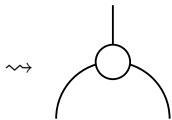


Quantum Latin squares

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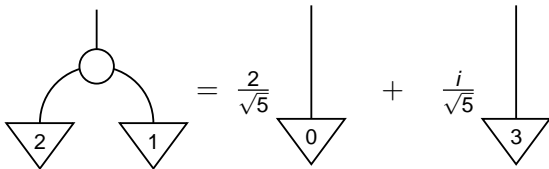
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Bases

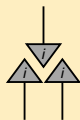

as tensors

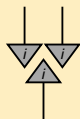

Definition (Basis tensors)

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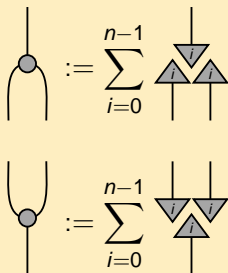
A grey dot will represent the computational basis.

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Basis tensors are uniquely characterized by the property that connected composites with the same boundary are equal.

Mutually unbiased bases (MUBs)

as characterised through spider tensors

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Definition (Mutually unbiased bases)

Given two orthonormal bases $|a_i\rangle$ and $|b_j\rangle$ for the n dimensional Hilbert space \mathcal{H} , they are mutually unbiased when:

$$|\langle a_i | b_j \rangle|^2 = \frac{1}{n}$$

$$\forall i, j, 0 \leq i, j < n - 1.$$

Mutually unbiased bases (MUBs)


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MUBs have a nice characterisation in terms of tensors.

Theorem

For orthonormal bases $\{ |b_i\rangle \}$ and $\{ |c_j\rangle \}$, the following are equivalent:

- the bases are mutually unbiased;

- the composite  is unitary (up to a constant).

Graphical characterisation

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Quantum Latin squares generalise Latin squares.

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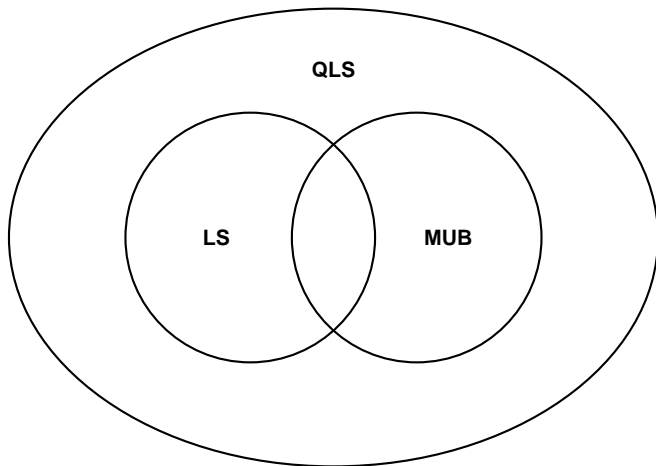
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Hence Quantum Latin squares generalise both Latin squares and mutually unbiased bases.

Venn Diagram 1



Definition

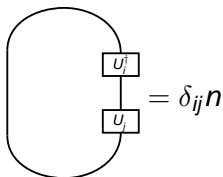
For a Hilbert space \mathcal{H} of dimension n , a *unitary error basis* is an n^2 family of unitary operators which form an orthogonal basis.

$$\text{Tr}(U_i^\dagger \circ U_j) = \delta_{ij}n$$

Unitary Error Bases

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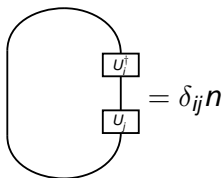
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The diagram shows a closed loop on the left side. On the right side, two rectangular boxes are stacked vertically, labeled u_j (top) and u_i (bottom). The lines of the loop connect the top of the u_j box to the top of the u_i box, and the bottom of the u_j box to the bottom of the u_i box. To the right of this loop is the equation $= \delta_{ij}n$.

Famous example: the Pauli matrices together with the identity.

Constructions of UEBs

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Input: Group (order n^2) with nice representation
- *QLS construction* [MV, 2015]
Input: Quantum Latin square and n Hadamards (order n)

An example

Take our example quantum Latin square:

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and the following family of Hadamard matrices:

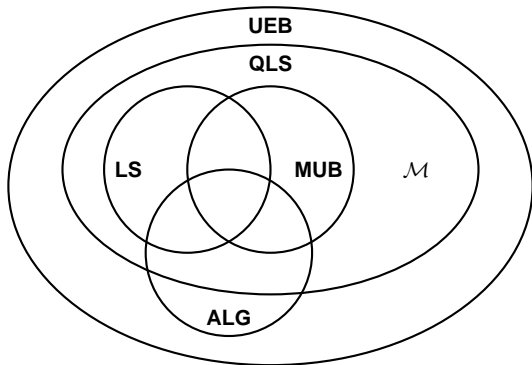
$$H_0 = H_1 = H_2 = H_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

We get the following UEB, \mathcal{M} :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & \frac{i}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{i}{\sqrt{5}} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{i}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \quad \begin{pmatrix} 0 & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{2i}{\sqrt{5}} & -\frac{i}{\sqrt{5}} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \frac{2i}{\sqrt{5}} & -\frac{i}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
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 \end{pmatrix}$$

Venn Diagram 2

Why our construction is great!



Theorem


The example UEB \mathcal{M} is not in LS, MUB or ALG.

UEBs built from tensors

QLS UEB


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 - ONB

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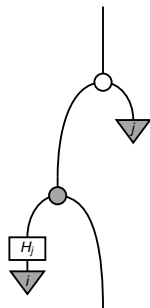
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


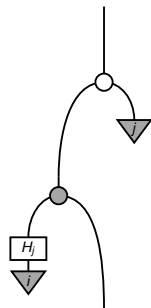
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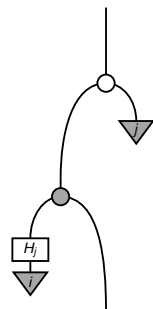
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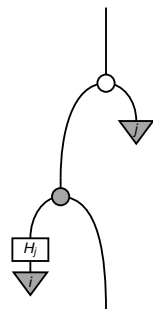
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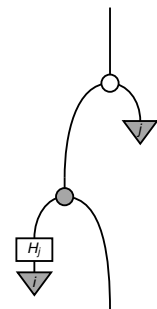
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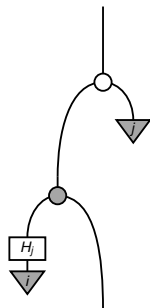
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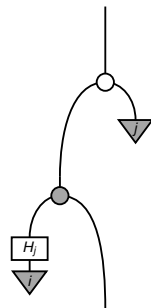


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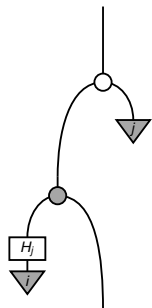
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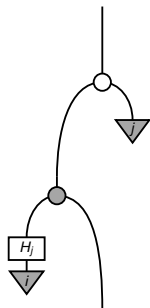


LS UEB

 - ONB

H_j - n Hadamards


 - LS




MUB UEB

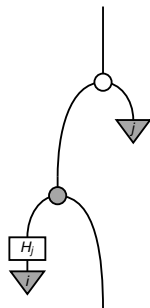
UEBs built from tensors

QLS UEB


 - ONB

H_j - n Hadamards


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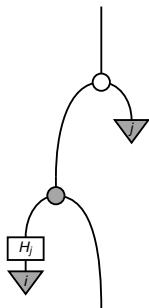


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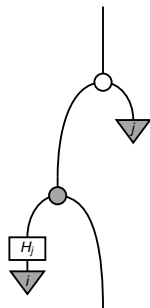
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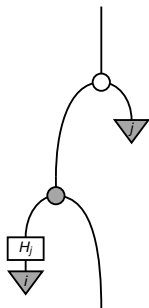


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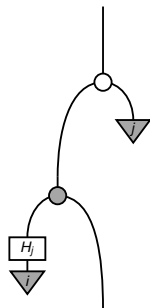
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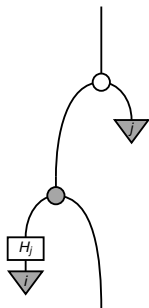


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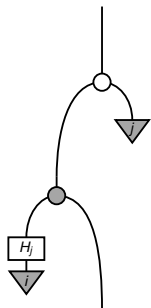
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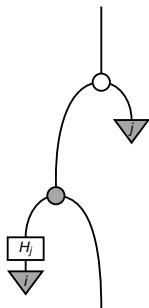


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H_j - n Hadamards


 - LS

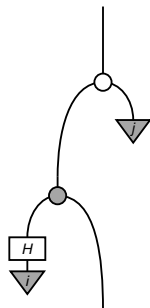


MUB UEB

 - ONB

H - Hadamard

 - ONB



Main Theorem

Theorem

Quantum Latin square bases are unitary error bases.

Main Theorem

Unitarity

First we prove that the U_{ij} are unitary.

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Proof.

Need to show that for all i, j : $U_{ij} \circ U_{ij}^\dagger = \mathbb{I}_n$.

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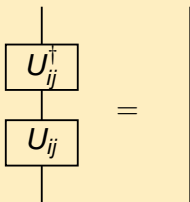
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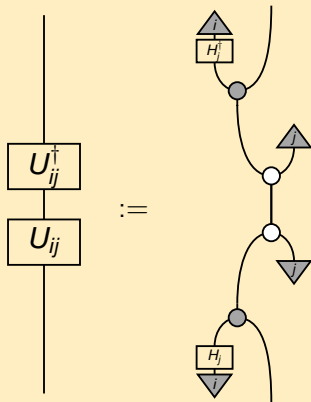


Main Theorem

Unitarity

Proof.

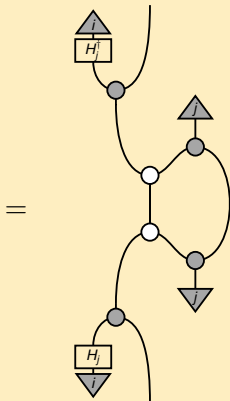
By definition:



Main Theorem

Unitarity

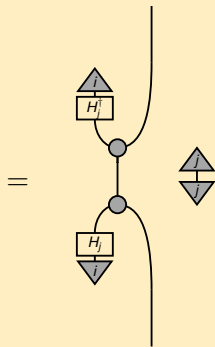
Proof.



Main Theorem

Unitarity

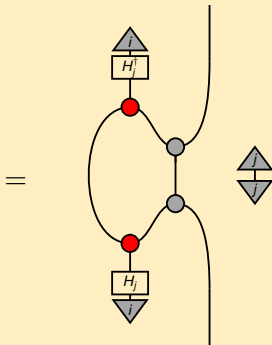
Proof.



Main Theorem

Unitarity

Proof.



Main Theorem

Unitarity

Proof.

$$= \begin{array}{c} \triangleup_i \\ | \\ \square_{H_j^i} \\ | \\ \square_{H_j} \\ | \\ \triangle\downarrow_j \end{array} \quad \Bigg| \quad \begin{array}{c} \triangleup_j \\ | \\ \triangle\downarrow_j \end{array}$$

Main Theorem

Unitarity

Proof.

$$= \begin{array}{c} \triangleup_i \\ \downarrow \\ \triangleleft_i \end{array} \quad \Bigg| \quad \begin{array}{c} \triangleup_j \\ \downarrow \\ \triangleleft_j \end{array}$$

Main Theorem

Unitarity

Proof.

=

Main Theorem

Unitarity

Proof.

The diagram shows a vertical line representing a quantum wire. On the left, an arrow points to a box labeled U_{ij}^\dagger above another box labeled U_{ij} . To the right of these boxes is an equals sign, followed by a single vertical line representing the identity operation.

$$\Rightarrow \begin{array}{c} \boxed{U_{ij}^\dagger} \\ | \\ \boxed{U_{ij}} \end{array} = |$$

Main Theorem

Orthogonality

Now we prove that the U_{ij} are an orthogonal basis.

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Need to show that for all i, j : $\text{Tr}(U_{pq}^\dagger \circ U_{ij}) = \delta_{ip}\delta_{jq}n$.

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The diagram shows a large vertical oval on the left. Two vertical lines extend from the top and bottom of this oval to the right. These lines pass through two rectangular boxes stacked vertically. The top box is labeled U_{pq}^\dagger and the bottom box is labeled U_{ij} . To the right of the bottom box, the expression $= \delta_{ip}\delta_{jq}n$ is written.

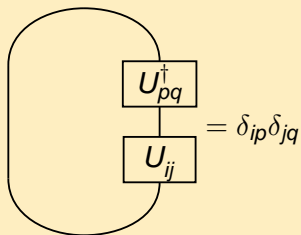
Main Theorem

Orthogonality

Now we prove that the U_{ij} are an orthogonal basis.

Proof.

Or disregarding normalization:

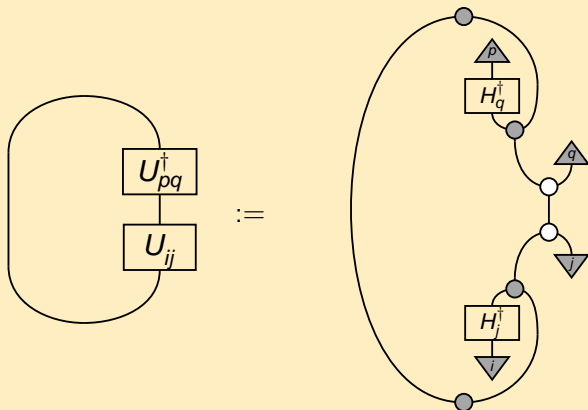
A diagram illustrating the trace of the product of two matrices. A large, rounded rectangle encloses two smaller rectangular boxes stacked vertically. The top box contains the expression U_{pq}^\dagger and the bottom box contains U_{ij} . A vertical line connects the two boxes, and a curved line on the left side of the large rectangle connects the top and bottom of the boxes, forming a closed loop. To the right of the boxes, an equals sign is followed by the expression $\delta_{ip}\delta_{jq}$.
$$U_{pq}^\dagger U_{ij} = \delta_{ip}\delta_{jq}$$

Main Theorem

Orthogonality

Proof.

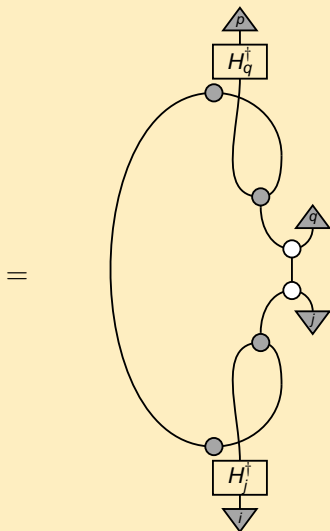
By definition:



Main Theorem

Orthogonality

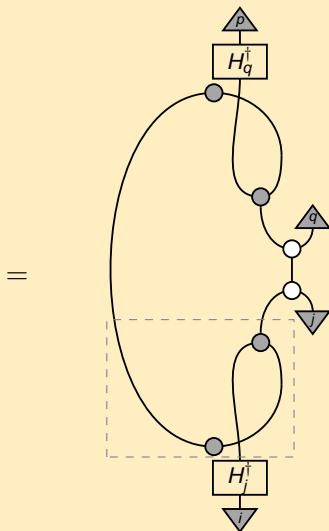
Proof.



Main Theorem

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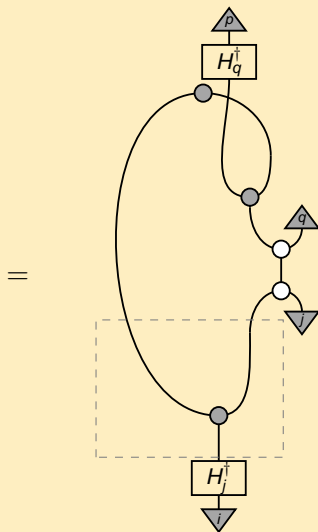


Main Theorem

Orthogonality

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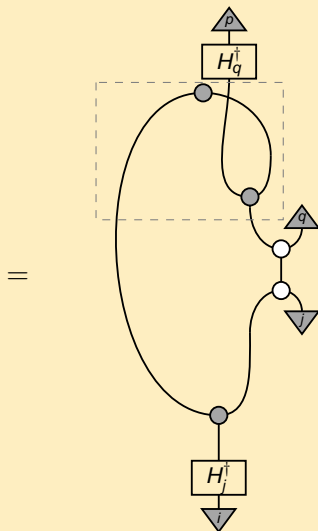
By spider merge



Main Theorem

Orthogonality

Proof.

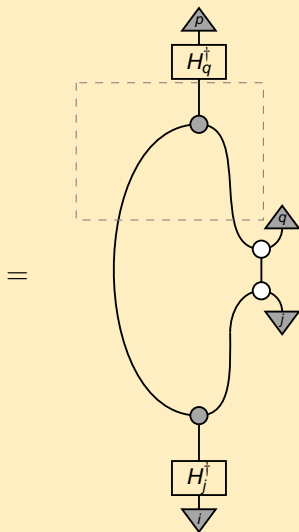


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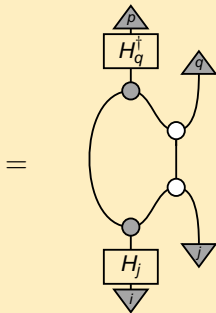
By spider merge



Main Theorem

Orthogonality

Proof.



Main Theorem

Orthogonality

Proof.

$$= \begin{array}{c} \triangle p \\ \downarrow \\ \square H_q^\dagger \\ \downarrow \\ \square H_j \\ \downarrow \\ \triangle i \end{array} \quad \begin{array}{c} \triangle q \\ \downarrow \\ \triangle j \end{array}$$

Main Theorem

Orthogonality

Proof.

$$= \begin{array}{c} \triangleup_p \\ \boxed{H_q^\dagger} \\ \boxed{H_j} \\ \triangleleft_i \end{array} \delta_{jq}$$

Main Theorem

Orthogonality

Proof.

$$= \begin{array}{c} \triangleup_p \\ \boxed{H_j^\dagger} \\ \boxed{H_j} \\ \triangleleft_i \end{array} \delta_{jq}$$

Main Theorem

Orthogonality

Proof.

$$= \begin{array}{c} \triangle \\ \rho \\ \hline i \\ \triangle \end{array} \delta_{jq}$$

Main Theorem

Orthogonality

Proof.

$$= \delta_{ip}\delta_{jq}$$

Main Theorem

Orthogonality

Proof.

$$\Rightarrow \left(\begin{array}{c} U_{pq}^\dagger \\ U_{ij} \end{array} \right) = \delta_{ip} \delta_{jq}$$

□