Quantum Latin Squares

and Unitary Error Bases

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Latin squares

Definition

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By sending $k \in \{0, \ldots, n-1\}$ to $|k\rangle \in \mathbb{C}^n$, we can turn a Latin square into an array of Hilbert space elements:

3	1	0	2
1	0	2	3
2	3	1	0
0	2	3	1

$\rightsquigarrow$

$\|3\rangle$	$\|1\rangle$	$\|0\rangle$	$\|2\rangle$
$\|1\rangle$	$\|0\rangle$	$\|2\rangle$	$\|3\rangle$
$\|2\rangle$	$\|3\rangle$	$\|1\rangle$	$\|0\rangle$
$\|0\rangle$	$\|2\rangle$	$\|3\rangle$	$\|1\rangle$

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For example:

$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$
$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$	$\frac{1}{\sqrt{5}}(i\|0\rangle + 2\|3\rangle)$	$\frac{1}{\sqrt{5}}(2\|0\rangle + i\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$
$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$	$\frac{1}{\sqrt{5}}(2\|0\rangle + i\|3\rangle)$	$\frac{1}{\sqrt{5}}(i\|0\rangle + 2\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$
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Here is our example quantum Latin square:

$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$
$\frac{1}{\sqrt{2}}(\lvert 1\rangle - \lvert 2\rangle)$	$\frac{1}{\sqrt{5}}(i\lvert 0\rangle + 2\lvert 3\rangle)$	$\frac{1}{\sqrt{5}}(2\lvert 0\rangle + i\lvert 3\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 1\rangle + \lvert 2\rangle)$
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$\|2\rangle$	$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$	$\frac{1}{\sqrt{5}}(2\|0\rangle + i\|3\rangle)$	$\frac{1}{\sqrt{5}}(i\|0\rangle + 2\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$
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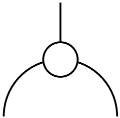

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Quantum Latin squares
as linear maps in Hilbert space

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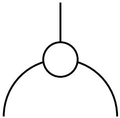

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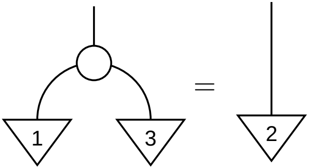

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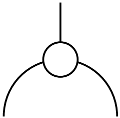

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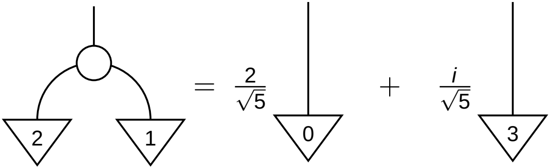

Definition (Basis tensors)

Given a basis for a Hilbert space, $|i\rangle, 0 \leq i < n-1$, we canonically define the following two linear maps:

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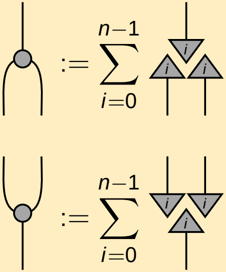

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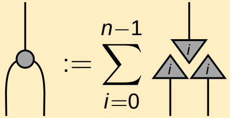

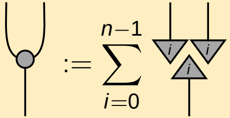

A grey dot will represent the computational basis.

Bases
as tensors

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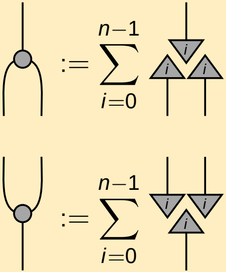

Basis tensors are uniquely characterized by the property that connected composites with the same boundary are equal.

Mutually unbiased bases (MUBs)

as characterised through spider tensors

MUBs have a nice characterisation in terms of tensors.

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Definition (Mutually unbiased bases)

Given two orthonormal bases $|a_i\rangle$ and $|b_j\rangle$ for the n dimensional Hilbert space $\mathcal{H}$, they are mutually unbiased when:

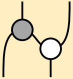
$$|\langle a_i|b_j\rangle|^2 = \frac{1}{n}$$

$\forall i,j\,,\ 0 \leq i,j < n-1$.

MUBs have a nice characterisation in terms of tensors.

Theorem

For orthonormal bases  *and* *, the following are equivalent:*

- *the bases are mutually unbiased;*
- *the composite*  *is unitary (up to a constant).*

Graphical characterisation

Let  be a linear map and  be a basis tensor;

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Quantum Latin squares

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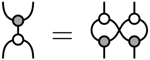

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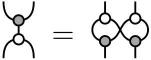

Quantum Latin squares generalise Latin squares.

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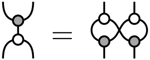

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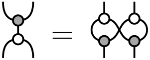

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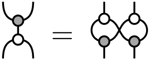

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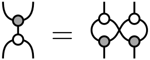

Mutually unbiased bases

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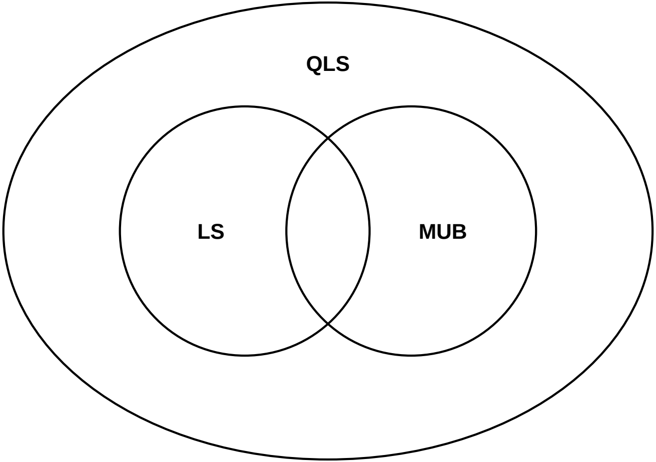
Hence Quantum Latin squares generalise both Latin squares and mutually unbiased bases.

Venn Diagram 1



Unitary Error Bases

Definition

For a Hilbert space $\mathcal{H}$ of dimension n, a *unitary error basis* is an n^2 family of unitary operators which form an orthogonal basis.

$$\mathrm{Tr}(U_i^\dagger \circ U_j) = \delta_{ij} n$$

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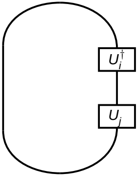

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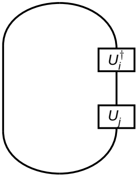

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Famous example: the Pauli matrices together with the identity.

Constructions of UEBs

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- *Algebraic construction* [Knill, 1996]
  Input: Group (order n^2) with nice representation
- *QLS construction* [MV, 2015]
  Input: Quantum Latin square and n Hadamards (order n)

An example

Take our example quantum Latin square:

$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$
$\frac{1}{\sqrt{2}}(\lvert 1\rangle - \lvert 2\rangle)$	$\frac{1}{\sqrt{5}}(i\lvert 0\rangle + 2\lvert 3\rangle)$	$\frac{1}{\sqrt{5}}(2\lvert 0\rangle + i\lvert 3\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 1\rangle + \lvert 2\rangle)$
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and the following family of Hadamard matrices:

$$H_0 = H_1 = H_2 = H_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

We get the following UEB, $\mathcal{M}$:

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 0 & \frac{i}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{i}{\sqrt{5}} & 0 \end{pmatrix}
\begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{i}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}
\begin{pmatrix} 0 & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{2i}{\sqrt{5}} & -\frac{i}{\sqrt{5}} & 0 \end{pmatrix}
\begin{pmatrix} 0 & \frac{2i}{\sqrt{5}} & -\frac{i}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{pmatrix}
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$$

$$
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$$

$$
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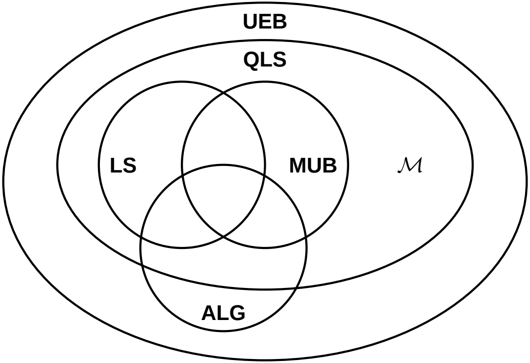
\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -1 & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

Venn Diagram 2

Why our construction is great!



Theorem

The example UEB $\mathcal{M}$ is not in LS, MUB or ALG.

UEBs built from tensors

QLS UEB

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 - ONB

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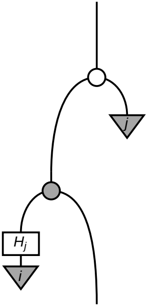

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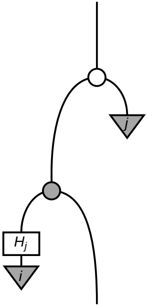

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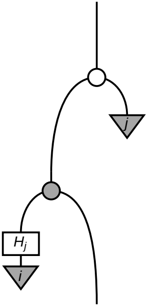

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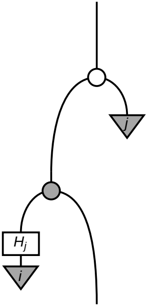

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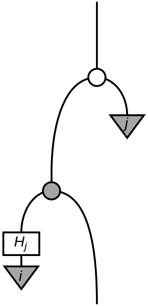


LS UEB

 - ONB

H_j - n Hadamards

 - LS

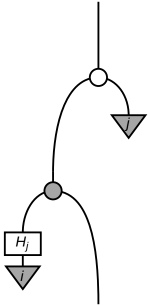
UEBs built from tensors

QLS UEB

- ONB

H_j - n Hadamards

- QLS

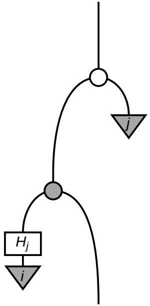


LS UEB

- ONB

H_j - n Hadamards

- LS



QLS UEB

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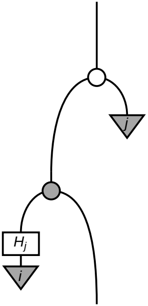

LS UEB

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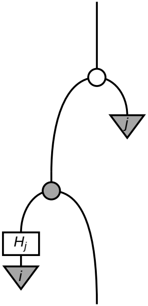


MUB UEB

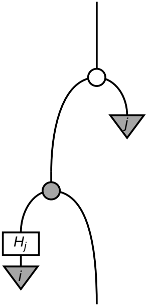
UEBs built from tensors

QLS UEB

 - ONB

H_j - n Hadamards

 - QLS



LS UEB

 - ONB

H_j - n Hadamards

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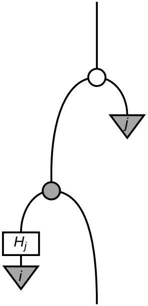

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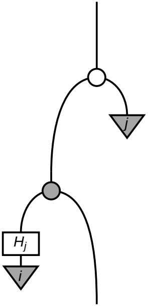

LS UEB

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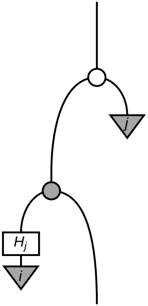

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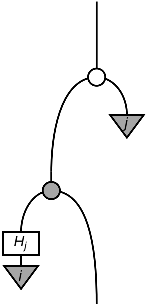

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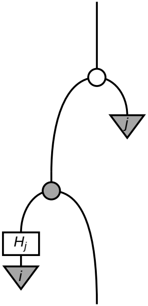

MUB UEB

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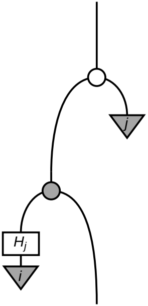

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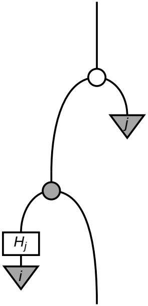

MUB UEB

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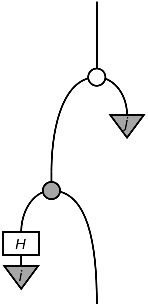

Main Theorem

Theorem

Quantum Latin square bases are unitary error bases.

First we prove that the U_{ij} are unitary.

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Proof.

Need to show that for all i, j: $U_{ij} \circ U_{ij}^{\dagger} = \mathbb{I}_n$.

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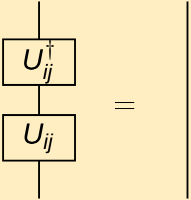
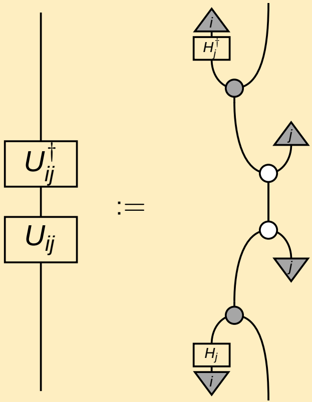


Main Theorem
Unitarity

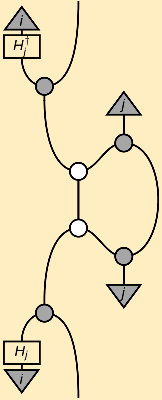
Proof.

By definition:



Proof.

$=$



Proof.

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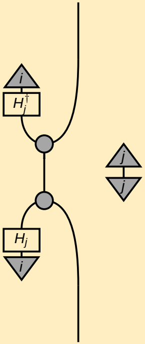

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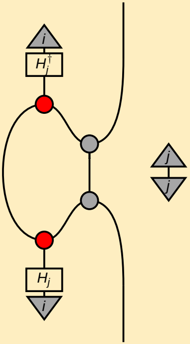

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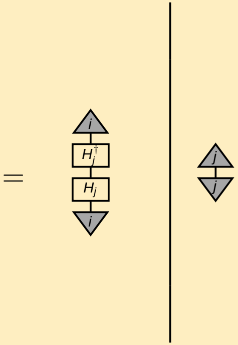
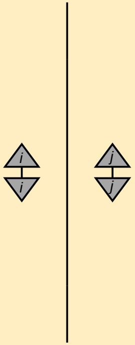


Proof.

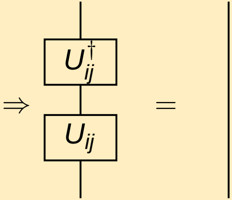
$=$ 

Proof.

$=$

Main Theorem
Unitarity

Proof.



Now we prove that the U_{ij} are an orthogonal basis.

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Need to show that for all i, j: $\mathrm{Tr}(U_{pq}^{\dagger} \circ U_{ij}) = \delta_{ip}\delta_{jq}n$.

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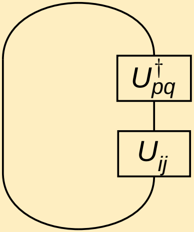
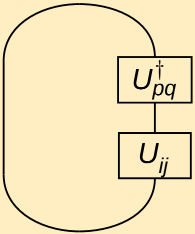


$$= \delta_{ip}\delta_{jq}n$$

Now we prove that the U_{ij} are an orthogonal basis.

Proof.

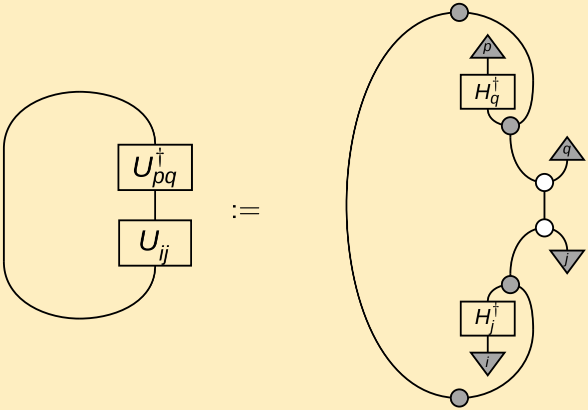
Or disregarding normalization:



$$= \delta_{ip}\delta_{jq}$$

Proof.

By definition:



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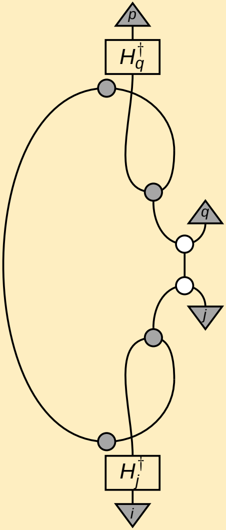
$=$



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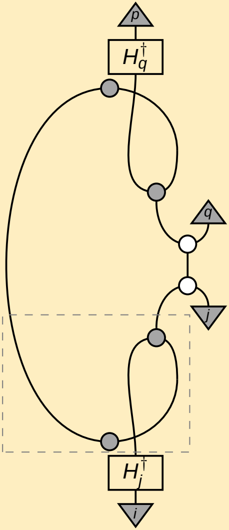
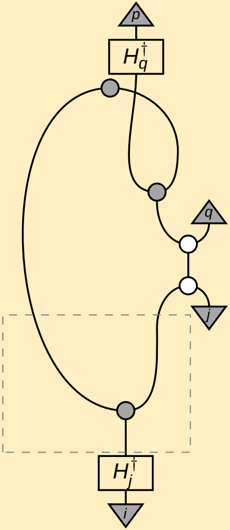


Proof.

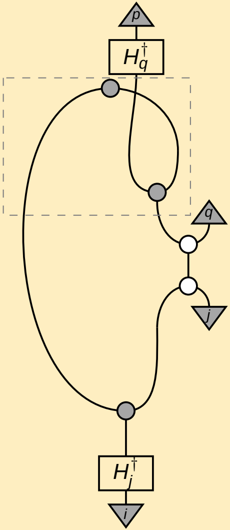
By spider merge

$=$



Proof.

$=$



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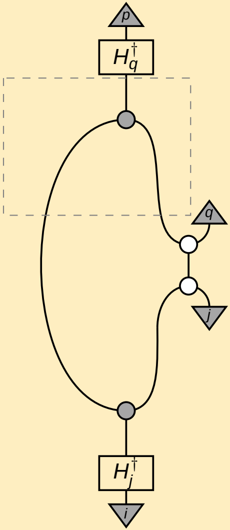
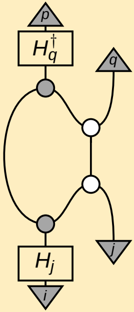


Proof.

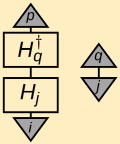
$=$



Main Theorem
Orthogonality

Proof.

$$=$$



Main Theorem
Orthogonality

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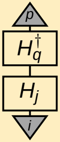
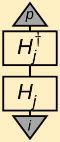
$=$



δ_{jq}

Proof.

$=$



δ_{jq}

Proof.

$=$ 

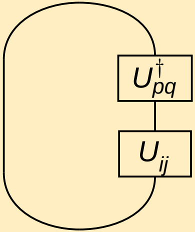
δ_{jq}

Proof.

$$= \delta_{ip}\delta_{jq}$$

Proof.

$\Rightarrow$



$= \delta_{ip}\delta_{jq}$

$\square$