Transversal logical gates

SPT phases

gapped boundaries

Beni Yoshida (Perimeter Institute)

*SPT = symmetry protected topological

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*SPT = symmetry protected topological

Transversal Logical Gates at QIP

- arXiv:1206.1609 Sergey Bravyi, Robert Koenig ⟵ QIP13
- arXiv:1304.3709 Adam Paetznick, Ben Reichardt ⟵ QIP14
- arXiv:1311.0879 Hector Bombin ⟵ QIP15
- arXiv:1408.1720 Fernando Pastawski, Beni Yoshida ⟵ QIP15
- arXiv:1503.02065 Alex Kubica, Beni Yoshida, Fernando Pastawski ⟵ Later
- arXiv:1503.07208 Beni Yoshida ⟵ This talk
- arXiv:1508.03468 Beni Yoshida ⟵ Merged
- arXiv:1509.03239 Sergey Bravyi, Andrew Cross ⟵ Later
- arXiv:1509.03626 Beni Yoshida ⟵ This talk

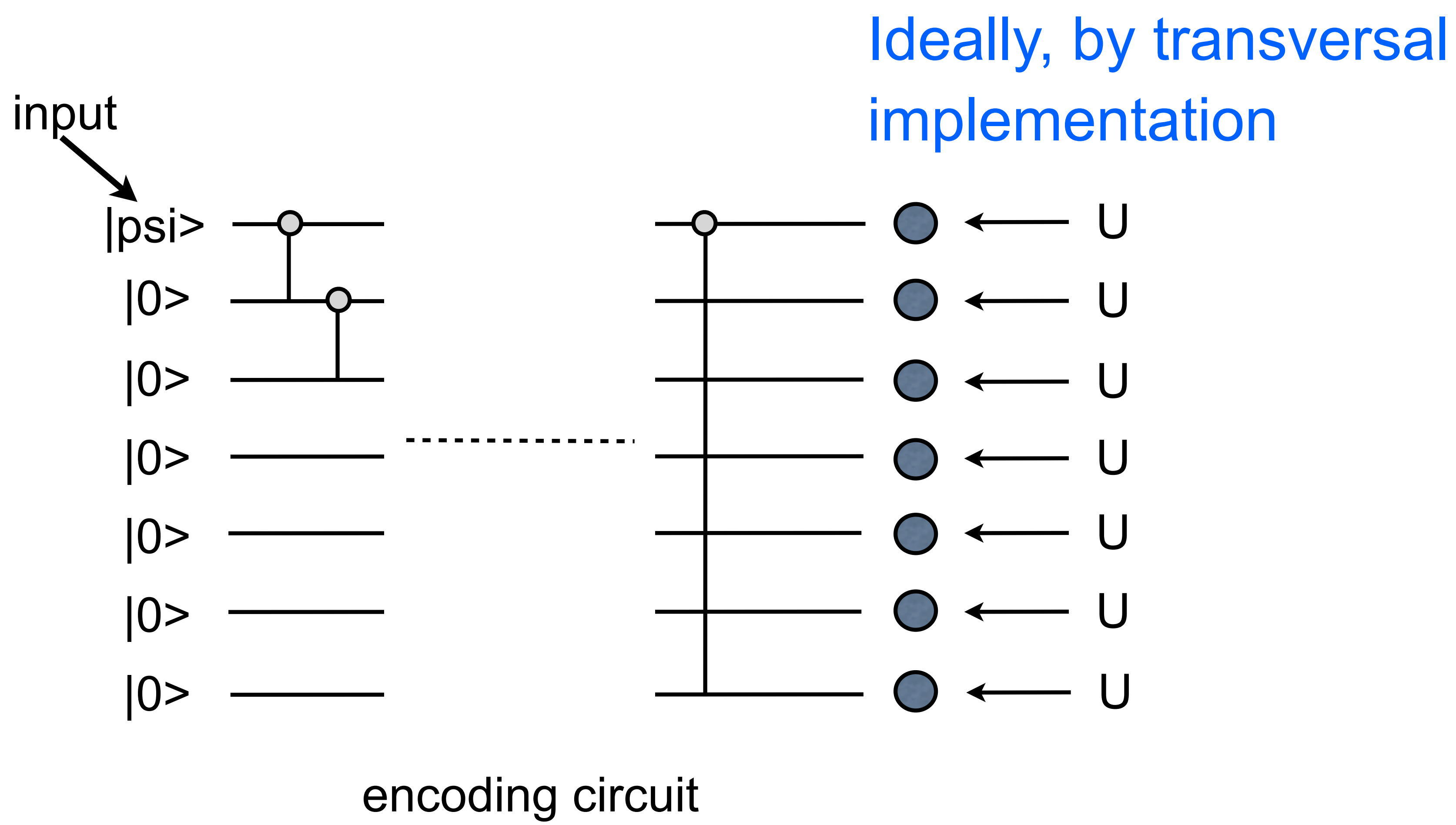
remark

arXiv:1508.03468 will not be covered in this talk.

Why transversal gates ?

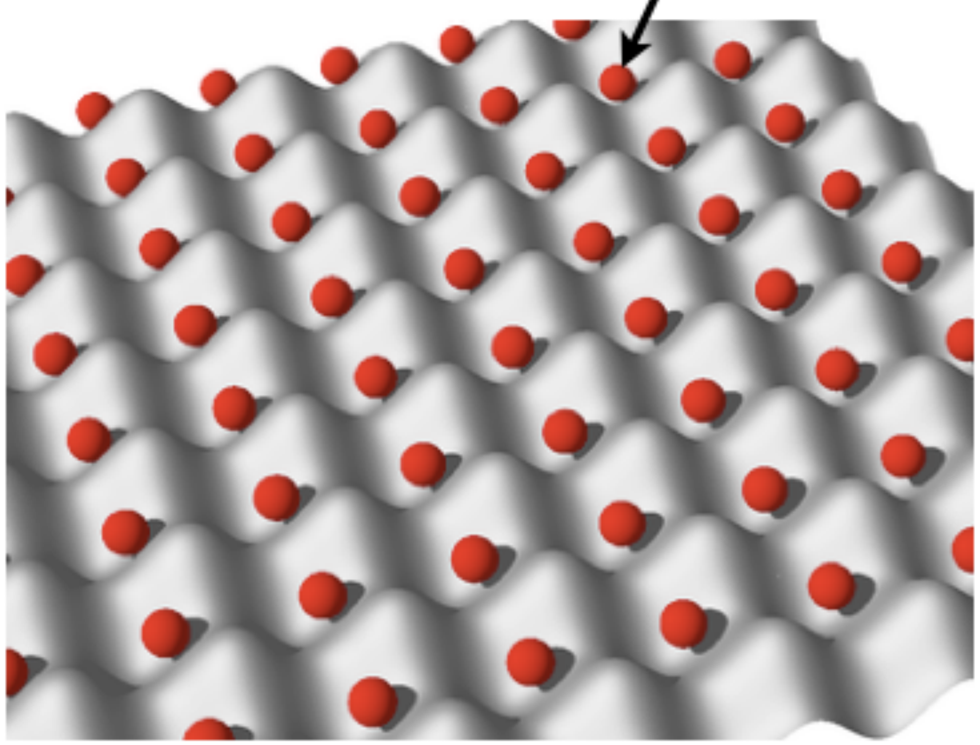
The problem(s)

- Given a quantum error-correcting code, how do we find transversal logical gates ?
- How do we design a quantum error-correcting code with useful transversal logical gates ?



Bravyi-Koenig theorem (2012)

Logical gate U : low-depth unitary gate (i.e. Local unitary)



D-dim lattice

Theorem

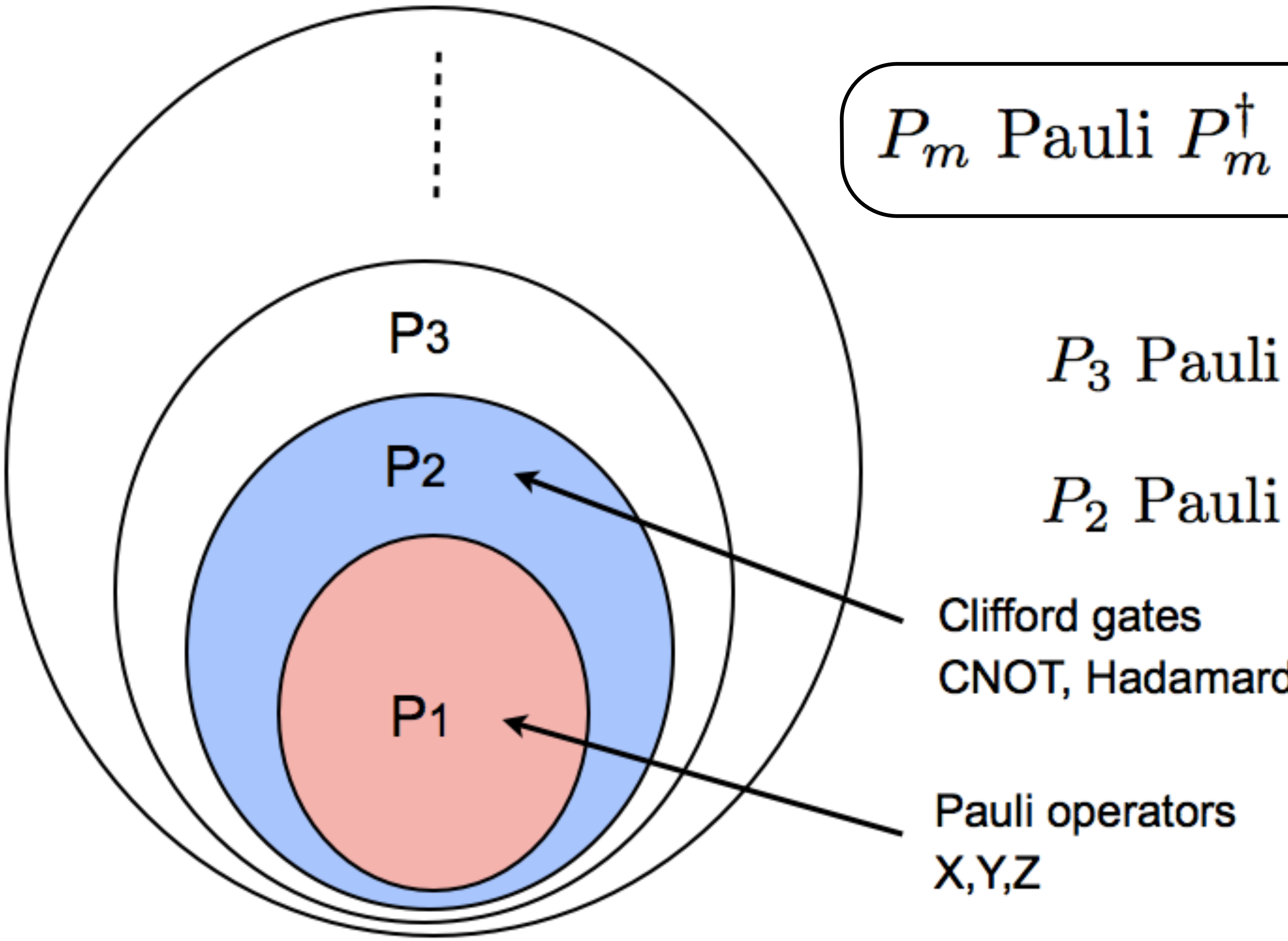
- For a stabilizer Hamiltonian in D dim, fault-tolerantly implementable gates are restricted to the D-th level of the Clifford hierarchy.

???

Clifford hierarchy (Gottesman & Chuang)

Sets of unitary transformations on N qubits

$$\boxed{P_m \text{ Pauli } P_m^\dagger = P_{m-1}}$$



$$P_3 \text{ Pauli } P_3^\dagger = P_2$$

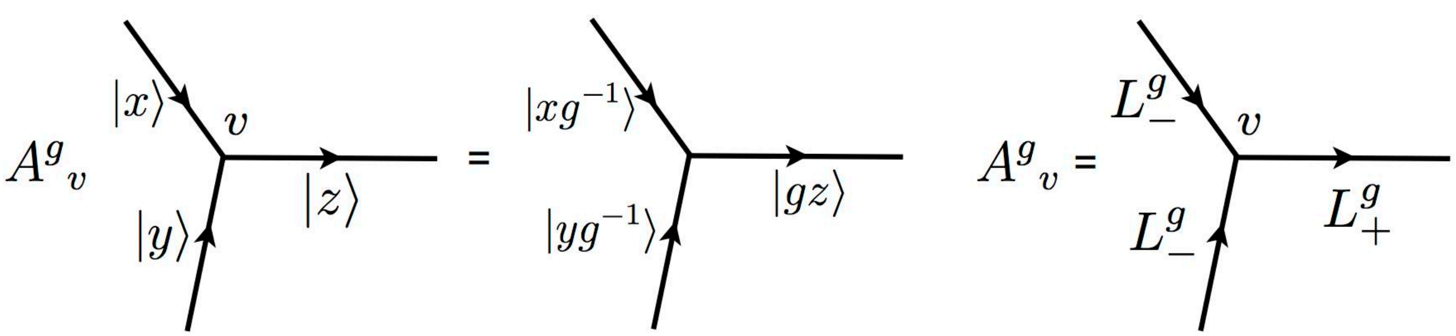
$$P_2 \text{ Pauli } P_2^\dagger = P_1$$

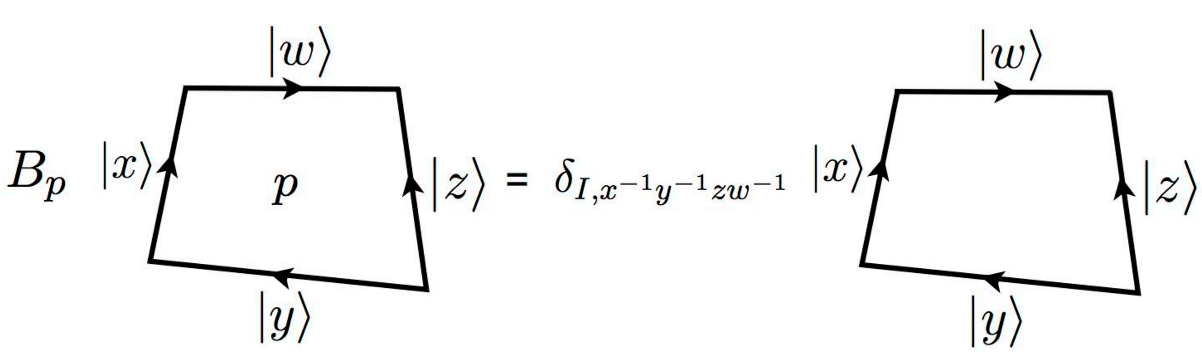
Pauli

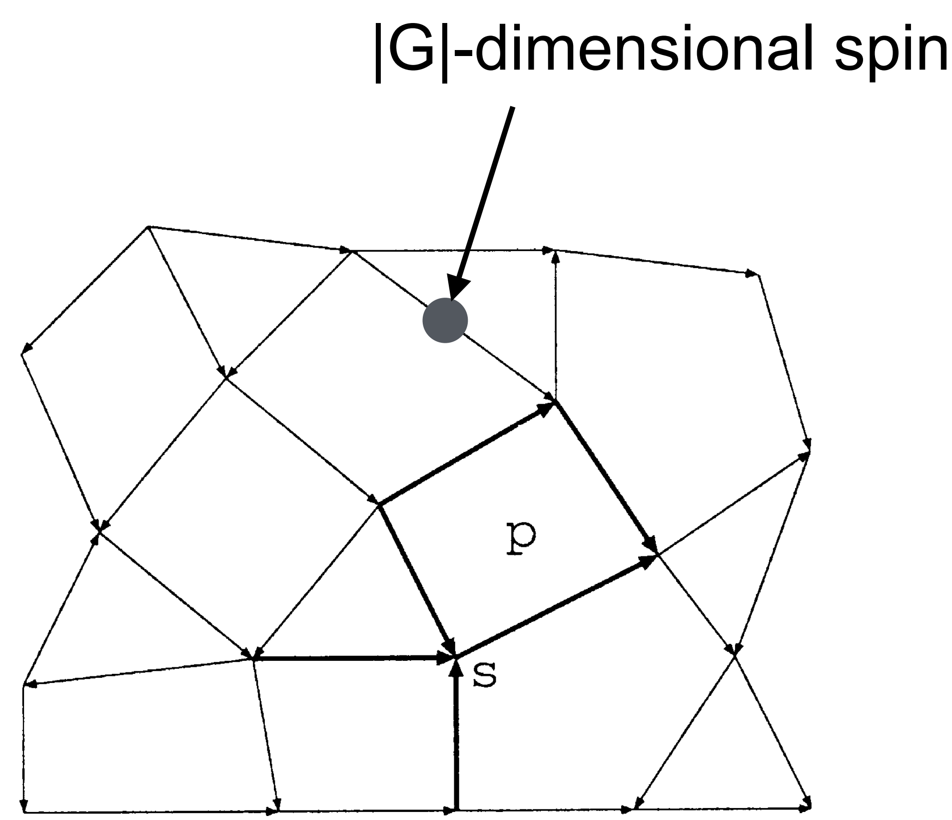
Quantum double model in d dimensions

- Quantum double model is a certain family of topological codes.
- Given a d-dimensional directed graph and a finite group G, one can define the quantum double model.

$$H_G = -\sum_v A_v - \sum_p B_p$$







*One can also add “twists”, which leads to the Dijkgraaf-Witten model.

Main result

A systematic framework for constructing logical gates

- We will consider d-cocycle functions over G by studying the group cohomology Hd(G,U(1)). $\omega_n(g_1,\ldots,g_n)$

- Using d-cocycle functions, we can provide a recipe of constructing a fault-tolerant logical gate for the d-dimensional quantum double model.

- If the cocycle function has a non-trivial sequence of slant products, then the logical gate is a non-trivial d-th level gate.

*Slant product : a map from n-cocycle to n-1 cocycle

0-cocyle

$$\omega_n \xrightarrow[i_{g_1}]{} \omega_n^{(g_1)} \xrightarrow[i_{g_2}]{} \omega_n^{(g_1,g_2)} \longrightarrow \cdots \longrightarrow \omega_n^{(g_1,g_2,\ldots,g_{n-1})} \xrightarrow[i_{g_n}]{} \omega_n^{(g_1,g_2,\ldots,g_{n-1},g_n)}$$

U(1) phase

n-cocyle (n-1)-cocyle

arXiv:1509.03626 BY

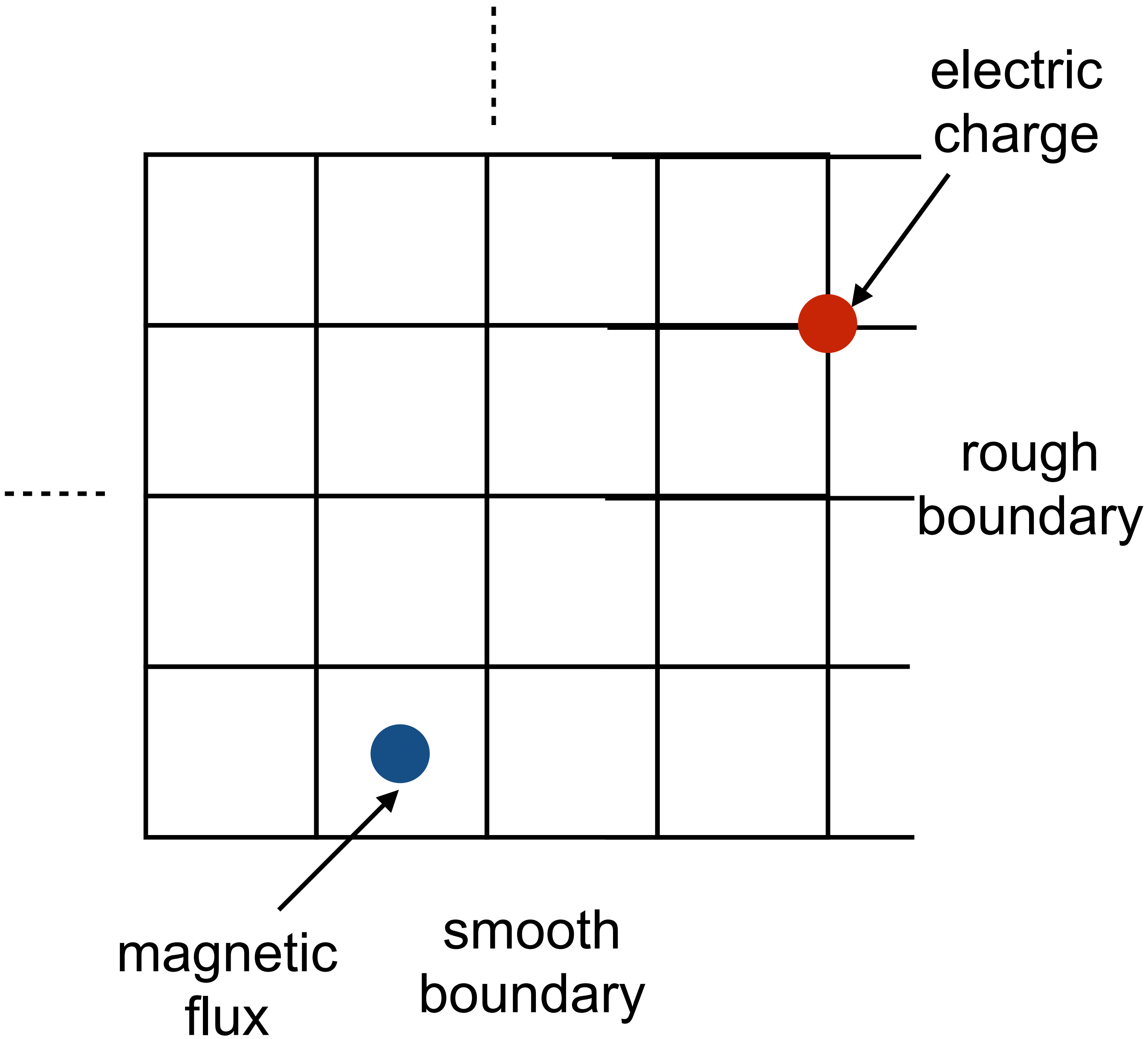
Transversal logical gates

SPT phases

gapped boundaries

Classification of gapped boundaries

- The Toric code has two types of boundaries (Bravyi-Kitaev 98)



Which anyons can *condense into a boundary ?

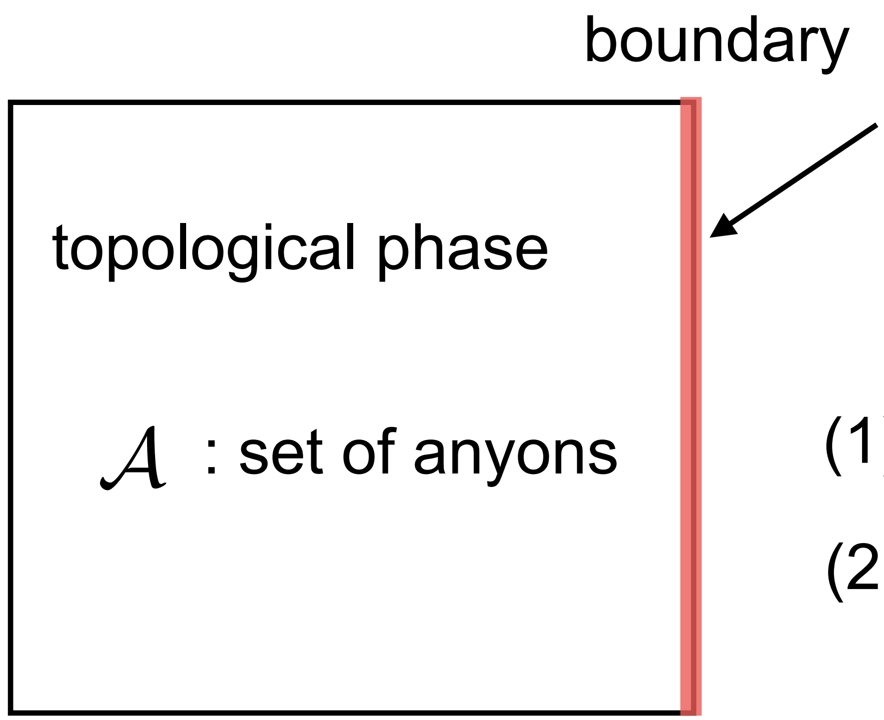
(i) rough boundary; electric charge

(ii) smooth boundary; magnetic flux

* can create and annihilate an anyon without involving others.

Lagrangian subgroup

- [Levin 13] Condensing anyons are characterized by Lagrangian subgroup.



Lagrangian subgroup

$$\mathcal{A}' \subset \mathcal{A}$$

(1) All the anyons in $\mathcal{A}'$ braid trivially with each other.

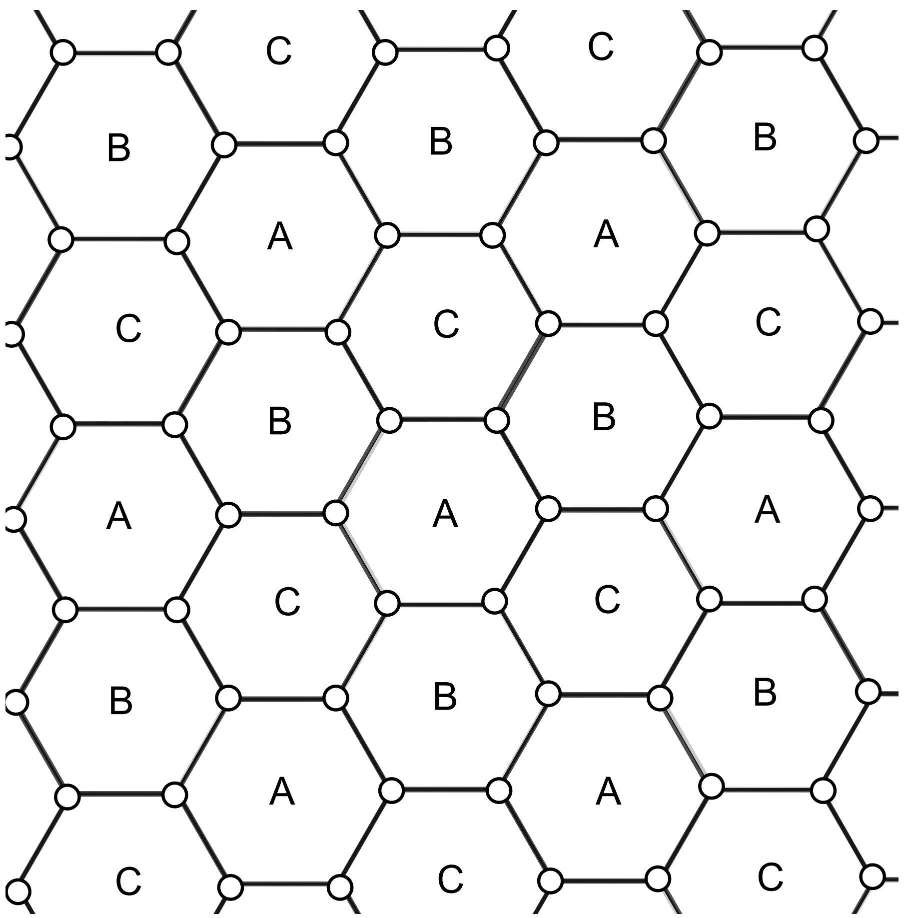
(2) Maximal subset.

- (Almost) complete classification of 2dim gapped boundaries

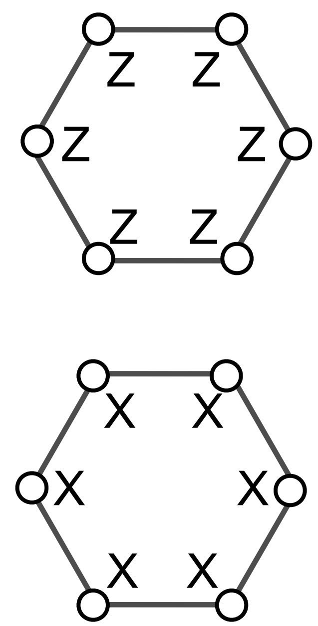
Cond-Mat High-Energy	Bais-Slingerland 09, Kapustin-Saulina 11, Levin-Gu 12, Levin 13, Wang-Wen 12, Barkeshli-Jian-Qi 13, Lan-Wang-Wen 15
Quant-Info	Bravyi-Kitaev 98, Bombin-MartinDelgado 08, Bombin 10, Beigi-Shor-Whalen 11
Math	Kitaev-Kong 12, Kong 13, Fuchs-Shweigert-Valentino 14

Gapped boundary and logical gate

• Fault-tolerant logical gates and gapped domain walls are closely related.



Topological color code



Hadamard logical gate
is transversal.

arXiv:1503.07208 BY

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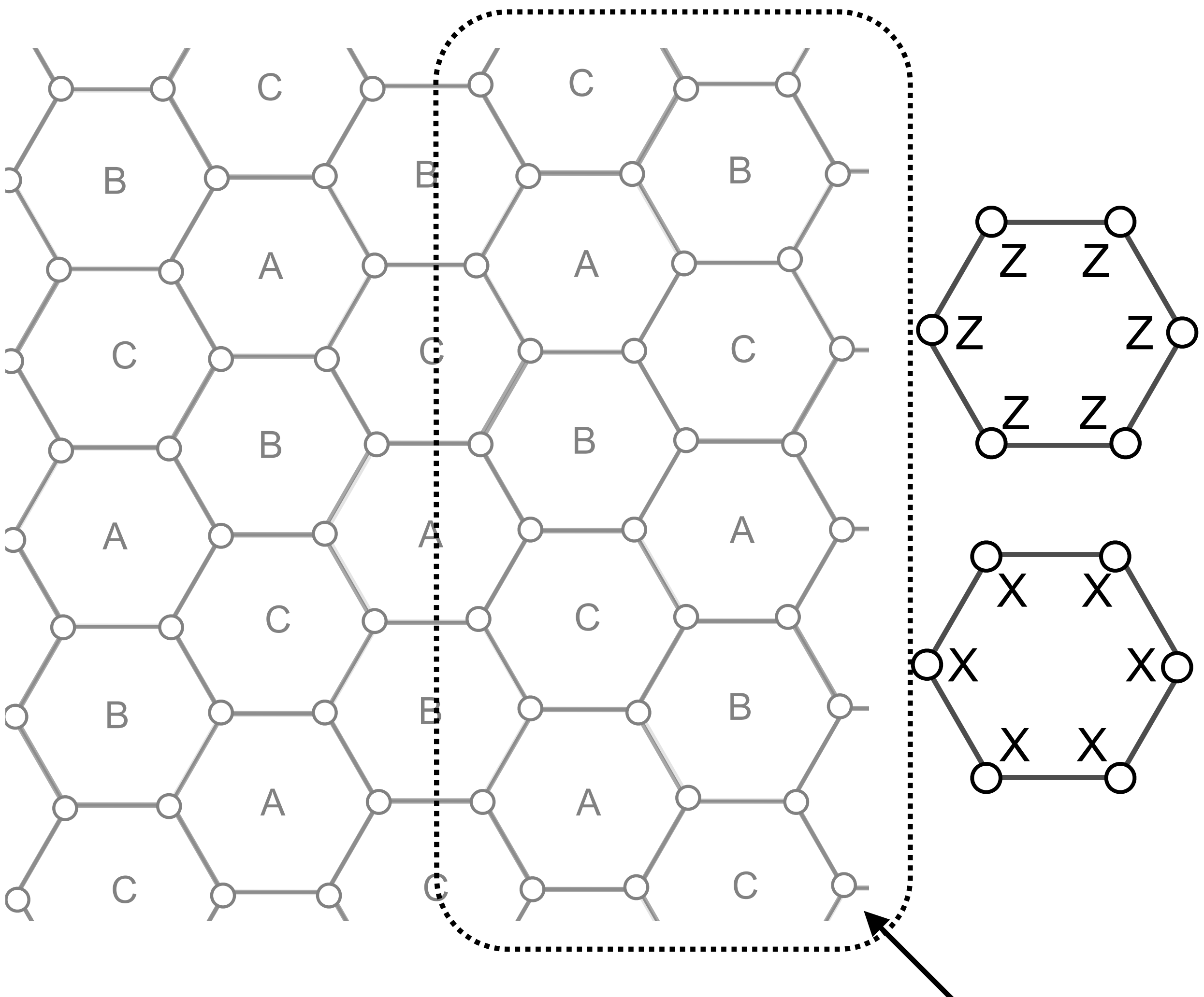

Hadamard logical gate
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Topological color code

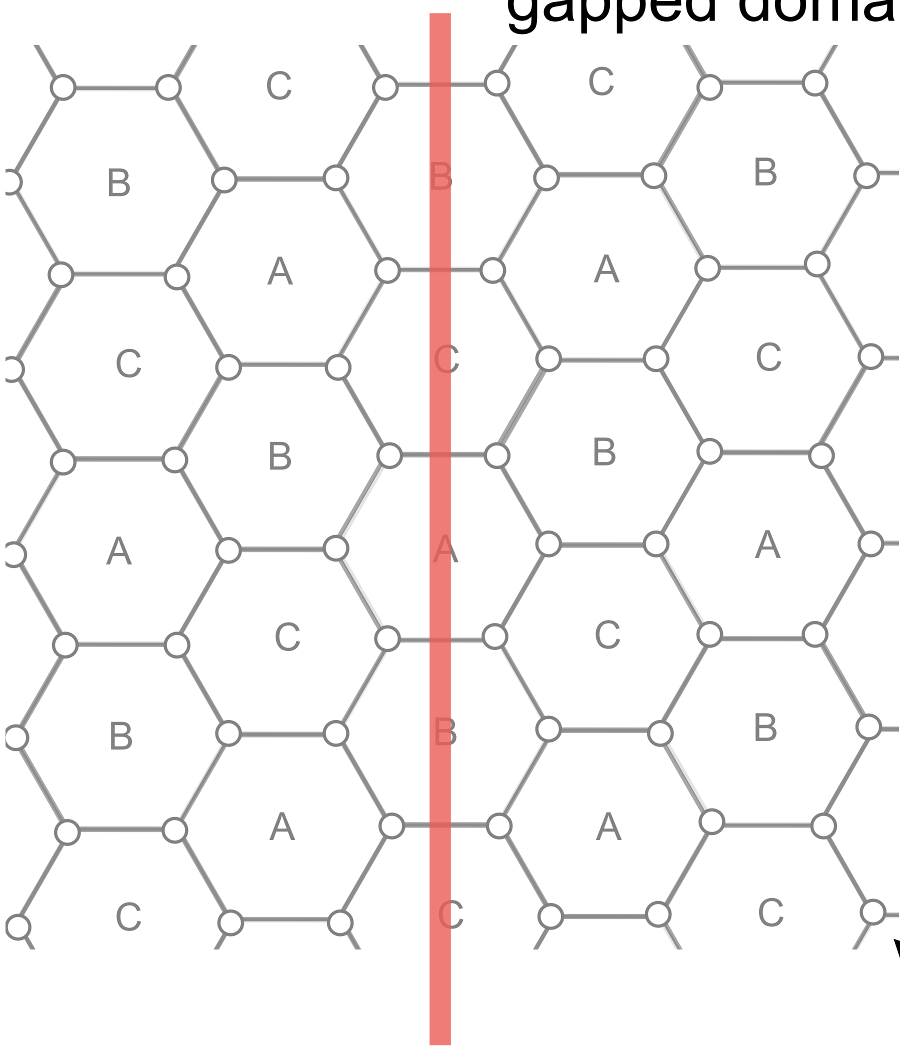
Apply Hadamard only
on the right hand side

arXiv:1503.07208 BY

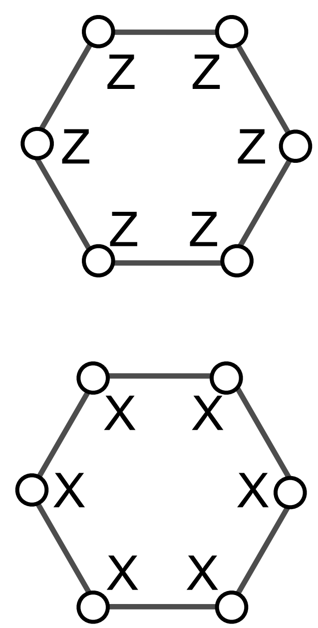
Gapped boundary and logical gate

- Fault-tolerant logical gates and gapped domain walls are closely related.

gapped domain wall



Topological color code



Apply Hadamard only on the right hand side

Hadamard logical gate is transversal.

e_A		m_A
e_B		m_B
e_C	domain wall $\longrightarrow$	m_C
m_A		e_A
m_B		e_B
m_C		e_C

arXiv:1503.07208 BY

Domain walls vs logical gates

Fact

- Given a fault-tolerant logical gate, one can construct a transparent domain wall in topological quantum code.

Conjecture

- There is a one-to-one correspondence between transparent domain walls and fault-tolerant logical gates in topological quantum field theory (TQFT).

- For Z_2*Z_2, there are 72 different domain walls. All of them have corresponding logical operations.

Another result

- We construct a gapped boundary / gapped domain wall in the d-dimensional quantum double model by using d-cocycle functions.

- In d>2, we can construct a gapped boundary where none of anyonic excitations can condense. (No electric charge/magnetic flux can condense)

- Anyons can condense into a boundary only if they are accompanied by superpositions of anyonic excitations.

"Lagrangian subgroup" needs to be modified.

Transversal logical gates

SPT phases

gapped boundaries

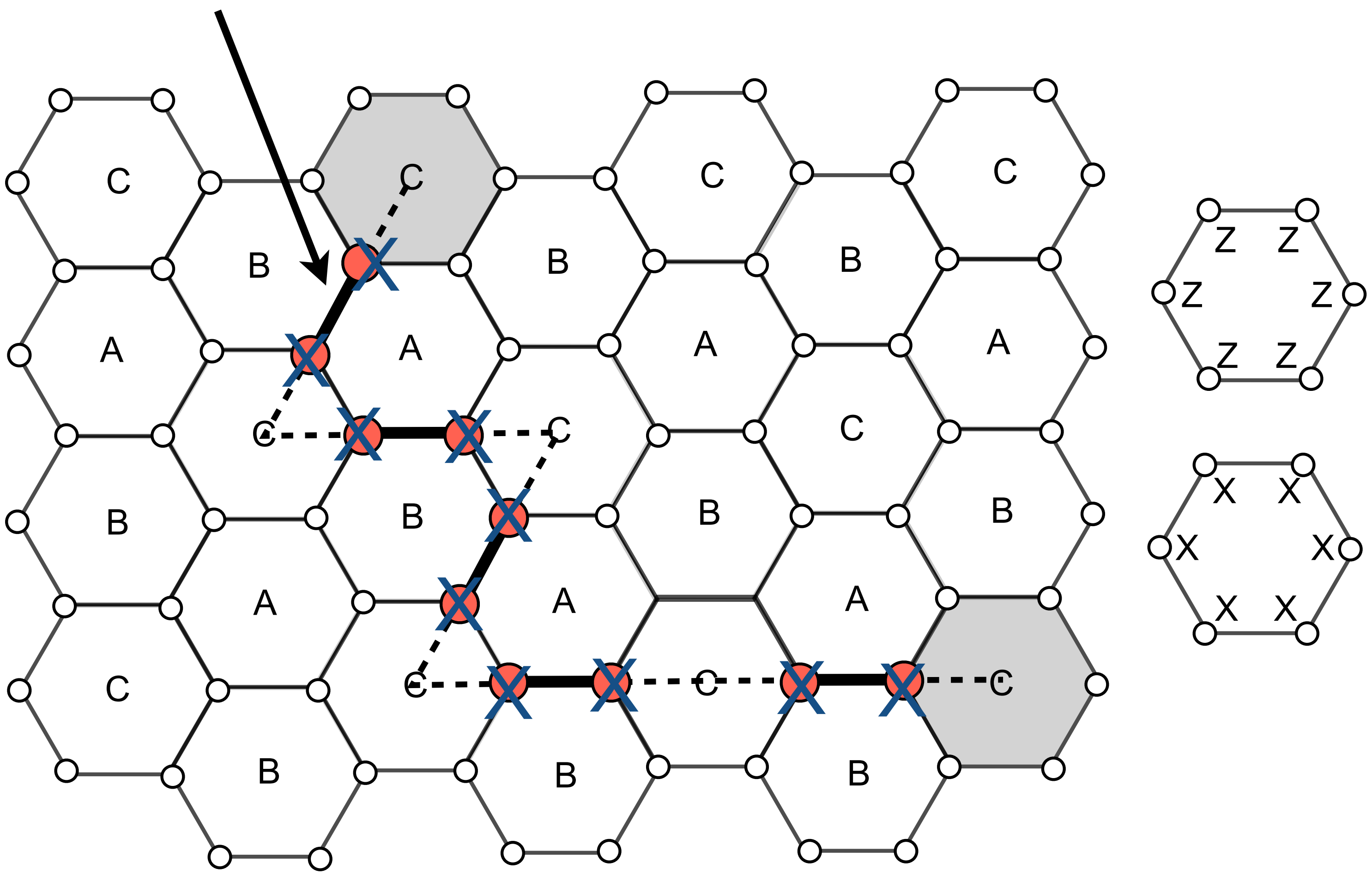
Topological color code (Bombin)

- The Hamiltonian is given by

$$H = -\sum_P S_P^{(X)} - \sum_P S_P^{(Z)}$$

defined on a three-colorable lattice

string operators



anyons in the color code

$e_A \leftarrow$ Pauli Z along A

$e_B \leftarrow$ Pauli Z along B

$e_C \leftarrow$ Pauli Z along C

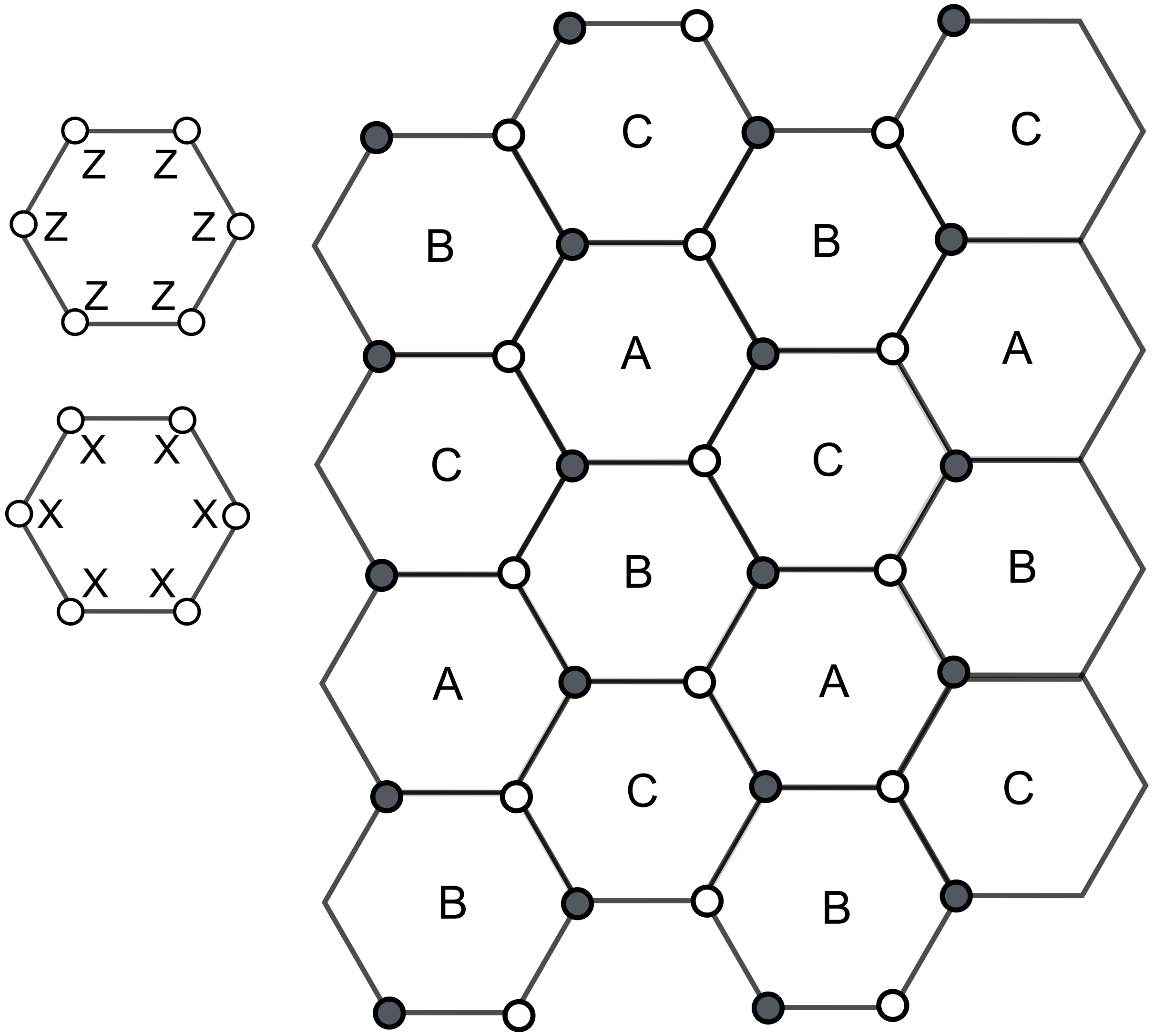
$m_A \leftarrow$ Pauli X along A

$m_B \leftarrow$ Pauli X along B

$m_C \leftarrow$ Pauli X along C

(equivalent to two copies of the toric code, BY2010)

Membrane operators in the color code



(1) Hadamard operator

$$\mathcal{H} \ : \ X \rightarrow Z \quad Z \rightarrow X$$

(2) R2 Phase operator

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix}$$

$$R_2 \ : \ X \rightarrow Y \quad Y \rightarrow -X$$

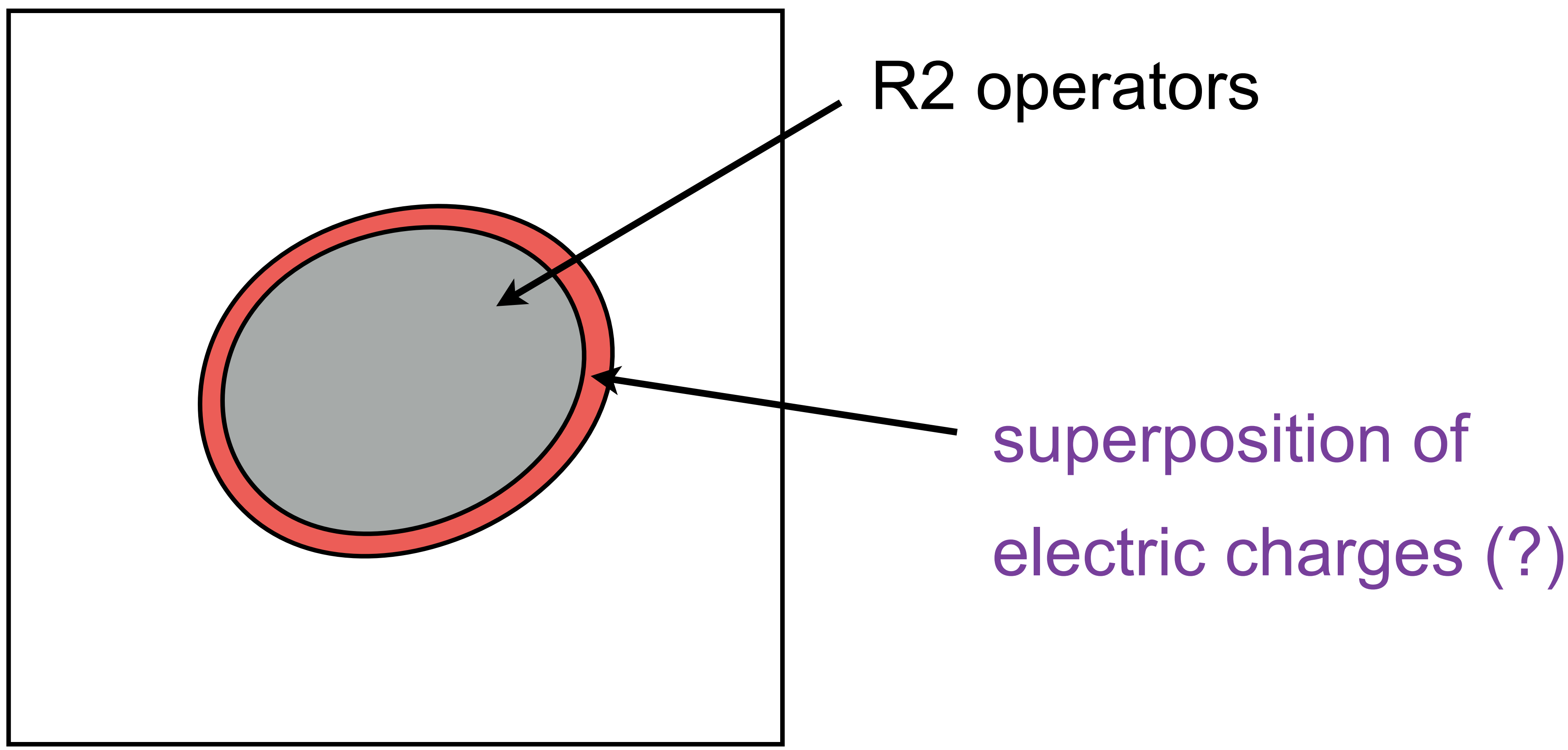
- Comment: Rm operators are transversal in m-dimensional color code…

$$R_m := \mathrm{diag}(1, \exp(i\pi/2^{m-1})).$$

String-like excitations ?

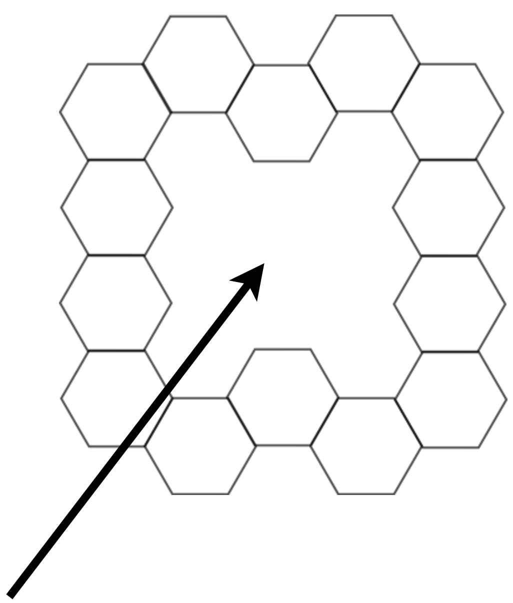
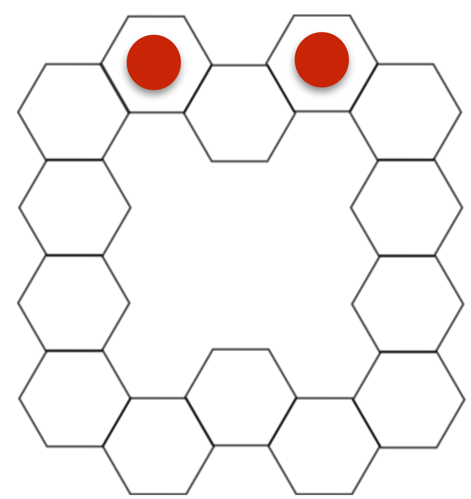
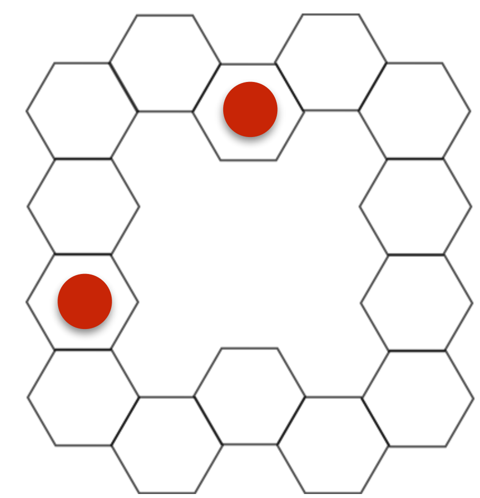
String logical operators → Point-like anyonic excitations

Membrane logical operators → String-like anyonic excitations(?)



Answer

- Loop-like excitations in the 2dim color code are "characterized" by 1dim Z2*Z2 SPT phase.

|psi > =  + $e^{i\theta_1}$  + $e^{i\theta_2}$ 

R2 operators applied

+ $\cdots$

Answer

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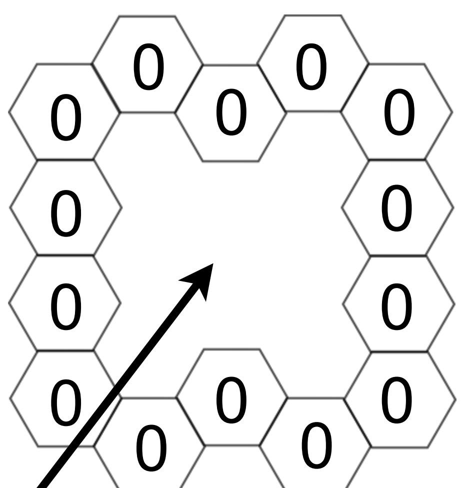
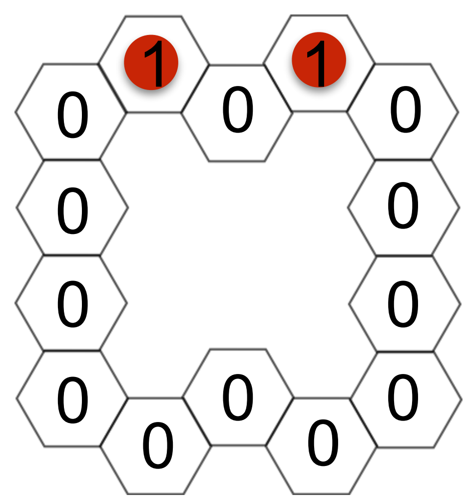
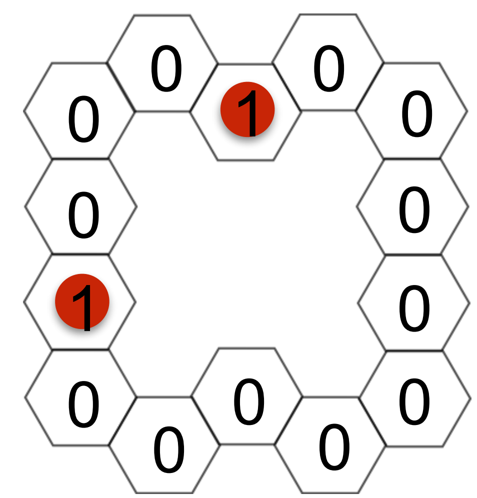
|psi > =  + $e^{i\theta_1}$  + $e^{i\theta_2}$ 

+ $\cdots$

R2 operators applied

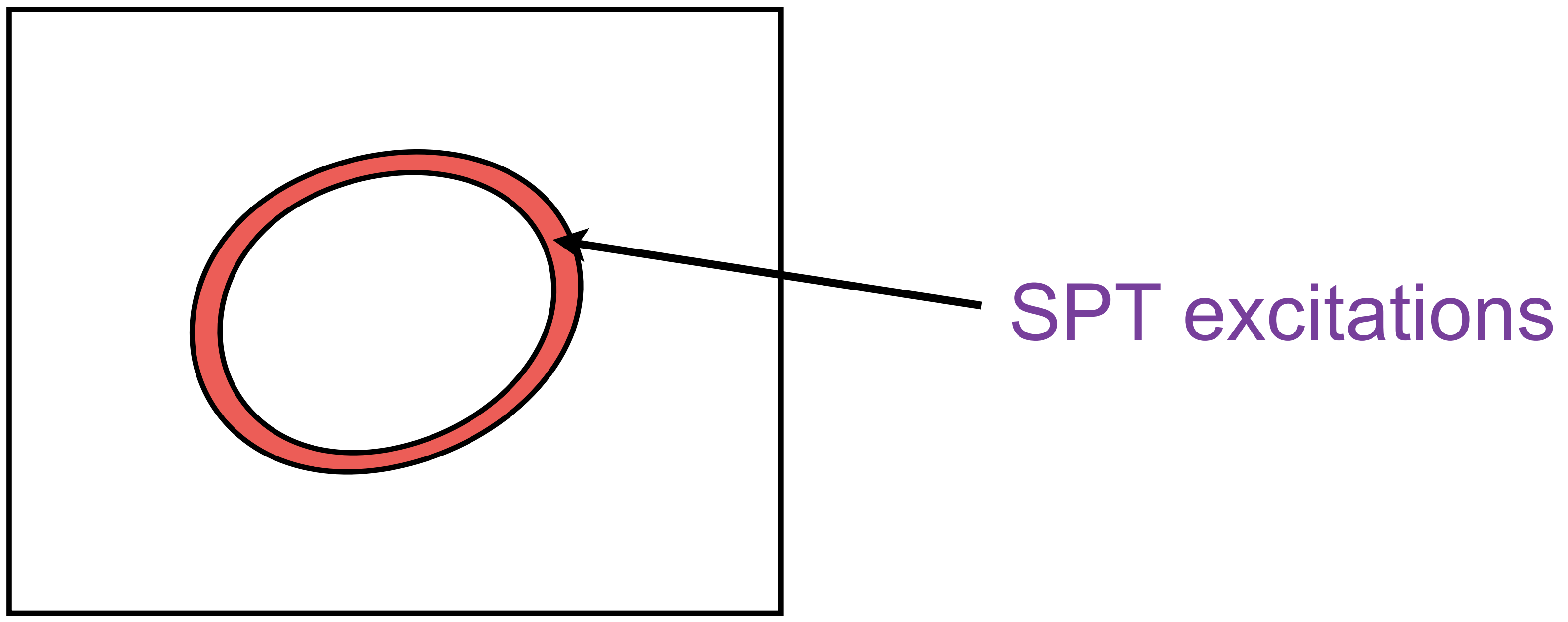
can be viewed as a one-dimensional wavefunction

Why SPT phases ?

- Origin of symmetries
  - Parity constraints of electric charges
  - 2dim color code = 2 copies of the toric code

    Electric charges from copy A and copy B get entangled to form a loop-like object.

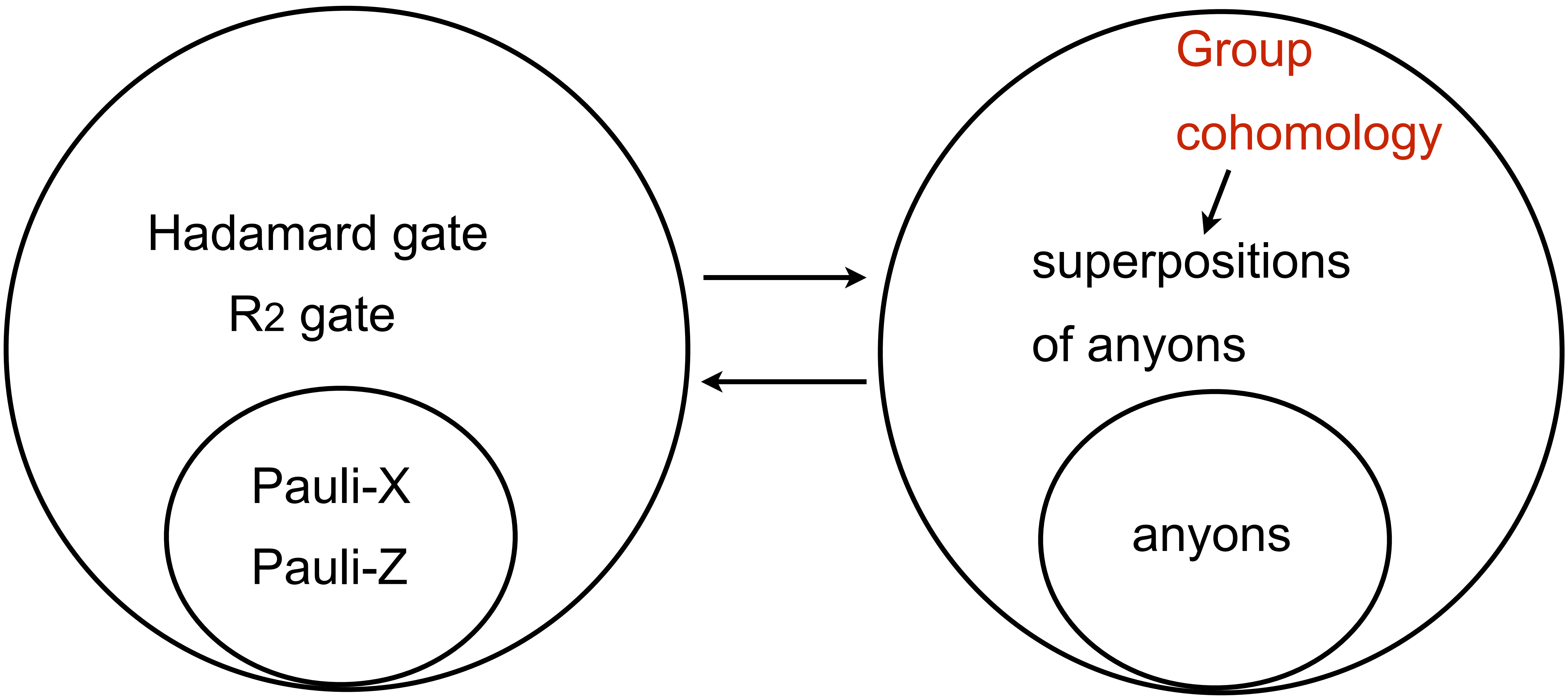
- Origin of non-triviality
  - Non-triviality of the gapped domain wall.



Toward classification of logical gates

Fault-tolerant logical gates

Possible excitations

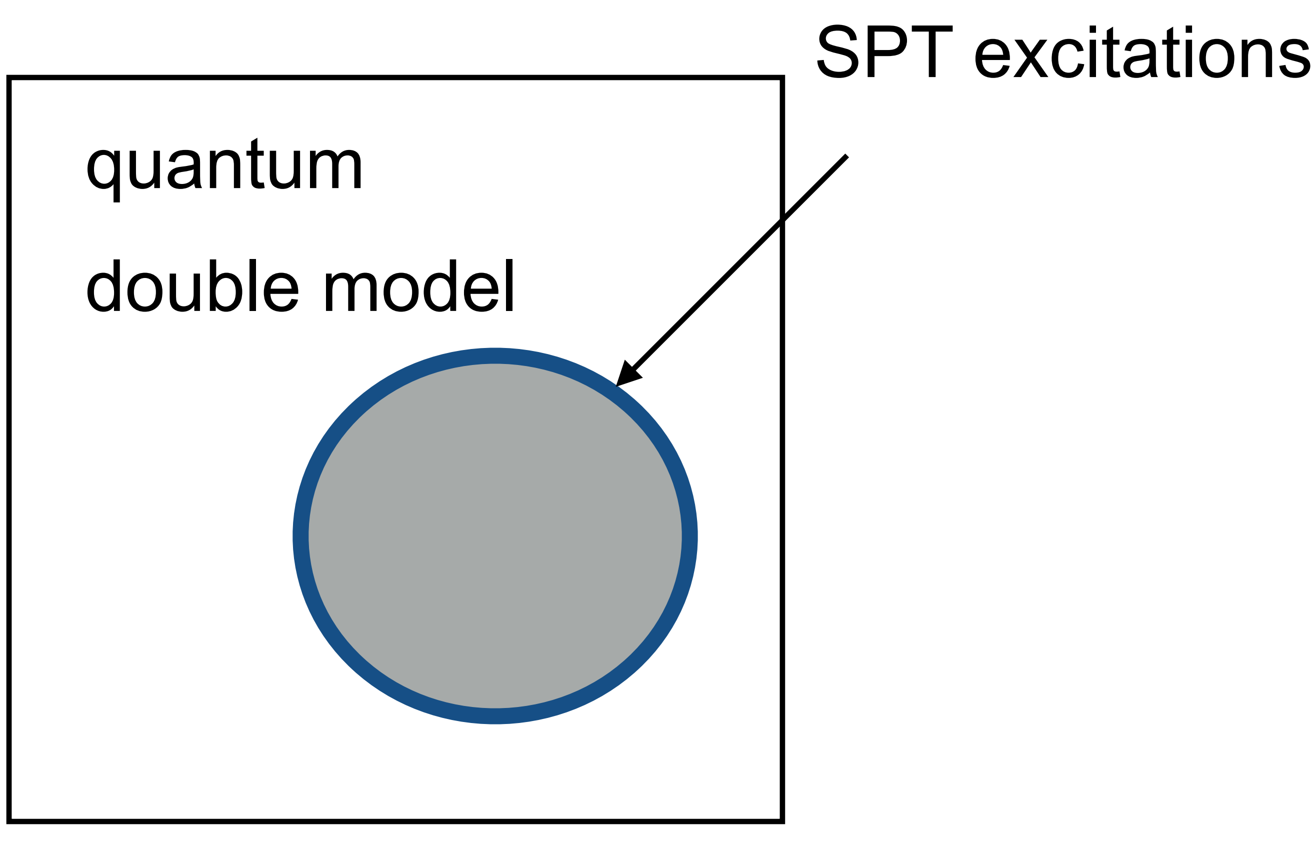


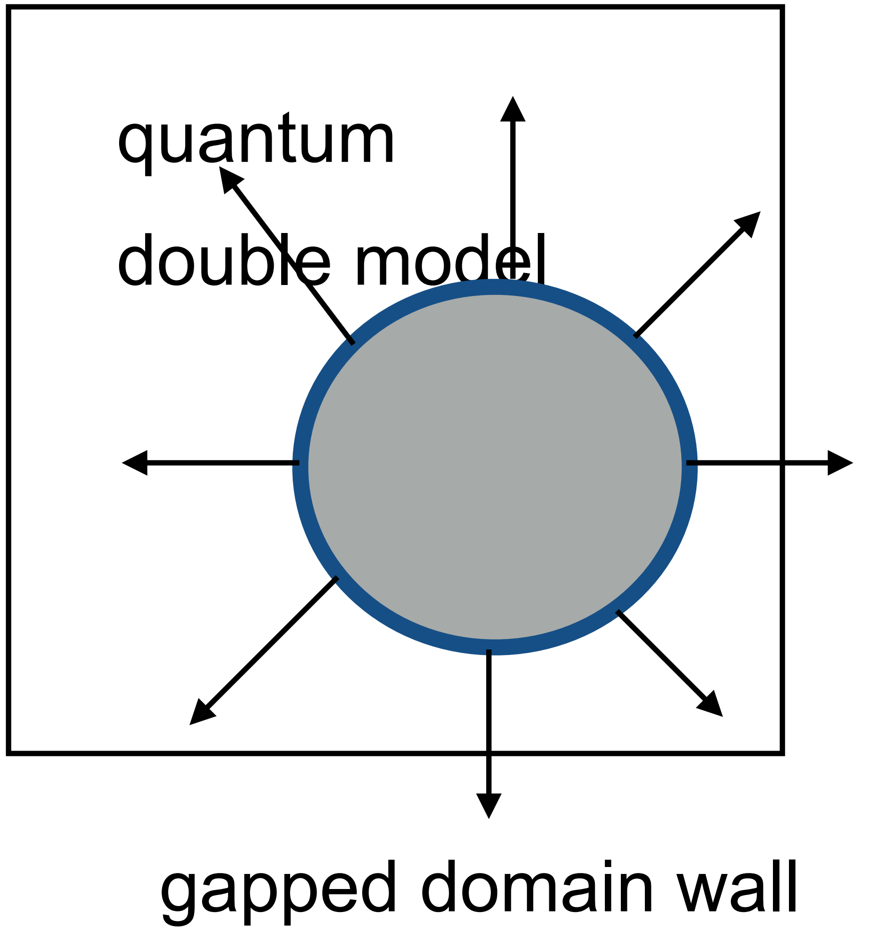
Key idea: sweeping SPT excitations

Sweep the domain wall over the entire system.

SPT phases are characterized by cocycle functions.

Logical actions are characterized by cocycle functions.





Topological color code ?

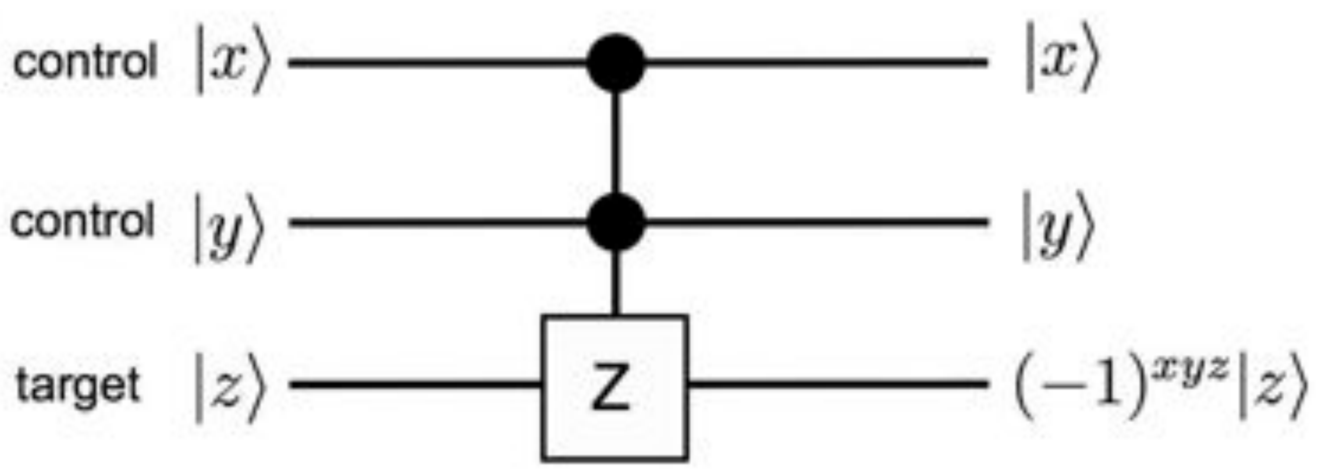
* The d-dimensional topological color code has a transversal Rd phase gate which belongs to the d-th level (outside of d-1 th level). (Bombin07)

* d-dimensional color code is equivalent to d copies of the d-dimensional toric code. (Kubica-BY-Pastawski 15)

i.e. the d-dimensional quantum double model with $\boxed{G = (\mathbb{Z}_2)^{\otimes d}}$

* There is a non-trivial d-cocycle: $\underline{\omega_d(g_1, \ldots, g_d) = (-1)^{g_1^{(1)} \cdots g_d^{(d)}}}$

* The corresponding gate is the d-qubit control-Z gate.



Overview of the results

Transversal logical gates

for d-dim quantum double model

SPT phases

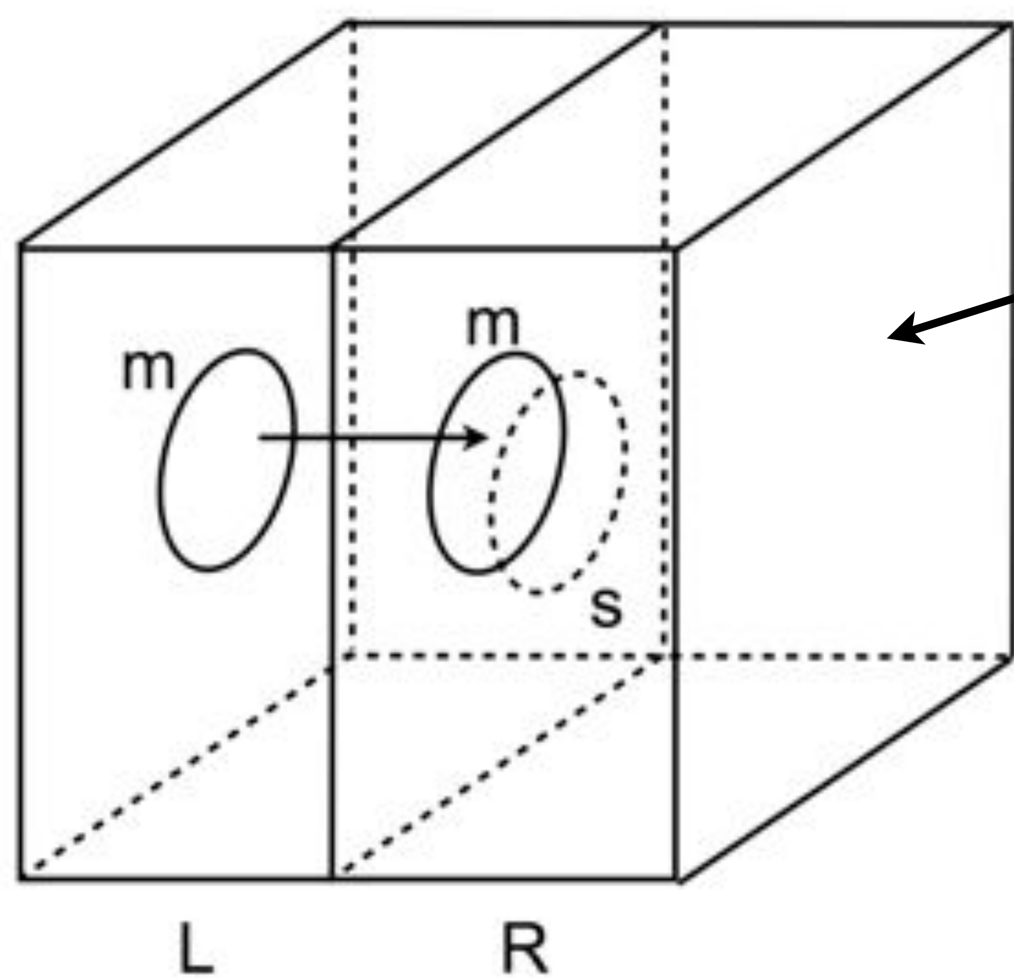
group cohomology
(d-cocycle)

gapped boundaries

beyond Lagrangian
subgroup

Domain wall in three-dimensions

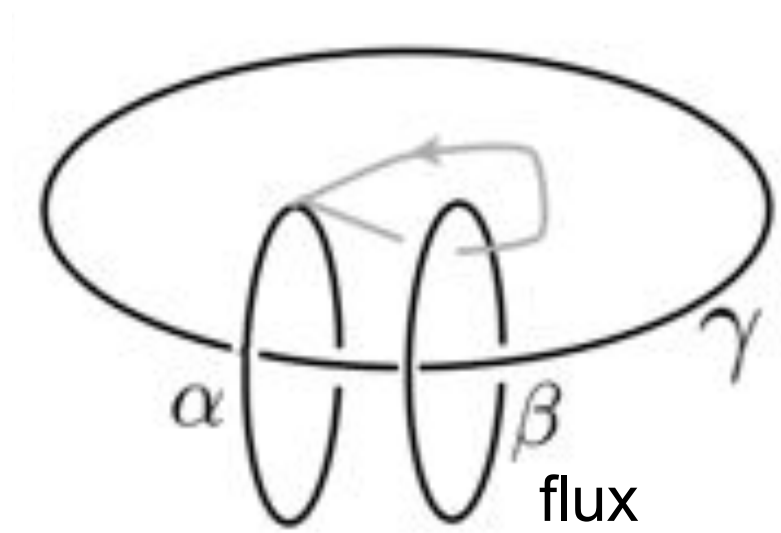
• magnetic flux becomes a composite of magnetic flux and superposition of electric charges (3dim color code)



R_3 operators only on the right hand side

Three-loop braiding statistics

- The three-dimensional color code exhibits non-trivial braiding statistics.



flux

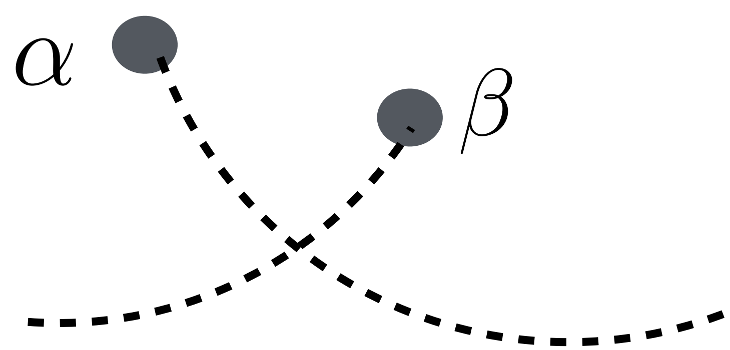
flux

SPT

The statistical angle can be computed by taking slant products of cocycle functions.

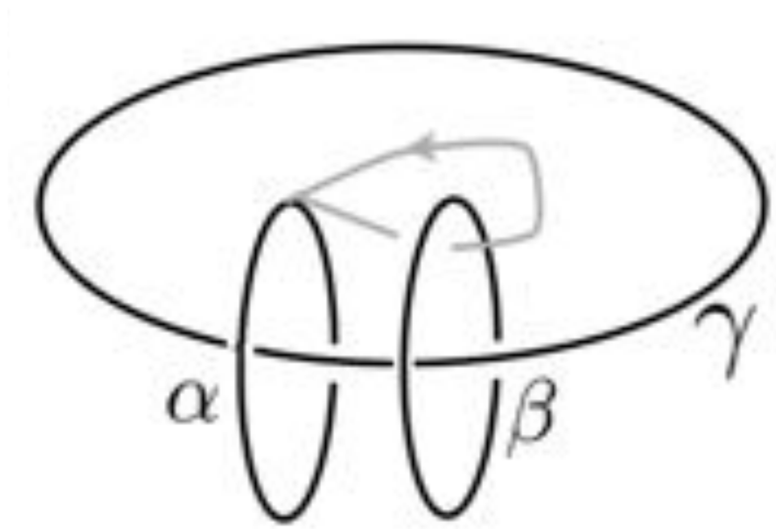
Multi-excitation braiding

- Two-particle braiding statistics can be studied by a group commutator



$$K(U_\alpha, U_\beta) = U_\alpha^\dagger U_\beta^\dagger U_\alpha U_\beta$$

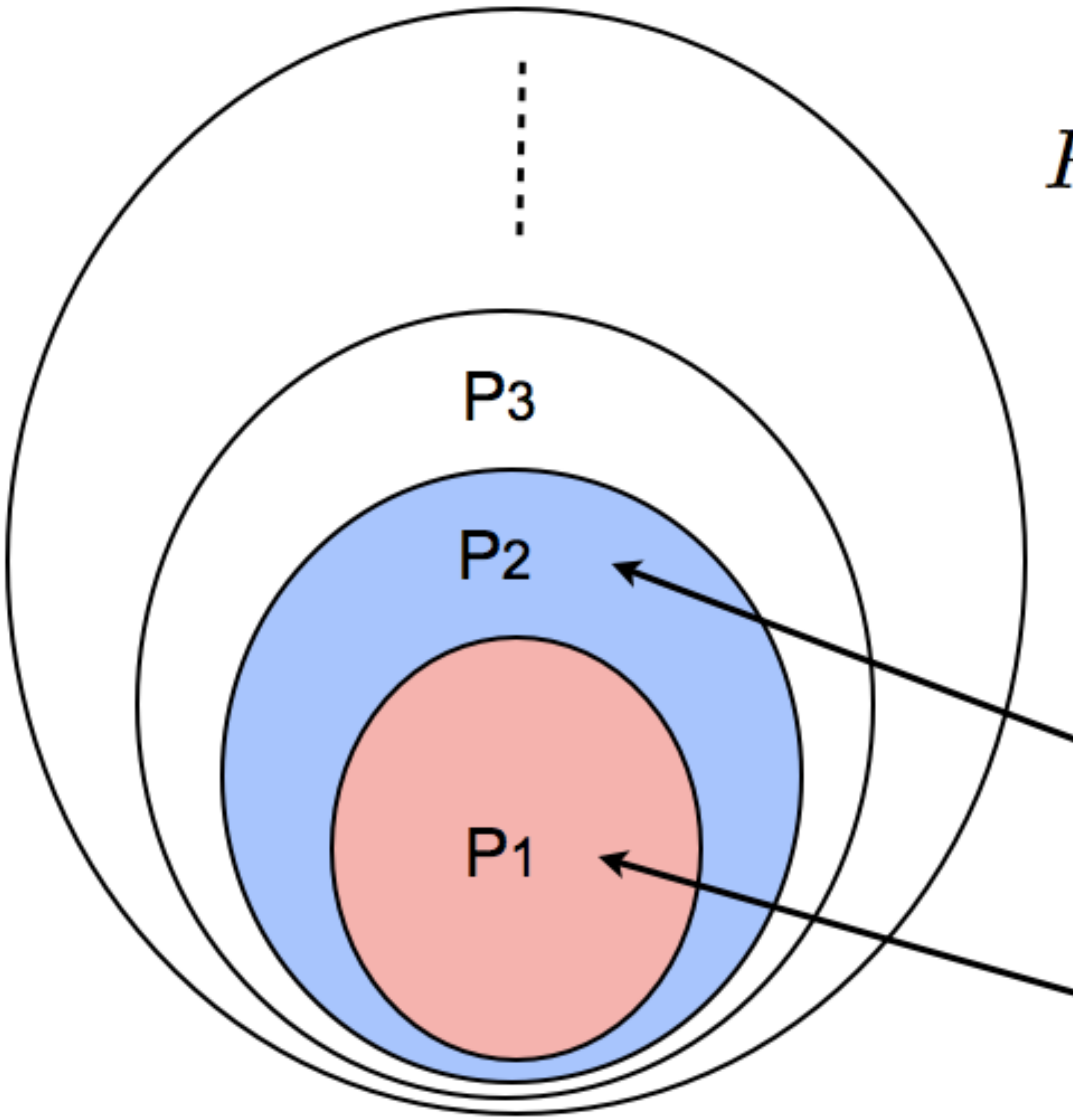
- Three-loop braiding statistics can be studied by a sequential group commutator



$$K(K(U_\alpha, U_\beta), U_\gamma) = (U_\alpha^\dagger U_\beta^\dagger U_\alpha U_\beta)^\dagger U_\gamma^\dagger (U_\alpha^\dagger U_\beta^\dagger U_\alpha U_\beta) U_\gamma.$$

Clifford hierarchy (Gottesman & Chuang)

Sets of unitary transformations on N qubits



$$P_m \text{ Pauli } P_m^\dagger = P_{m-1}$$

$$P_3 \text{ Pauli } P_3^\dagger = P_2$$

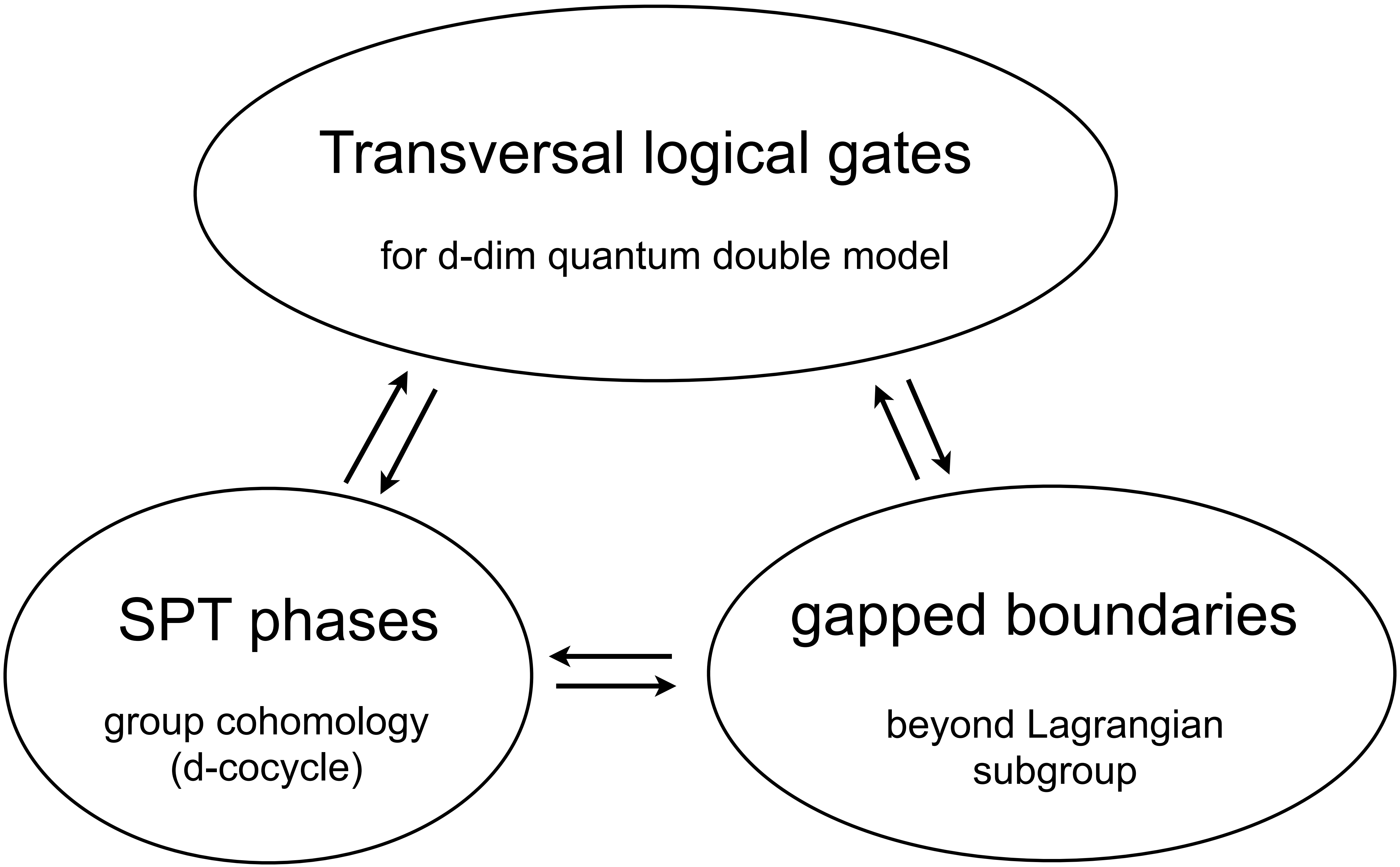
$$P_2 \text{ Pauli } P_2^\dagger = P_1$$

Pauli

Clifford gates
CNOT, Hadamard, R_2

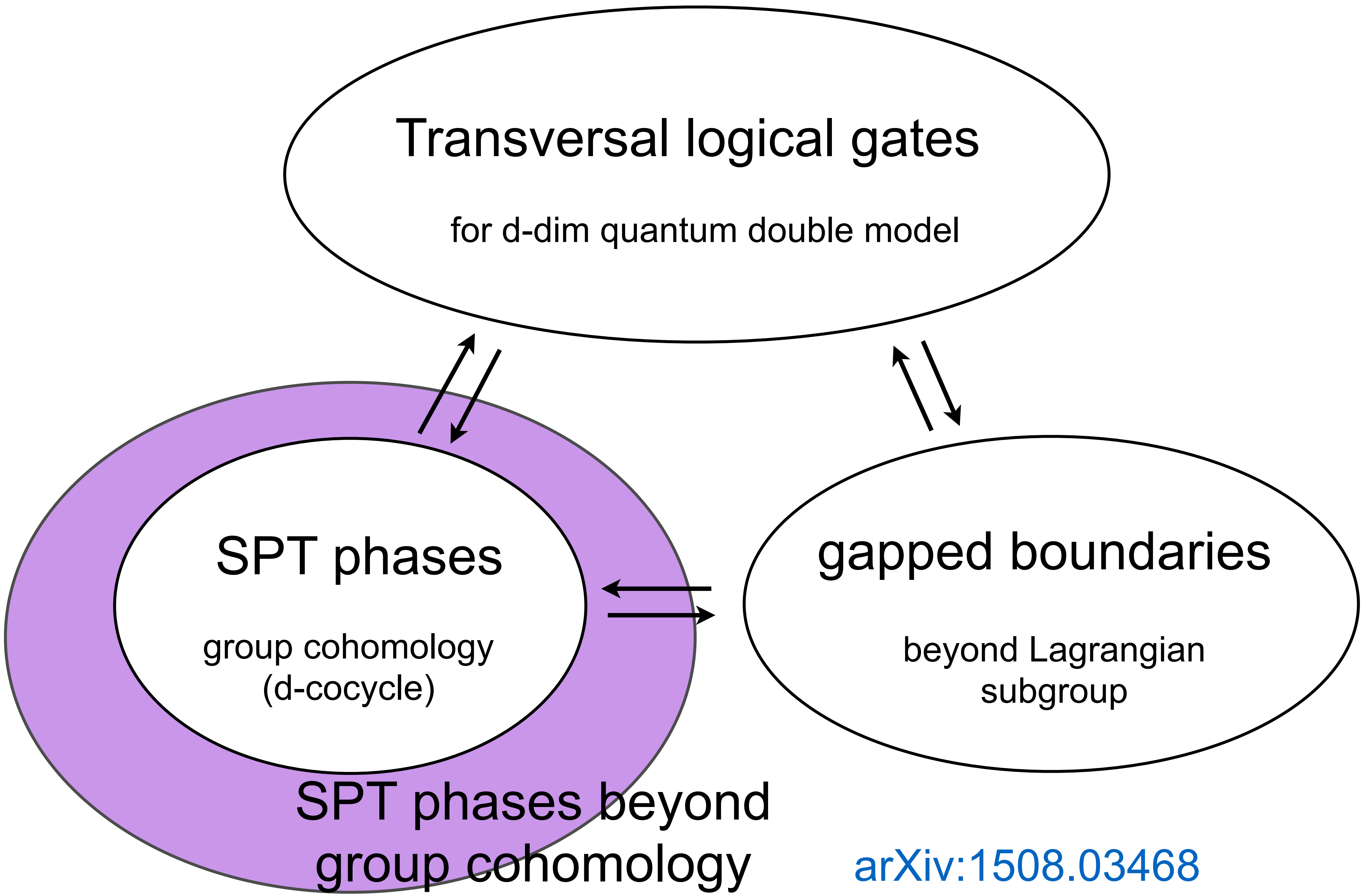
Pauli operators
X,Y,Z

Overview of the results



- arXiv:1503.07208
- arXiv:1509.03626

Overview of the results



arXiv:1508.03468