Quantum conditional mutual information and approximate Markov chains

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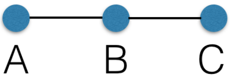

$$I(A : C|B)$$

Correlation measure between A and C from point of view of B

Objective:

$$\boxed{\text{Structure of states on } A \otimes B \otimes C \text{ with } I(A : C|B) \leq \epsilon}$$

Outline:

1. Definition and properties of conditional mutual information
2. How to ensure that $I(A : C|B) \leq \epsilon$?
3. What is the right operational property?
4. Statement and overview of the proof

Entropy and conditioning

Entropy: measure of uncertainty in a system

Shannon entropy for distribution p_X:

$$H(X) = -\sum_x p_X(x) \log p_X(x) \in [0, \log|X|]$$

Quantum von Neumann entropy for density operator ρ_A:

$$H(A) = -\operatorname{tr}(\rho_A \log \rho_A) \in [0, \log|A|]$$

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Multiple systems: State ρ_{AB} acting on $A \otimes B$

Conditional entropy of A from B's viewpoint

$$H(A|B)_\rho = H(AB)_\rho - H(B)_\rho$$

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Conditional entropy of A from B's viewpoint

$$H(A|B)_\rho = H(AB)_\rho - H(B)_\rho$$

Interpretation:

- Classical B: $\rho_{AB} = \sum_b \rho_A(b) \otimes p(b)|b\rangle\langle b|$

$$H(A|B)_\rho = \sum_b p(b) H(A)_{\rho(b)}$$

- Quantum B: More subtle
  $H(A|B)_\rho$ can be negative when ρ entangled

$$-\log |A| \leq H(A|B)_\rho \leq \log |A|$$

Mutual information and conditioning

- State ρ_{AC} acting on $A \otimes C$
  Mutual Information:

$$I(A:C)_\rho = H(C)_\rho - H(C|A)_\rho$$

  - Classical ρ: $0 \leq I(A:C)_\rho \leq \min\{\log|A|, \log|C|\}$
  - Quantum ρ: $0 \leq I(A:C)_\rho \leq 2\min\{\log|A|, \log|C|\}$

- State ρ_{ABC} acting on $A \otimes B \otimes C$
  Conditional Mutual Information:

$$I(A:C|B)_\rho = H(C|B)_\rho - H(C|AB)_\rho$$

  - Classical B: $\rho_{ABC} = \sum_b \rho_{AC}(b) \otimes p(b)|b\rangle\langle b|_B$

$$I(A:C|B)_\rho = \sum_b p(b) I(A:C)_{\rho(b)} \in [0, \min\{\log|A|, \log|C|\}]$$

  - Quantum B: More subtle

$$0 \leq I(A:C|B)_\rho \leq 2\min\{\log|A|, \log|C|\}$$

Useful property

Additivity property make it a very useful measure:

$\boxed{\textbf{Chain rule}}$

$$
\begin{aligned}
&I(A_1 \dots A_n : C|B) \\
&\qquad = I(A_1 : C|B) + I(A_2 : C|BA_1) + \cdots + I(A_n : C|BA_1 \dots A_{n-1})
\end{aligned}
$$

Correlations can be decomposed into parts

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Correlations can be decomposed into parts

Some applications:

- Direct sum results in communication complexity [Talk Braverman et al. tomorrow]
- Entanglement measures (squashed entanglement) [Christandl, Winter, 2003]
- de Finetti-type statements [Raghavendra, Tan, 2011] [Brandao, Harrow, 2013]
- ...

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Typical argument:

1. Total correlation between $A_1 \dots A_n$ and C bounded:
$$I(A_1 \dots A_n : C|B) \leq 2 \log |C|$$
2. Correlation has to be spread:
$$\frac{1}{n} \sum_{i=1}^{n} I(A_i : C|BA_1 \dots A_{i-1}) \leq \frac{2 \log |C|}{n}$$

Sample application

Intuition: loosing one bit can be replaced with some advice

Theorem

For any joint distribution $P_{X_1 \dots X_n YZ}$ where Y is a bit. There exists a subset $L \subset \{1, \dots, n\}$ of length $|L| \leq 1/\epsilon$

$$\text{for all } i, \quad \mathrm{P}_{\mathrm{guess}}(X_i|ZX_L) \geq \mathrm{P}_{\mathrm{guess}}(X_i|ZY) - \sqrt{(2\ln 2)\epsilon}$$

Interpretation: Y can be replaced by a small number of bits of $X_1 \dots X_n$

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Algorithm to construct L

$L \leftarrow \emptyset$
while $\exists i \in \{1, \dots, n\}$ st $I(X_i : Y | Z X_L) > \epsilon$
  $L \leftarrow L \cup \{i\}$

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Claim 1: The algorithm terminates in $< 1/\epsilon$ steps

Claim 2: When algorithm terminates, $\mathrm{P}_{\text{guess}}(X_i|ZX_L) \gtrapprox_\epsilon \mathrm{P}_{\text{guess}}(X_i|ZY)$

Sample application (Proof of claim 1)

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For any joint distribution $P_{X_1 \ldots X_n YZ}$ where Y is a bit. There exists a subset $L \subset \{1, \ldots, n\}$ of length $|L| \leq 1/\epsilon$

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Claim 1: The algorithm terminates in at most $1/\epsilon$ steps
$L = \{i_1, \ldots, i_\ell\}$

$$I(X_L : Y|Z) = \sum_{p=1}^{\ell} I(X_{i_p} : Y|ZX_{i_1 \ldots i_{p-1}}) \geq \ell \cdot \epsilon$$

But $I(X_L : Y|Z) \leq 1$ because Y is one bit

So $\ell \leq \frac{1}{\epsilon}$ $\square$

Sample application (Proof of claim 2)

Theorem

For any joint distribution $P_{X_1 \dots X_n YZ}$ where Y is a bit. There exists a subset $L \subset \{1, \dots, n\}$ of length $|L| \leq 1/\epsilon$

$$\textit{for all } i, \quad \mathrm{P}_{\mathrm{guess}}(X_i|ZX_L) \geq \mathrm{P}_{\mathrm{guess}}(X_i|ZY) - \sqrt{(2\ln 2)\epsilon}$$

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Claim 2: When algorithm terminates, $\mathrm{P}_{\mathrm{guess}}(X_i|ZX_L) \gtrapprox_\epsilon \mathrm{P}_{\mathrm{guess}}(X_i|ZY)$
For all i,

$$\begin{aligned}
\epsilon \geq I(X_i : Y|ZX_L) &= \mathop{\mathbb{E}}_{zx_L} \left\{ I(X_i : Y)_{P_{X_i Y|zx_L}} \right\} \\
&\geq \mathop{\mathbb{E}}_{zx_L} \left\{ \frac{1}{2\ln 2} \left\| P_{X_i Y|zx_L} - P_{X_i|zx_L} \times P_{Y|zx_L} \right\|_1^2 \right\} \\
&\geq \frac{1}{2\ln 2} \left(\mathop{\mathbb{E}}_{zx_L} \left\{ \left\| P_{X_i|zx_L y} - P_{X_i|zx_L} \right\|_1 \right\} \right)^2 \\
&\geq \frac{1}{2\ln 2} \left(\mathrm{P}_{\mathrm{guess}}(X_i|ZX_L Y) - \mathrm{P}_{\mathrm{guess}}(X_i|ZX_L) \right)^2
\end{aligned}$$

$$\mathrm{P}_{\mathrm{guess}}(X_i|ZX_L) \geq \mathrm{P}_{\mathrm{guess}}(X_i|ZX_L Y) - \sqrt{(2\ln 2)\epsilon} \geq \mathrm{P}_{\mathrm{guess}}(X_i|ZY) - \sqrt{(2\ln 2)\epsilon} \quad \square$$

Sample application: quantum systems

Wanted

For any **quantum** *density operator* $\rho_{X_1 \ldots X_n CB}$ *where* C *is a qubit. There exists a subset* $L \subset \{1, \ldots, n\}$ *of length* $|L| \leq 2/\epsilon$

$$\text{for all } i, \quad \mathrm{P}_{\text{guess}}(X_i | B X_L) \geq \mathrm{P}_{\text{guess}}(X_i | BC) - \sqrt{(2 \ln 2)\epsilon}$$

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Claim 2: When algorithm terminates, $\mathrm{P}_{\mathrm{guess}}(X_i|BX_L) \gtrapprox_\epsilon \mathrm{P}_{\mathrm{guess}}(X_i|BC)$? quantum ?
We have for all i, $I(X_i : C|BX_L) \leq \epsilon$. But how to conclude?

$$\epsilon \geq I(X_i : Y|ZX_L) = \mathop{\mathbb{E}}_{zx_L}\left\{ I(X_i : Y)_{P_{X_iY|zx_L}} \right\}$$? quantum ?

$$\geq \mathop{\mathbb{E}}_{zx_L}\left\{ \frac{1}{2\ln 2} \left\| P_{X_iY|zx_L} - P_{X_i|zx_L} \times P_{Y|zx_L} \right\|_1^2 \right\}$$? quantum ?

$$\geq \frac{1}{2\ln 2}\left(\mathrm{P}_{\mathrm{guess}}(X_i|ZX_LY) - \mathrm{P}_{\mathrm{guess}}(X_i|ZX_L)\right)^2$$

Structure of states with small QCMI: $\epsilon = 0$ case

Theorem (Strong subadditivity [Lieb, Ruskai, 1973])

For all quantum states ρ, $I(A:C|B)_\rho \geq 0$

Rewritten: $H(A|B) + H(C|B) \geq H(AC|B)$

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Theorem (QCMI and Markov chains [Petz, 1988])

$$I(A:C|B)_\rho = 0 \quad \Leftrightarrow \quad \exists \mathcal{T} : B \to BC, \ (\mathcal{I}_A \otimes \mathcal{T}_{BC\leftarrow B})(\rho_{AB}) = \rho_{ABC}$$

Interpretation: C can be generated by acting only on B (without acting on A)

Structure of $\mathcal{T}$: $\mathcal{T}_{BC\leftarrow B}(\gamma) = \rho_{BC}^{1/2}\rho_B^{-1/2}(\gamma \otimes \mathrm{id}_C)\rho_B^{-1/2}\rho_{BC}^{1/2}$

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ρ_{ABC} is a **quantum Markov chain:** $\exists \mathcal{T} : B \to BC,\ (\mathcal{I}_A \otimes \mathcal{T}_{BC \leftarrow B})(\rho_{AB}) = \rho_{ABC}$

Illustration of Markov chain condition

$I(A_1 : A_3 | A_2) = 0$

$I(A_1 A_2 : A_4 | A_3) = 0$

$\ddots$

$I(A_1 \ldots A_{n-2} : A_n | A_{n-1}) = 0$

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Then

$$
\begin{aligned}
\rho_{A_1 A_2 A_3} &= (\mathcal{I}_{A_1} \otimes \mathcal{T}_{A_2 A_3 \leftarrow A_2})(\rho_{A_1 A_2}) \\
\rho_{A_1 A_2 A_3 A_4} &= (\mathcal{I}_{A_1 A_2} \otimes \mathcal{T}_{A_3 A_4 \leftarrow A_3})(\rho_{A_1 A_2 A_3}) \\
&\vdots \\
\rho_{A_1 \ldots A_n} &= (\mathcal{I}_{A_1 \ldots A_{n-2}} \otimes \mathcal{T}_{A_{n-1} A_n \leftarrow A_{n-1}})(\rho_{A_1 \ldots A_{n-1}}) \\
\Longrightarrow \quad \rho_{A_1 \ldots A_n} &= \left(\mathcal{T}_{A_{n-1} A_n \leftarrow A_{n-1}} \circ \cdots \circ \mathcal{T}_{A_2 A_3 \leftarrow A_2}\right) (\rho_{A_1 A_2})
\end{aligned}
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I(A1…An-2:An|An-1)=0

Then

$$
\begin{aligned}
\rho_{A_1A_2A_3} &= (\mathcal{I}_{A_1} \otimes \mathcal{T}_{A_2A_3\leftarrow A_2})(\rho_{A_1A_2}) \\
\rho_{A_1A_2A_3A_4} &= (\mathcal{I}_{A_1A_2} \otimes \mathcal{T}_{A_3A_4\leftarrow A_3})(\rho_{A_1A_2A_3}) \\
&\vdots \\
\rho_{A_1\ldots A_n} &= (\mathcal{I}_{A_1\ldots A_{n-2}} \otimes \mathcal{T}_{A_{n-1}A_n\leftarrow A_{n-1}})(\rho_{A_1\ldots A_{n-1}}) \\
\Longrightarrow \rho_{A_1\ldots A_n} &= \left(\mathcal{T}_{A_{n-1}A_n\leftarrow A_{n-1}} \circ \cdots \circ \mathcal{T}_{A_2A_3\leftarrow A_2}\right)(\rho_{A_1A_2})
\end{aligned}
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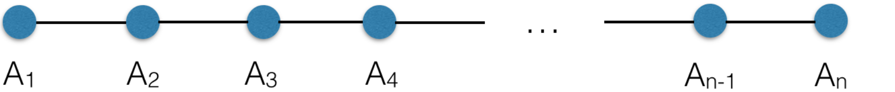
Efficient representation of $\rho_{A_1\ldots A_n}$: only $O(n)$ bits (compare to $2^{\Omega(n)}$ in general)

Illustration of Markov chain condition



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$I(A_1 \ldots A_{n-2} : A_n | A_{n-1}) = 0$

Then

$$\begin{aligned} \rho_{A_1A_2A_3} &= (\mathcal{I}_{A_1} \otimes \mathcal{T}_{A_2A_3 \leftarrow A_2})(\rho_{A_1A_2}) \\ \rho_{A_1A_2A_3A_4} &= (\mathcal{I}_{A_1A_2} \otimes \mathcal{T}_{A_3A_4 \leftarrow A_3})(\rho_{A_1A_2A_3}) \\ &\vdots \\ \rho_{A_1\ldots A_n} &= (\mathcal{I}_{A_1\ldots A_{n-2}} \otimes \mathcal{T}_{A_{n-1}A_n \leftarrow A_{n-1}})(\rho_{A_1\ldots A_{n-1}}) \\ \implies \rho_{A_1\ldots A_n} &= \left(\mathcal{T}_{A_{n-1}A_n \leftarrow A_{n-1}} \circ \cdots \circ \mathcal{T}_{A_2A_3 \leftarrow A_2}\right)(\rho_{A_1A_2}) \end{aligned}$$

Efficient representation of $\rho_{A_1\ldots A_n}$: only $O(n)$ bits (compare to $2^{\Omega(n)}$ in general)

Hope: if $I(A_1 \ldots A_{i-1} : A_{i+1} | A_i) \leq \epsilon$

$\implies$ then efficient approximate representation of $\rho_{A_1\ldots A_n}$

Small QCMI: $\epsilon > 0$ (Known lower bounds on QCMI)

Theorem (Remainder term for SSA [Carlen, Lieb, 2014] see also [Zhang, Wu 2014])

$$I(A:C|B)_\rho \geq \operatorname{tr}\left[\sqrt{\rho_{ABC}} - \exp\left(\frac{1}{2}\log\rho_{AB} - \frac{1}{2}\log\rho_B + \frac{1}{2}\log\rho_{BC}\right)\right]^2$$

Problem: Term $\exp(\log + \log)$ difficult to interpret **operationally**

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Problem: Term $\exp(\log + \log)$ difficult to interpret **operationally**

Theorem (Faithful squashed entanglement [Brandao, Christandl, Yard, 2010])

$$I(A:C|B)_\rho \geq \min_{\sigma_{AC}\ \textit{separable}} \frac{1}{8\ln 2}\|\rho_{AC} - \sigma_{AC}\|^2_{LOCC}$$

Problem: Bound is independent of B, value 0 when A or C classical

QCMI used to quantify entanglement: $E_{sq}(A:C)_\rho = \inf_{\rho_{ABC}} \frac{1}{2} I(A:C|B)_\rho$

Small QCMI: $\epsilon > 0$ case

ρ_{ABC} is a **quantum Markov chain:** $\exists \mathcal{T} : B \to BC, \ (\mathcal{I}_A \otimes \mathcal{T}_{BC\leftarrow B})(\rho_{AB}) = \rho_{ABC}$

Candidate conjecture 1:

$$I(A:C|B)_\rho \leq \epsilon \quad \Rightarrow \quad \rho_{ABC} \approx_{f(\epsilon)} \omega_{ABC}, \text{ with } \omega_{ABC} \text{ Markov chain}$$

Counterexamples [Ibinson, Linden, Winter, 2006] and [Christandl, Schuch, Winter, 2012] $\to f$ has to depend on dimensions

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Candidate conjecture 2:
[Li, Winter, 2012], [Kim, 2013], [Zhang, 2013], [Berta, Seshadreesan, Wilde, 2014]

$$I(A:C|B)_\rho \le \epsilon \quad \Rightarrow \quad \exists \mathcal{T} : B \to BC,\ (\mathcal{I}_A \otimes \mathcal{T})(\rho_{AB}) \approx_\epsilon \rho_{ABC}$$

with $\mathcal{T}(\gamma) = \rho_{BC}^{1/2} \rho_B^{-1/2} (\gamma \otimes \mathrm{id}_C) \rho_B^{-1/2} \rho_{BC}^{1/2}$

Remarks:

- **Conj. 1** and **Conj. 2** are true for **classical** states
- General **quantum** case: **Conj. 2** does **not** imply **Conj. 1**

Main result

A proof of a variant of **Conj. 2**

Theorem

For any ρ_{ABC}, there exists $\mathcal{T} : B \to BC$ such that

$$I(A:C|B)_\rho \geq -2\log F(\rho_{ABC}, \mathcal{T}_{BC\leftarrow B}(\rho_{AB}))$$

Remarks:

- $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$ is the fidelity
- Implies $I(A:C|B)_\rho \geq \frac{1}{4\ln 2}\|\rho_{ABC} - \mathcal{T}_{BC\leftarrow B}(\rho_{AB})\|_1^2$

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Remarks:

- $F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$ is the fidelity
- Implies $I(A : C|B)_\rho \geq \frac{1}{4 \ln 2}\|\rho_{ABC} - \mathcal{T}_{BC\leftarrow B}(\rho_{AB})\|_1^2$
- Properties of the map $\mathcal{T}$

$$\mathcal{T}_{BC\leftarrow B}(\gamma) = V_{BC}\rho_{BC}^{1/2}\rho_B^{-1/2}U_B(\gamma \otimes \mathrm{id}_C)U_B^\dagger\rho_B^{-1/2}\rho_{BC}^{1/2}V_{BC}^\dagger$$

Structure of states ρ_{ABC} with $I(A : C|B)_\rho \leq \epsilon$

$\implies$ states for which C can be approximately reconstructed from B

Back to our sample applications (replacing lost C)

Main result: $I(A:C|B)_\rho \geq \frac{1}{4\ln 2}\|\rho_{ABC} - \mathcal{T}_{BC\leftarrow B}(\rho_{AB})\|_1^2$

Theorem

For any **quantum** *density operator* $\rho_{X_1 \ldots X_n CB}$ *where* C *is a qubit. There exists a subset* $L \subset \{1, \ldots, n\}$ *of length* $|L| \leq 1/\epsilon$

$$\text{for all } i, \quad \mathrm{P}_{\mathrm{guess}}(X_i|BX_L) \geq \mathrm{P}_{\mathrm{guess}}(X_i|BC) - \sqrt{(4\ln 2)\epsilon}$$

Algorithm to construct L

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  $L \leftarrow L \cup \{i\}$

Claim 1: The algorithm terminates in at most $2/\epsilon$ steps quantum ✓

Claim 2: When algorithm terminates, $\mathrm{P}_{\mathrm{guess}}(X_i|BX_L) \gtrapprox_\epsilon \mathrm{P}_{\mathrm{guess}}(X_i|BC)$

We have for all i, $I(X_i : C|BX_L) \leq \epsilon$

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Main result: $I(A:C|B)_\rho \geq \frac{1}{4\ln 2}\|\rho_{ABC} - \mathcal{T}_{BC\leftarrow B}(\rho_{AB})\|_1^2$

Theorem

For any **quantum** *density operator* $\rho_{X_1\ldots X_n CB}$ *where* C *is a qubit. There exists a subset* $L \subset \{1,\ldots,n\}$ *of length* $|L| \leq 1/\epsilon$

$$\text{for all } i, \quad \mathrm{P}_{\mathrm{guess}}(X_i|BX_L) \geq \mathrm{P}_{\mathrm{guess}}(X_i|BC) - \sqrt{(4\ln 2)\epsilon}$$

Algorithm to construct L

$L \leftarrow \emptyset$
while $\exists i \in \{1,\ldots,n\}$ st $I(X_i : C|BX_L) > \epsilon$
  $L \leftarrow L \cup \{i\}$

Claim 1: The algorithm terminates in at most $2/\epsilon$ steps   quantum ✓
Claim 2: When algorithm terminates, $\mathrm{P}_{\mathrm{guess}}(X_i|BX_L) \gtrapprox_\epsilon \mathrm{P}_{\mathrm{guess}}(X_i|BC)$

We have for all i, $I(X_i : C|BX_L) \leq \epsilon$
Apply **main result**: $\rho_{X_iCBX_L} \approx_\delta \mathcal{T}_{BX_LC\leftarrow BX_L}(\rho_{X_iBX_L})$ with $\delta = \sqrt{(4\ln 2)\epsilon}$
Strategy for guessing X_i from B and X_L:

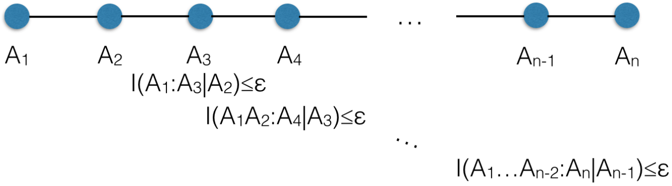
1. Construct the state $\mathcal{T}_{BX_LC\leftarrow BX_L}(\rho_{X_iBX_L})$
2. Pretend the state was $\rho_{X_iCBX_L}$ and use its optimal guessing strategy

$$\mathrm{P}_{\mathrm{guess}}(X_i|BX_L)_\rho \geq \mathrm{P}_{\mathrm{guess}}(X_i|BX_LC)_\rho - \delta \geq \mathrm{P}_{\mathrm{guess}}(X_i|BC)_\rho - \delta$$

Back to our sample applications (long chain)

Main result: $I(A:C|B)_\rho \geq \frac{1}{4\ln 2}\|\rho_{ABC} - \mathcal{T}_{BC\leftarrow B}(\rho_{AB})\|_1^2$

Let $\rho_{A_1\ldots A_n}$ be of the form



Let $\delta = \sqrt{(4\ln 2)\epsilon}$, then

$$\begin{aligned}
\rho_{A_1\ldots A_n} &\approx_\delta \mathcal{T}_{A_{n-1}A_n\leftarrow A_{n-1}}(\rho_{A_1\ldots A_{n-1}}) \\
&\approx_{2\delta} \mathcal{T}_{A_{n-1}A_n\leftarrow A_{n-1}}(\mathcal{T}_{A_{n-2}A_{n-1}\leftarrow A_{n-2}}(\rho_{A_1\ldots A_{n-2}})) \\
&\;\vdots \\
&\approx_{(n-2)\delta} \left(\mathcal{T}_{A_{n-1}A_n\leftarrow A_{n-1}} \circ \cdots \circ \mathcal{T}_{A_2A_3\leftarrow A_2}\right)(\rho_{A_1A_2})
\end{aligned}$$

If $\epsilon \ll \frac{1}{n^2}$, good approximation of $\rho_{A_1\ldots A_n}$ in $O(n)$ memory

Main result: proof sketch

Statement to prove:

$$\exists \mathcal{T} : B \to BC, \quad F(\rho_{ABC}, \mathcal{T}_{BC\leftarrow B}(\rho_{AB})) \geq 2^{-\frac{1}{2} I(A:C|B)}$$

1. Easy special case: **flat marginals** $\rho_B = \frac{\Pi_B}{r_B}$ and $\rho_{BC} = \frac{\Pi_{BC}}{r_{BC}}$

$$F(\rho_{ABC}, \rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}) = \sqrt{\frac{r_{BC}}{r_B}} F(\rho_{ABC}, \Pi_{BC}\Pi_B\rho_{AB}\Pi_B\Pi_{BC})$$
$$\geq 2^{-\frac{1}{2}(H(BC)_\rho - H(B)_\rho)} 2^{-\frac{1}{2}D(\rho_{ABC}\|\rho_{AB}\otimes \mathrm{id}_C)} = 2^{-\frac{1}{2}I(A:C|B)_\rho}$$

2. General case $\to \approx$ flat marginals: **study $\rho^{\otimes n}$** and consider types

$$I(A:C|B)_\rho = \frac{I(A^n : C^n|B^n)_{\rho^{\otimes n}}}{n}$$

   Obtain $\mathcal{T}^n_{B^nC^n\leftarrow B^n}$ such that $F(\rho_{ABC}^{\otimes n}, \mathcal{T}^n(\rho_{AB}^{\otimes n})) \geq 2^{-\frac{1}{2}I(A^n:C^n|B^n)_{\rho^{\otimes n}}}$

3. If $\mathcal{T}^n_{B^nC^n\leftarrow B^n} = \mathcal{T}^{\otimes n}_{BC\leftarrow B}$, done.
   For that, **de Finetti reduction**: $\mathcal{T}^n \leq \mathrm{poly}(n) \int \mathcal{T}^{\otimes n} d\mathcal{T}$

Map of known proofs

$$I(A:C|B)_\rho = \frac{1}{n} I(A:C|B)_{\rho^{\otimes n}}$$

$$\geq$$

[FR15]: Properties of fidelity $\mathbb{D} = -2\log F$

[BHOS15]: Quantum state redistribution $\mathbb{D} = D$

[STH16]: Properties of pinching map $\mathbb{D} = D$

$$\frac{1}{n}\mathbb{D}(\rho_{ABC}^{\otimes n} \| \mathcal{T}^n_{B^n C^n \leftarrow B^n}(\rho_{AB}^{\otimes n}))$$

$$\geq$$

[FR15]: de Finetti reduction

[BT15]: SDP duality

[STH16]: Minimax theorem

$$\frac{1}{n}\mathbb{D}(\rho_{ABC}^{\otimes n} \| \mathcal{T}^{\otimes n}_{BC \leftarrow B}(\rho_{AB}^{\otimes n})) = \mathbb{D}(\rho_{ABC} \| \mathcal{T}_{BC \leftarrow B}(\rho_{AB}))$$

Map of known proofs

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Proof not following this scheme: see next talk

Conclusion

- Conditional mutual information useful for its additivity properties
- **Main result:**

$$I(A:C|B)_\rho \leq \epsilon$$
$$\Rightarrow \ \rho_{ABC} \text{ approximately satisfies Markov chain condition}$$

- Can show a similar upper bound on $I(A:C|B)$:

$$\frac{1}{4\ln 2}\|\rho_{ABC} - \mathcal{T}_{BC\leftarrow B}(\rho_{AB})\|_1^2 \leq I(A:C|B)_\rho \leq 7\log d_A\sqrt{\|\rho_{ABC} - \mathcal{T}_{BC\leftarrow B}(\rho_{AB})\|_1}$$

Conclusion

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- **Open questions:**
  - Many natural improvements: next talk
  - More applications of recoverability: direct sum communication complexity? lower bounds using restricted norms (LOCC, ...)?