Anchoring games for parallel repetition

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CalTech

Roadmap

Games and parallel repetition

Feige's Counterexample

The anchoring transformation and our results

Proof ideas

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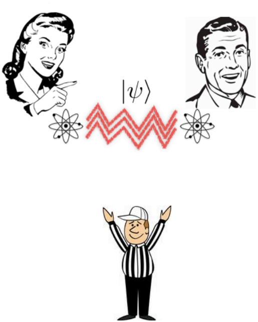
Two player games

Game G: distribution $\mu(x, y)$, predicate $V(x, y, a, b)$



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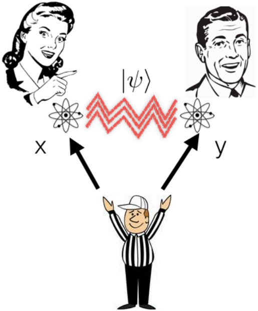

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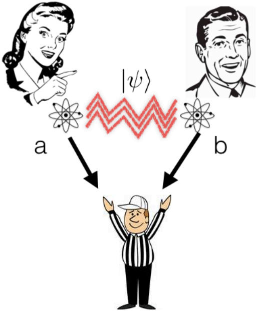

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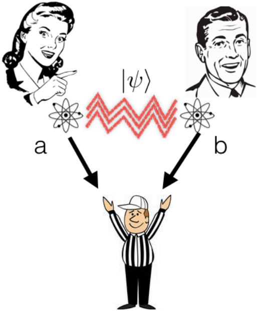

- Referee samples questions $(x,y) \sim \mu$
- Alice gets x, Bob gets y
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- Players **win** iff $V(x,y,a,b) = 1$

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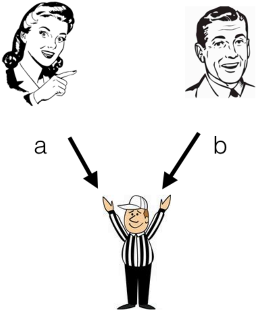

- **Classical value**: $\mathrm{val}(G) =$ maximum winning probability when Alice and Bob's strategies are *local* (i.e. no entanglement).
  - Can assume strategies are **deterministic**.

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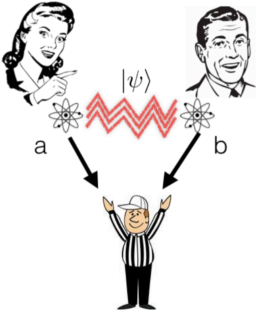

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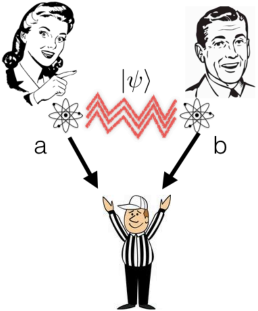

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  - Can assume strategies are **deterministic**.
- **Entangled value**: $\mathrm{val}^\star(G)$ = maximum winning probability when Alice and Bob are *quantumly entangled*.
- **Entanglement can help**: There exist G with $\mathrm{val}(G) < \mathrm{val}^\star(G)$ (e.g. CHSH).

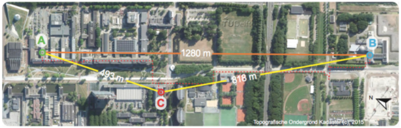
Why games?

Theoretical Computer Science

- Local checking of proofs (classical and quantum), hardness of constraint satisfaction problems, cryptography,...

Testing non-locality in quantum mechanics

- Bell inequality *violations* correspond to $\mathrm{val}^{\star}(G) > \mathrm{val}(G)$
- The recent Loophole-Free Bell Test by [Hensen, et al.] is a two player game in action



Device independent information processing

- Certified random number generation, QKD, delegated quantum computation,...

Testing non-locality in one round

- Consider the Greenberger-Horne-Zeilinger (GHZ) game
  - Three players
  - $\mathrm{val}(GHZ) = 3/4$
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- This is called **gap amplification**.

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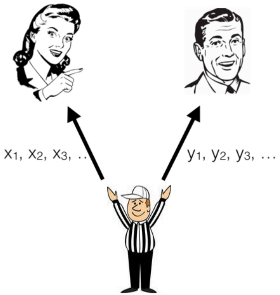
  - Play each GHZ game independently using entangled strategy.
- $\mathrm{val}(\mathrm{GHZ}^n) \overset{?}{\leq} (3/4)^n$

Parallel repetition of games



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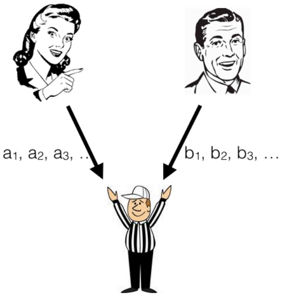

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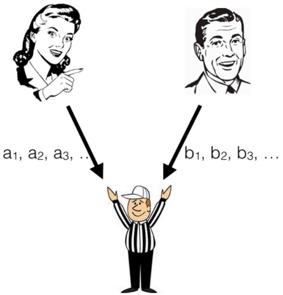

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The Parallel Repetition Question: How does $\mathrm{val}(G^n)$ relate to $\mathrm{val}(G)$ and n?

Parallel repetition of games

- In 1995, Ran Raz proved the **Parallel Repetition Theorem**
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- **Open questions**: Does the Parallel Repetition Theorem hold for:
  - Entangled games?
  - Multiplayer games?
- **Our result**: We change the problem by solving the **gap amplification** problem for entangled games and multiplayer games by introducing a technique called **anchored parallel repetition**.

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Next: Feige's Counterexample G where $\frac{1}{2} = \mathrm{val}(G) = \mathrm{val}(G^2)$.

Feige's Counterexample

- Alice and Bob get uniform and independent bits $x, y \in \{0, 1\}$.
- Alice and Bob both output a statement of the form

"**[Alice/Bob]'s input bit is [$0/1$]**"

- Players win iff their statements agree and are true.

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Examples

Alice gets $x = 0$, Bob gets $y = 1$.

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$$\mathrm{val}(G) = 1/2$$

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The repeated game G^2

- Alice gets $(x_1, x_2) \in \{0,1\}^2$
- Bob gets $(y_1, y_2) \in \{0,1\}^2$
- Alice and Bob have to output two statements of the form:

"[Alice/Bob]'s input bit in G_1 is [$0/1$]"

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- Would expect $\mathrm{val}(G^2) = \mathrm{val}(G)^2 = \frac{1}{4}$.

Feige's Counterexample

Strategy for G^2

Alice says:

"**Alice's input in G_1 is x_1**"

"**Bob's input in G_2 is x_1**"

Bob says:

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- $\Pr[\text{Win } G^2] = \Pr[\text{Win } G_2 | \text{Win } G_1] \times \Pr[\text{Win } G_1] = \frac{1}{2}$.

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- $\Pr[\text{Win } G^2] = \Pr[\text{Win } G_2 | \text{Win } G_1] \times \Pr[\text{Win } G_1] = \frac{1}{2}.$
- **Winning G_2 is correlated with winning G_1!**

$$\mathrm{val}(G^2) = \mathrm{val}(G)$$

Parallel repetition of games

Non-product strategies makes the parallel repetition of games non-trivial!

- The difficulty of non-product strategies is pervasive
  - Additivity conjectures in quantum information
  - Hardness amplification of proof systems
  - Direct sum/product theorems in complexity theory

Parallel repetition of games

Parallel Repetition Theorem [Raz '95, Holenstein '07]

For a two-player game G with $\mathrm{val}(G) = 1 - \varepsilon \geq 1/2$,

$$\mathrm{val}(G^n) = (1 - \varepsilon^3)^{\Omega(n/s)}.$$

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Two major open questions since then:

Does Raz's parallel repetition theorem extend to

1. More than two players?
2. Entangled players?

Answers known for special classes of games.

- **Quantum parallel repetition**
  - XOR games [Cleve, et al. ’08]
  - Unique games [Kempe-Regev-Toner ’08]
  - Free games [Chailloux-Scarpa ’14, Jain-Pereszlényi-Yao ’14, Chung-Wu-Y. ’15]
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General case is still wide open!

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$$G \to G_\perp \to G_\perp^n$$

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For all G, $G_\perp$ is an equivalent game that obeys quantum and multiplayer parallel repetition theorems.

- Technique of *changing the game* for gap amplification is inspired by [Feige-Kilian '94].

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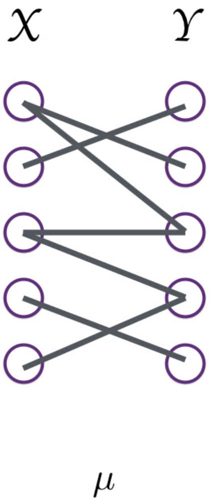
Anchoring transformation

- A two player game G can be viewed as a **graph game**.
- $(x, y) \sim \mu$ is a uniformly random edge from a bipartite graph.



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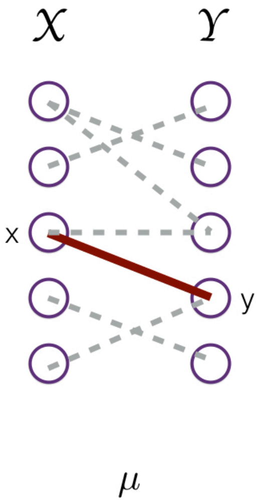

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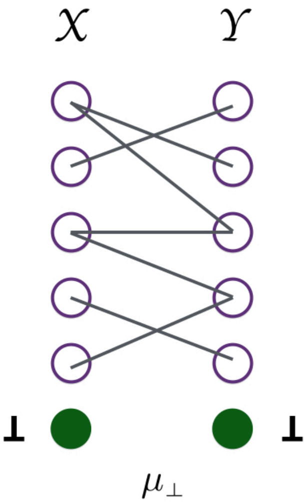
- $G \to G_\perp$: Add **anchor questions**, and connect all other questions to them.



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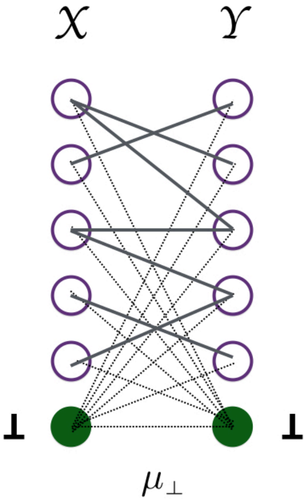

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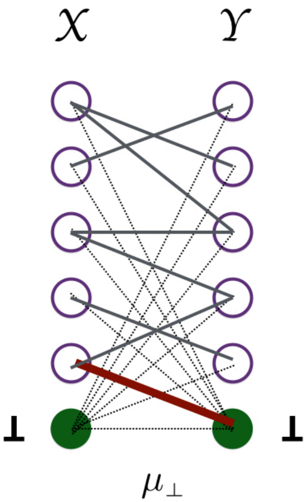

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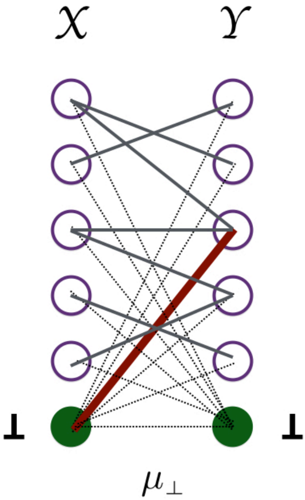

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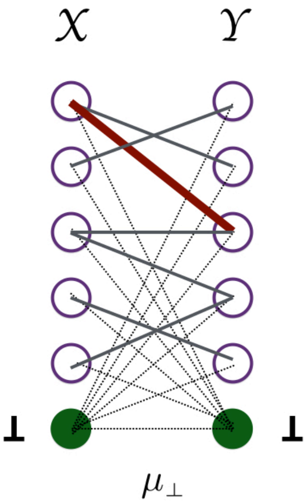
- If either player receives "$\perp$", then players **automatically win**. Otherwise, the referee checks according to G.



Anchored games

G: k-player game with question distribution μ.

In $G_\perp$:

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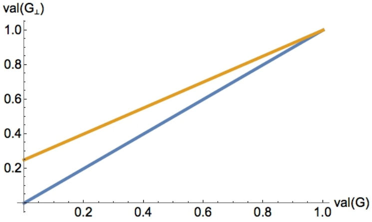
- If any player receives "$\perp$", players win automatically.
- Otherwise, referee checks answers according to G.



$$\mathrm{val}(G_\perp) = p_\alpha \cdot \mathrm{val}(G) + (1 - p_\alpha)$$

$$\mathrm{val}^\star(G_\perp) = p_\alpha \cdot \mathrm{val}^\star(G) + (1 - p_\alpha)$$

$$p_\alpha = (1-\alpha)^k$$

Our results

Theorem: Multiplayer parallel repetition

If G is a k-player game with $\mathrm{val}(G) = 1 - \varepsilon$, then

$$\mathrm{val}(G_{\perp}^{n}) = (1 - \varepsilon^{3})^{\Omega(n/s)}.$$

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Theorem: Quantum parallel repetition

If G is a two-player game with $\mathrm{val}^{\star}(G) = 1 - \delta$, then

$$\mathrm{val}^{\star}(G_{\perp}^{n}) = (1 - \delta^{8})^{\Omega(n/s)}.$$

s denotes answer lengths of the players.

Applications: Testing non-locality

- Greenberger-Horne-Zeilinger (GHZ) game
  - Three players
  - $\mathrm{val}(GHZ) = 3/4$
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  - $\mathrm{val}(GHZ) = 3/4$
  - $\mathrm{val}^\star(GHZ) = 1$
- **Using our multiplayer parallel repetition theorem**:
  - $\mathrm{val}^\star(\mathrm{GHZ}_\perp^n) = 1$
  - $\mathrm{val}(\mathrm{GHZ}_\perp^n) = e^{-\Omega(n)}$.

Applications: Complexity theory

- Suppose it is NP-hard (or QMA-hard) to distinguish between: given G,
  - $\mathrm{val}^\star(G) = 1$, or
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- Suppose it is NP-hard (or QMA-hard) to distinguish between: given G,
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- **Using our quantum parallel repetition theorem**: Then for all $\varepsilon > 0$ it is NP-hard (or QMA-hard) to distinguish between: given G
  - $\mathrm{val}^{\star}(G) = 1$
  - $\mathrm{val}^{\star}(G) = \varepsilon$.

Roadmap

Games and parallel repetition

Feige's Counterexample

The anchoring transformation and our results

Proof ideas

Why does anchoring help?

An anchored game $G_\perp$ is *easier* than G. How could analyzing $G_\perp^n$ be helpful?

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Recall that the difficulty comes from non-product strategies.

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Intuition: Anchor questions randomly "disrupt" careful coordination of players!

- E.g., if Alice's question in G_2 is "$\perp$", she cannot use it to play G_1.
- But Bob doesn't know she received "$\perp$" in G_2!

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Contradiction.

Proof strategy

Fix a *supergood* strategy for G^n.

- Entangled state: $|\psi\rangle$
- For all $\mathbf{x} = (x_1, \ldots, x_n)$, Alice uses POVM $\{A_{\mathbf{x}}(\mathbf{a})\}_{\mathbf{a}}$.
- For all $\mathbf{y} = (y_1, \ldots, y_n)$, Bob uses POVM $\{B_{\mathbf{y}}(\mathbf{b})\}_{\mathbf{b}}$.

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Claim: If $\mathrm{val}^\star(G^n) \gg \mathrm{val}^\star(G)^n$, there exist many i such that

$$\Pr[\text{Win } G_i \,|\, \text{Win } G_1, \ldots, G_{i-1}] > \mathrm{val}^\star(G) + \delta.$$

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Goal: Alice and Bob try to simulate playing G_i *conditioned* on winning games $G_1, \ldots, G_{i-1}$.

Proof strategy

Strategy for G_i:

1. Alice gets x_i, Bob gets y_i.
2. Using local operations Alice and Bob generate the entangled state $|\Phi_{x_i,y_i}\rangle$.
3. Alice and Bob measure the state to obtain answers a_i and b_i.

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Strategy for G_i:

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Intuition: The state $|\Phi_{x_i,y_i}\rangle$ represents the behavior of Alice and Bob in G^n conditioned on winning games $G_1,\ldots,G_{i-1}$ and inputs (x_i,y_i).

What is $|\Phi_{x_i,y_i}\rangle$?

Imagine the following experiment: Alice and Bob play G^n using the optimal entangled strategy, but their inputs $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are in **superposition**.

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Initially, the state of Alice and Bob is

$$|\Phi\rangle = \sum_{\mathbf{x},\mathbf{y}} \sqrt{\mu(\mathbf{x},\mathbf{y})}\ |\mathbf{x}\rangle \otimes |\psi\rangle \otimes |\mathbf{y}\rangle$$

Recall that $\mu(\mathbf{x},\mathbf{y}) = \mu(x_1, y_1) \times \cdots \times \mu(x_n, y_n)$.

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Controlled by their question registers, they apply their measurements, and coherently store their answers.

$$|\Phi\rangle = \sum_{\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}} \sqrt{\mu(\mathbf{x},\mathbf{y})}\ |\mathbf{x}\rangle|\mathbf{a}\rangle \otimes |\psi_{\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}}\rangle \otimes |\mathbf{y}\rangle|\mathbf{b}\rangle$$

$$|\psi_{\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}}\rangle = \left(\sqrt{A_{\mathbf{x}}(\mathbf{a})} \otimes \sqrt{B_{\mathbf{y}}(\mathbf{b})}\right)|\psi\rangle.$$

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After conditioning the state on winning $G_1, \dots, G_{i-1}$ and (x_i, y_i) we have

$$|\Phi_{x_i,y_i}\rangle \propto \sum_{\substack{\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}: \\ V(x_j,y_j,a_j,b_j)=1\ \forall j<i \\ x_i,y_i}} \sqrt{\mu(\mathbf{x},\mathbf{y})}\ |\mathbf{x}\rangle|\mathbf{a}\rangle \otimes |\psi_{\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}}\rangle \otimes |\mathbf{y}\rangle|\mathbf{b}\rangle$$

In other words: $|\Phi_{x_i,y_i}\rangle$ is the *post-measurement state* of Alice and Bob in the game G^n, conditioned on questions (x_i, y_i) and the having won games $G_1, \dots, G_{i-1}$.

Why is $|\Phi_{x_i,y_i}\rangle$ useful?

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- **It's not clear how to locally sample $|\Phi_{x_i,y_i}\rangle$!**
- This is the primary challenge of proving a quantum parallel repetition theorem.

Anchoring our way to parallel repetition

We carefully define $|\Phi_{x_i,y_i}\rangle$ so that

1. it allows Alice and Bob to win G_i with high probability, and
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Key Lemma

On average over $(x_i, y_i) \sim \mu$,

$$|\Phi_{x_i,y_i}\rangle \approx U_{x_i} \otimes V_{y_i} |\Phi_{\perp,\perp}\rangle$$

for some unitaries U_{x_i}, V_{y_i}.

Ingredients in our Key Lemma

- We bound

$$I(X_i;B)_\Phi \le \varepsilon \qquad \text{and} \qquad I(Y_i;A)_\Phi \le \varepsilon$$

- By Pinsker, this implies that Bob's side of entanglement in Φ is nearly independent of X_i, and Alice's side of Φ is nearly independent of Y_i.
- (**The Church of the Larger Hilbert Space**) Uhlmann's Theorem implies there exist unitaries U_{x_i} and V_{y_i} so that

$$|\Phi_{x_i,\perp}\rangle \approx U_{x_i} \otimes \mathbb{I}|\Phi_{\perp,\perp}\rangle \qquad \text{and} \qquad |\Phi_{\perp,y_i}\rangle \approx \mathbb{I} \otimes V_{y_i}|\Phi_{\perp,\perp}\rangle$$

- Using the fact that the "$\perp$" questions anchor all other questions, we can stitch together these two statements to get

$$|\Phi_{x_i,y_i}\rangle \approx U_{x_i} \otimes V_{y_i}|\Phi_{\perp,\perp}\rangle$$

Theorem: Quantum parallel repetition

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Conclusion

What we showed: We give the first gap amplification results for general games in the quantum and multiplayer settings.

Future directions

- Other applications of anchoring?
  - Direct sums in communication complexity?
  - Additivity questions in quantum information
- Prove a general quantum parallel repetition theorem
  - First, prove that $\mathrm{val}^\star(G^n)$ **goes to 0 as $n \to \infty$** for all G when $\mathrm{val}^\star(G) < 1$.
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Thank you! Any questions?