Additivity in Classical and Quantum Information Theory

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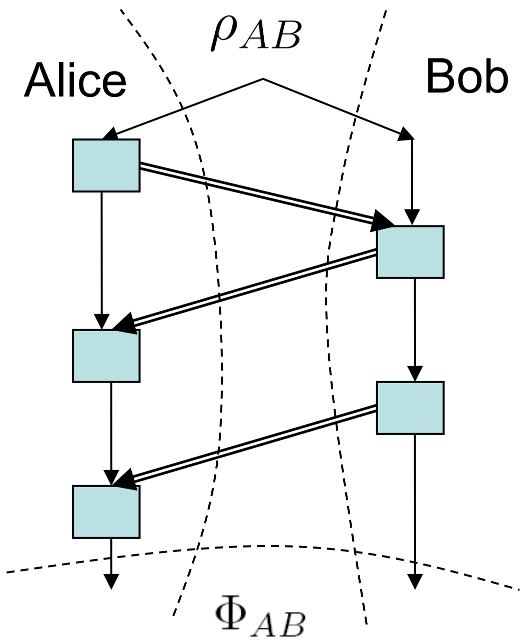
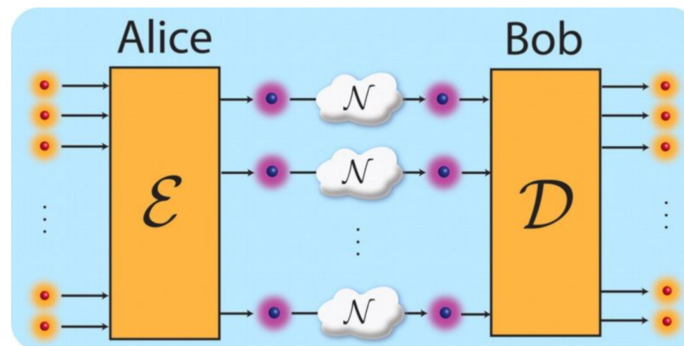
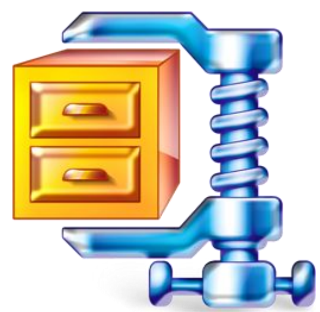
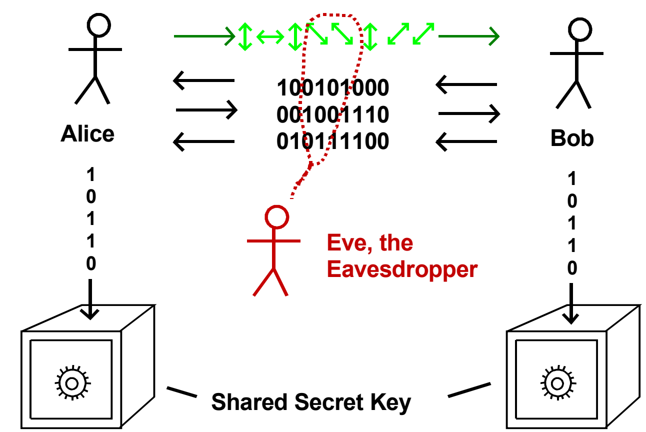
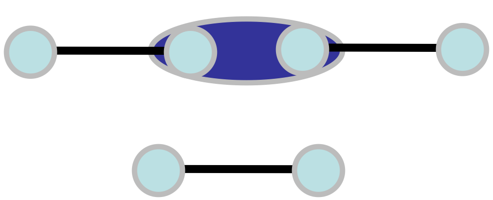
Information theory: optimal rates in sending, storing, processing data

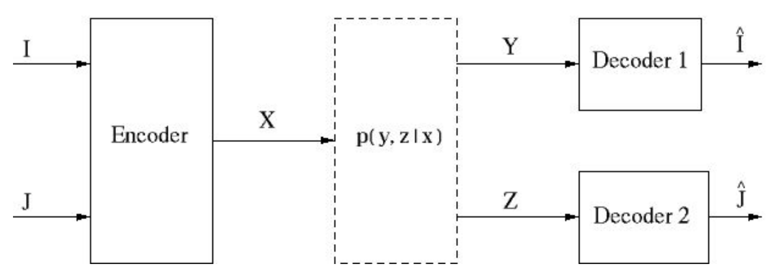






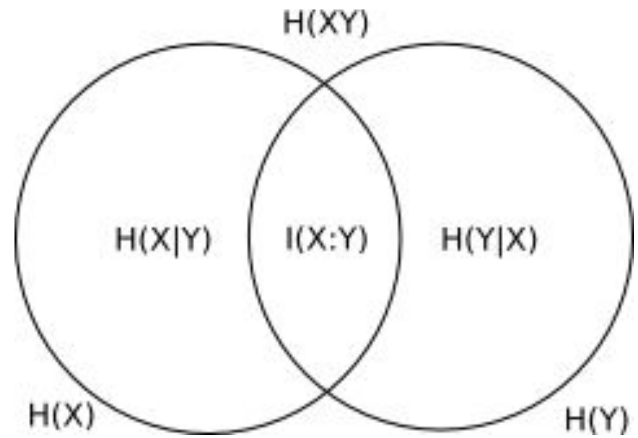






Entropy formulas quantify the answers

- $H(X) = -\sum_x p_x \log p_x$
- $H(\rho) = -\mathrm{Tr}\, \rho \log \rho$



- Optimal Compression: H(X)
- Schumacher Compression: $H(\rho)$
- Classical Channel capacity: max I(X;Y)
  $I(X;Y) = H(X) + H(Y) - H(XY)$
- Quantum Communication: max {H(B) – H(E)}
- Private capacity: max {I(V;B)-I(V;E)}

Additivity lets us calculate answers

C( + )

= C()+C()

Classical Capacity of Classical Channel

Nonadditivity is the rule Especially quantumly

- Good: Better rates, e.g., for classical and quantum communication.

- Bad:
  Mostly don't know capacities, distillable entanglement, etc.

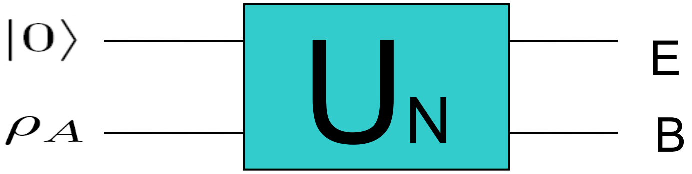
  Have upper and lower bounds that are far apart.

Outline

1. Entropy formulas and their additivity proofs
2. All the uniformly additive formulas under standard decouplings
3. Standard decoupling is typical
4. Completely coherent information: a new additive quantity
5. Observation:classical-quantum correspondence

Entropy formulas

- Quantum channel: unitary interaction with a inaccessible environment



- Entropy formula : linear combination of entropies

$$f_\alpha(U_\mathcal{N}, \phi_{V_1 \ldots V_n A}) = \sum_{s \in \mathcal{P}(V_1 \ldots V_n BE)} \alpha_s H(s)_\rho$$

with $\rho_{V_1 \ldots V_n BE} = (I \otimes U_\mathcal{N}) \phi_{V_1 \ldots V_n A} (I \otimes U_\mathcal{N}^\dagger)$

- Maximized version: $f_\alpha(U_\mathcal{N}) = \max_{\phi_{V_1 \ldots V_n A}} f_\alpha(U_\mathcal{N}, \phi_{V_1 \ldots V_n A})$
- Additivity: $f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}) = f_\alpha(U_{\mathcal{N}_1}) + f_\alpha(U_{\mathcal{N}_2})$

Additivity Proofs

$$f_\alpha(U_{\mathcal{N}}) = \max_{\phi_{V_1 \ldots V_n A}} f_\alpha(U_{\mathcal{N}}, \phi_{V_1 \ldots V_n A})$$

$$f_\alpha(U_{\mathcal{N}}, \phi_{V_1 \ldots V_n A}) = \sum_{s \in \mathcal{P}(V_1 \ldots V_n BE)} \alpha_s H(s)_\rho$$

- Enough to show subadditive:

additive: $\boxed{f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}) = f_\alpha(U_{\mathcal{N}_1}) + f_\alpha(U_{\mathcal{N}_2})}$

"$\geq$" is obvious $\Updownarrow$

subadditive: $\boxed{f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}) \leq f_\alpha(U_{\mathcal{N}_1}) + f_\alpha(U_{\mathcal{N}_2})}$

$\Updownarrow$

$$\boxed{f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}, \phi_{12}^*) \leq f_\alpha(U_{\mathcal{N}_1}, \phi_1^*) + f_\alpha(U_{\mathcal{N}_2}, \phi_2^*)}$$

Standard Additivity Proof

- Additivity proofs: two key steps

  1) Decoupling: $$\phi_{12} \begin{array}{l} \nearrow \tilde{\phi}_1 \\ \searrow \hat{\phi}_2 \end{array}$$

  2) Apply entropy inequalities to show

$$f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}, \phi_{12}) \leq f_\alpha(U_{\mathcal{N}_1}, \tilde{\phi}_1) + f_\alpha(U_{\mathcal{N}_2}, \hat{\phi}_2)$$

- We call f_α uniformly (sub)-additive under the given decoupling. The set of all such formulas are called the additive cone.

A canonical example

- Entanglement assisted capacity:

$$C_{ea}(\mathcal{N}) = \mathbf{max}_{\phi_{VA}} I(V;B)$$

1) Decoupling

$$\phi_{VA_1A_2} \rightarrow \hat{\phi}_{\hat{V}A_2} = \phi_{VB_1|A_2}$$

$$\phi_{VA_1A_2} \rightarrow \tilde{\phi}_{\tilde{V}A_1} = \phi_{VA_1}$$

2) Entropy inequality

$$\begin{aligned} I(V;B_1B_2) &= I(V;B_1) + I(V;B_2|B_1) \\ &= I(V;B_1) + I(VB_1;B_2) - I(B_1;B_2) \\ &\leq I(V;B_1) + I(VB_1;B_2) \leq C_{ea}(\mathcal{N}_1) + C_{ea}(\mathcal{N}_2) \end{aligned}$$

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Decoupling

- We focus on "standard decoupling".

$$
\begin{tikzcd}
 & \phi_{\tilde{V}_1\dots\tilde{V}_nA_1} \\
\phi_{V_1\dots V_nA_1A_2} \arrow[ur] \arrow[dr] \arrow[d, dotted, bend right, "\mathcal{N}_1"'] \arrow[d, dotted, "\mathcal{N}_2"] & \\
(\phi_{V_1\dots V_nB_1E_1A_2},\ \phi_{V_1\dots V_nB_2E_2A_1}) & \phi_{\hat{V}_1\dots\hat{V}_nA_2}
\end{tikzcd}
$$

$$\tilde{V}_i = \tilde{M}_2V_i\,, \quad \tilde{M}_2 \in \mathcal{P}(B_2E_2); \quad \hat{V}_i = \hat{M}_1V_i\,, \quad \hat{M}_1 \in \mathcal{P}(B_1E_1)$$

- Example

$$\tilde{V}_1 = V_1B_2,\ \tilde{V}_2 = V_2E_2,\ \tilde{V}_3 = V_3$$

$$\hat{V}_1 = V_1,\ \hat{V}_2 = V_2B_1E_1,\ \hat{V}_3 = V_3$$

Entropy Inequalities

- Strong subadditivity:

  $I(A;B|C) = H(AC)+ H(BC)-H(ABC)-H(C)>=0$

  [H(A)>=0, H(A)+H(B)-H(AB)>=0, H(AB)+H(A)-H(B)>=0, H(AB)+H(AC)-H(B)-H(C)>=0]

- There may be more, but we don’t know them! (Classically, there is more: H(A|B)>=0, Non-Shannon inequalities.)

- Luckily, we don't need them ^_^

Zero Auxiliary Variable

$$f_\alpha(\mathcal{N}, \phi_A) = \alpha_B H(B) + \alpha_E H(E) + \alpha_{BE} H(BE)$$

$$\text{Decoupling: } \phi_{A_1 A_2} \to (\phi_{A_1}, \phi_{A_2})$$

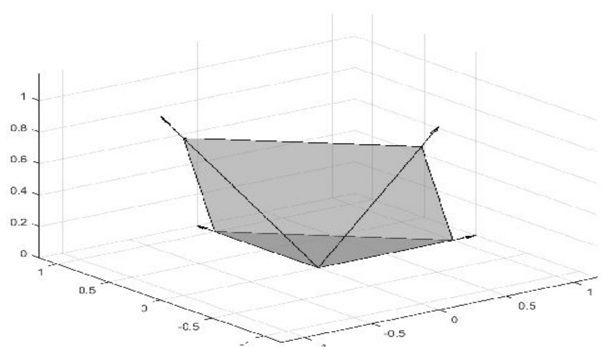
$$\Pi^{\varnothing} := \{f_\alpha \mid f_\alpha(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}, \phi_{A_1 A_2}) \leq f_\alpha(U_{\mathcal{N}_1}, \phi_{A_1}) + f_\alpha(U_{\mathcal{N}_2}, \phi_{A_2})\}$$

Result: $\boxed{\text{full characterization of } \Pi^{\varnothing}}$

$\Pi^{\varnothing}$ Rays

$$f_\alpha = \lambda_1 H(B) + \lambda_2 H(E) + \lambda_3 H(B|E) + \lambda_4 H(E|B)$$

$$\lambda_i \geq 0$$



$\Pi^{\varnothing}$ Faces

$$\alpha_B + \alpha_{BE} \geq 0$$
$$\alpha_E + \alpha_{BE} \geq 0$$
$$\alpha_B + \alpha_E + \alpha_{BE} \geq 0$$
$$\alpha_{BE} \geq 0.$$

Anything inside the cone is uniformly additive. Outside the cone, there is A state that makes f_α not subadditive.

One Auxiliary Variable

At first, consider

$$f_{\alpha^V}(\mathcal{N}, \phi_{VA}) = \alpha_V H(V) + \alpha_{BV} H(BV) + \alpha_{EV} H(EV) + \alpha_{BEV} H(BEV)$$

Fix a standard decoupling:

$\tilde{V} \in \{V, B_2V, E_2V, B_2E_2V\}$ and

$\hat{V} \in \{V, B_1V, E_1V, B_1E_1V\}$

These are labeled by (a,b) $a,b = 0...3$

for each decoupling (a,b), define the additive cone:

$$\Pi^{V,(a,b)} :=$$

$$\{f_{\alpha^V} \mid f_{\alpha^V}(U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2}, \phi_{VA_1A_2}) \leq f_{\alpha^V}(U_{\mathcal{N}_1}, \phi_{\tilde{V}A_1}) + f_{\alpha^V}(U_{\mathcal{N}_2}, \phi_{\hat{V}A_2})\}$$

We give a full characterization of $\Pi^{V,(a,b)}$

One Auxiliary Variable

The additive cone $\prod^{V,(a,b)}$

case	(a,b)	$\hat{M}_1$	$\tilde{M}_2$	equivalents	Additive Cone	Extreme Rays
1.	(3,3)	B_1E_1	B_2E_2	(0,0)	$\alpha_V + \alpha_{BV} + \alpha_{EV} \geq 0$ $\alpha_V + \alpha_{BV} \geq 0$ $\alpha_V + \alpha_{EV} \geq 0$ $\alpha_V \geq 0$	$-H(E\|BV)$ $-H(E\|V)$ $-H(B\|EV)$ $-H(B\|V)$
2.	(3,2)	B_1E_1	E_2	$(2,3),(3,1)$ $(1,3),(1,0),(0,1)$ $(2,0),(0,2)$	$\alpha_{BV} \leq 0$ $\alpha_V + \alpha_{BV} \geq 0$	$-H(BE\|V)$ $\pm H(B\|EV)$ $-H(B\|V)$
3.	(3,0)	B_1E_1	ϕ	(0,3)	$\alpha_{EV} \leq 0$ $\alpha_{BV} \leq 0$	$H(E\|BV)$ $-H(E\|V)$ $\pm H(BE\|V)$
4.	(1,1)	B_1	B_2	(2,2)	$\alpha_{EV} = 0$ $\alpha_V \geq 0$ $\alpha_{BEV} \geq 0$	$-H(B\|V)$ $H(E\|BV)$
5.	(1,2)	B_1	E_2	(2,1)	$\alpha_{BEV} \geq 0$ $\alpha_V \geq 0$	$\pm[H(EV) - H(BV)]$ $H(E\|BV)$ $-H(E\|V)$

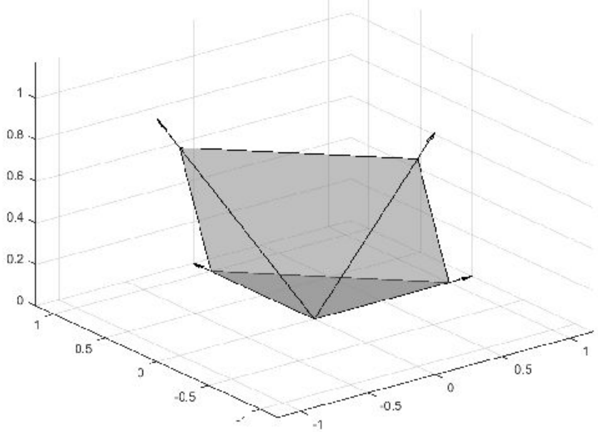
One Auxiliary Variable

$$f_\alpha = f_{\alpha^\varnothing} + f_{\alpha^V}$$

$$f_{\alpha^\varnothing} := \alpha_B H(B) + \alpha_E H(E) + \alpha_{BE} H(BE)$$
$$f_{\alpha^V} := \alpha_V H(V) + \alpha_{BV} H(BV) + \alpha_{EV} H(EV) + \alpha_{BEV} H(BEV)$$

Result: $\boxed{f_\alpha(U_\mathcal{N}) \text{ U.A. w.r.t. } (a,b) \quad \text{iff} \quad f_{\alpha^\varnothing} \in \Pi^\varnothing \;\&\; f_{\alpha^V} \in \Pi^{V,(a,b)}}$

$\Pi^\varnothing$



$f_{\alpha^\varnothing}$

$\Pi^{V,(a,b)}$

case	(a,b)	$\hat{M}_1$	$\tilde{M}_2$	equivalents	Additive Cone	Extreme Rays
1.	(3,3)	B_1E_1	B_2E_2	(0,0)	$\alpha_V + \alpha_{BV} + \alpha_{EV} \geq 0$ $\alpha_V + \alpha_{BV} \geq 0$ $\alpha_V + \alpha_{EV} \geq 0$ $\alpha_V \geq 0$	$-H(E\|BV)$ $-H(E\|V)$ $-H(B\|EV)$ $-H(B\|V)$
2.	(3,2)	B_1E_1	E_2	$(2,3),(3,1)$ $(1,3),(1,0),(0,1)$ $(2,0),(0,2)$	$\alpha_{BV} \leq 0$ $\alpha_V + \alpha_{BV} \geq 0$	$-H(BE\|V)$ $\pm H(B\|EV)$ $-H(B\|V)$
3.	(3,0)	B_1E_1	ϕ	(0,3)	$\alpha_{EV} \leq 0$ $\alpha_{BV} \leq 0$	$H(E\|BV)$ $-H(E\|V)$ $\pm H(BE\|V)$
4.	(1,1)	B_1	B_2	(2,2)	$\alpha_{EV} = 0$ $\alpha_V \geq 0$ $\alpha_{BEV} \geq 0$	$-H(B\|V)$ $H(E\|BV)$
5.	(1,2)	B_1	E_2	(2,1)	$\alpha_{BEV} \geq 0$ $\alpha_V \geq 0$	$\pm[H(EV) - H(BV)]$ $H(E\|BV)$ $-H(E\|V)$

f_{α^V}

Many Auxiliary Variables (of number n)

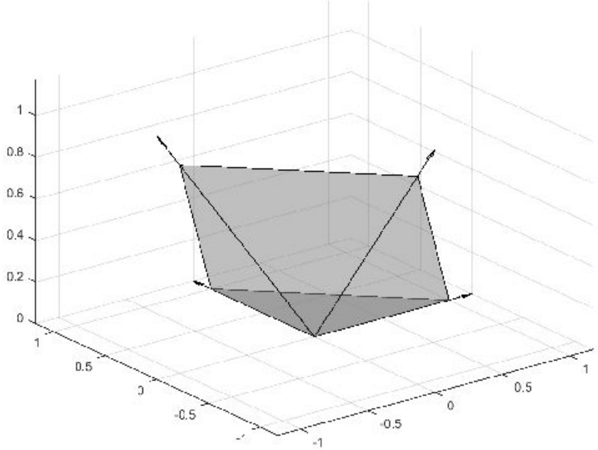
$$f_\alpha = \sum_{S \in \mathcal{P}(V_1 \ldots V_n)} f_{\alpha^S}$$

$$f_{\alpha^S} := \alpha_S H(S) + \alpha_{BS} H(BS) + \alpha_{ES} H(ES) + \alpha_{BES} H(BES)$$

(e.g., when n=2, $f_\alpha = f_{\alpha^\varnothing} + f_{\alpha^{V_1}} + f_{\alpha^{V_2}} + f_{\alpha^{V_1 V_2}}$)

Result: $f_\alpha(U_\mathcal{N})$ U.A. w.r.t. $(a_1, b_1) \ldots (a_n, b_n)$ iff $f_{\alpha^S} \in \Pi^{S,(a_S, b_S)}$

$$\Pi^\varnothing$$



$f_{\alpha^\varnothing}$

$$\Pi^{V,(a,b)}$$

case	(a,b)	$\hat{M}_1$	$\tilde{M}_2$	equivalents	Additive Cone	Extreme Rays
1.	(3,3)	B_1E_1	B_2E_2	(0,0)	$\alpha_V + \alpha_{BV} + \alpha_{EV} \geq 0$ $\alpha_V + \alpha_{BV} \geq 0$ $\alpha_V + \alpha_{EV} \geq 0$ $\alpha_V \geq 0$	$-H(E\|BV)$ $-H(E\|V)$ $-H(B\|EV)$ $-H(B\|V)$
2.	(3,2)	B_1E_1	E_2	$(2,3), (3,1)$ $(1,3), (1,0), (0,1)$ $(2,0), (0,2)$	$\alpha_{BV} \leq 0$ $\alpha_V + \alpha_{BV} \geq 0$	$-H(BE\|V)$ $\pm H(B\|EV)$ $-H(B\|V)$
3.	(3,0)	B_1E_1	ϕ	(0,3)	$\alpha_{EV} \leq 0$ $\alpha_{BV} \leq 0$	$H(E\|BV)$ $-H(E\|V)$ $\pm H(BE\|V)$
4.	(1,1)	B_1	B_2	(2,2)	$\alpha_{EV} = 0$ $\alpha_V \geq 0$ $\alpha_{BEV} \geq 0$	$-H(B\|V)$ $H(E\|BV)$
5.	(1,2)	B_1	E_2	(2,1)	$\alpha_{BEV} \geq 0$ $\alpha_V \geq 0$	$\pm[H(EV) - H(BV)]$ $H(E\|BV)$ $-H(E\|V)$

f_{α^S} , $S \neq \varnothing$

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1. Entropy formulas and their additivity proofs
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Non-standard Decouplings

$$\begin{tikzcd}
 & \phi_{\tilde{V}_1\ldots\tilde{V}_n A_1} \\
\phi_{V_1\ldots V_n A_1 A_2} \arrow[ur] \arrow[dr] \arrow[d, dotted, "\mathcal{N}_1"', "\mathcal{N}_2"] & \\
(\phi_{V_1\ldots V_n B_1 E_1 A_2},\ \phi_{V_1\ldots V_n B_2 E_2 A_1}) & \phi_{\hat{V}_1\ldots\hat{V}_n A_2}
\end{tikzcd}$$

- Standard decoupling (a special relabeling)

$$\tilde{V}_i = \tilde{M}_2 V_i\,, \quad \tilde{M}_2 \in \mathcal{P}(B_2E_2); \quad \hat{V}_i = \hat{M}_1 V_i\,, \quad \hat{M}_1 \in \mathcal{P}(B_1E_1)$$

- Consistent Decoupling (general relabeling)

$$\tilde{V}_i \in \mathcal{P}(V_1 \ldots V_n B_2 E_2) \text{ with } \tilde{V}_i \cap \tilde{V}_j = \varnothing;$$

$$\hat{V}_i \in \mathcal{P}(V_1 \ldots V_n B_1 E_1) \text{ with } \hat{V}_i \cap \hat{V}_j = \varnothing.$$

example:

$$\tilde{V}_1 = V_2 B_2, \tilde{V}_2 = V_3 E_2, \tilde{V}_3 = V_1$$

$$\hat{V}_1 = V_2 V_3, \hat{V}_2 = B_1, \hat{V}_3 = \varnothing$$

Non-standard Decouplings

Result: Among consistent decouplings, standard ones suffice.

$(\forall)$ $f_\alpha(U_\mathcal{N}, \phi_{V_1 \ldots V_n A})$ being uniformly subadditive w.r.t. a consistent decoupling,

$(\exists)$ $f_\beta(U_\mathcal{N}, \varphi_{V_1 \ldots V_m A})$ with $m \leq n$, being uniformly subadditive w.r.t. a standard decoupling, such that

$$\max_{\phi_{V_1 \ldots V_n A}} f_\alpha(U_\mathcal{N}, \phi_{V_1 \ldots V_n A}) = \max_{\varphi_{V_1 \ldots V_m A}} f_\beta(U_\mathcal{N}, \varphi_{V_1 \ldots V_m A}).$$

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Completely Coherent Information

case	(a,b)	$\hat{M}_1$	$\tilde{M}_2$	equivalents	Additive Cone	Extreme Rays
1.	(3,3)	B_1E_1	B_2E_2	(0,0)	$\alpha_V + \alpha_{BV} + \alpha_{EV} \geq 0$ $\alpha_V + \alpha_{BV} \geq 0$ $\alpha_V + \alpha_{EV} \geq 0$ $\alpha_V \geq 0$	$-H(E\|BV)$ $-H(E\|V)$ $-H(B\|EV)$ $-H(B\|V)$
2.	(3,2)	B_1E_1	E_2	$(2,3),(3,1)$ $(1,3),(1,0),(0,1)$ $(2,0),(0,2)$	$\alpha_{BV} \leq 0$ $\alpha_V + \alpha_{BV} \geq 0$	$-H(BE\|V)$ $\pm H(B\|EV)$ $-H(B\|V)$
3.	(3,0)	B_1E_1	ϕ	(0,3)	$\alpha_{EV} \leq 0$ $\alpha_{BV} \leq 0$	$H(E\|BV)$ $-H(E\|V)$ $\pm H(BE\|V)$
4.	(1,1)	B_1	B_2	(2,2)	$\alpha_{EV} = 0$ $\alpha_V \geq 0$ $\alpha_{BEV} \geq 0$	$-H(B\|V)$ $H(E\|BV)$
5.	(1,2)	B_1	E_2	(2,1)	$\alpha_{BEV} \geq 0$ $\alpha_V \geq 0$	$\pm[H(EV) - H(BV)]$ $H(E\|BV)$ $-H(E\|V)$

Completely Coherent Information

case	(a,b)	$\hat{M}_1$	$\tilde{M}_2$	equivalents	Additive Cone	Extreme Rays
1.	(3,3)	B_1E_1	B_2E_2	(0,0)	$\alpha_V + \alpha_{BV} + \alpha_{EV} \geq 0$ $\alpha_V + \alpha_{BV} \geq 0$ $\alpha_V + \alpha_{EV} \geq 0$ $\alpha_V \geq 0$	$-H(E\|BV)$ $-H(E\|V)$ $-H(B\|EV)$ $-H(B\|V)$
2.	(3,2)	B_1E_1	E_2	$(2,3), (3,1)$ $(1,3), (1,0), (0,1)$ $(2,0), (0,2)$	$\alpha_{BV} \leq 0$ $\alpha_V + \alpha_{BV} \geq 0$	$-H(BE\|V)$ $\pm H(B\|EV)$ $-H(B\|V)$
3.	(3,0)	B_1E_1	ϕ	(0,3)	$\alpha_{EV} \leq 0$ $\alpha_{BV} \leq 0$	$H(E\|BV)$ $-H(E\|V)$ $\pm H(BE\|V)$
4.	(1,1)	B_1	B_2	(2,2)	$\alpha_{EV} = 0$ $\alpha_V \geq 0$ $\alpha_{BEV} \geq 0$	$-H(B\|V)$ $H(E\|BV)$
5.	(1,2)	B_1	E_2	(2,1)	$\alpha_{BEV} \geq 0$ $\alpha_V \geq 0$	$\boxed{\pm[H(EV) - H(BV)]}$ $H(E\|BV)$ $-H(E\|V)$

Completely Coherent Information

$$I^{cc}(\mathcal{N}) = \max_{\phi_{VA}}[H(VB) - H(VE)]$$

properties:

- Symmetric in B $\longleftrightarrow$ E .
- Lower bound for cost of swapping B and E.

  [J. Oppenheim and A. Winter, arXiv:quant-ph/0511082]
- Upper bound for simultaneous quantum communication rate to B and E.
- For degradable channels, $I^{cc}(N) = Q(N) = Q^{(1)}(N)$.
- WANT: operational meaning.

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A Classical-Quantum Coincidence

- we can do this whole game for classical entropy formulas too.
- we get exactly the same set of uniformly additive functions.
- Could have been more, since there are more classical inequalities: $H(X|Y) >= 0$.
- But uniform additivity only uses strong subadditivity.

Open Questions

- Additivity other than uniform additivity?
- More general decouplings?

$$\begin{tikzcd} & \tilde{\phi}_1 \\ \phi_{12} \arrow[ur] \arrow[dr] & \\ & \hat{\phi}_2 \end{tikzcd}$$

- Completely coherent information: operational meaning?
- Understand classical-quantum correspondence better.

Thank you!