Majorization and entropy at the output of bosonic Gaussian channels

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Remark: *while this talk is mainly focused on the results of Ref. [1] (output state majorization), it will also contain a general overview of other related optimization theorems which have been recently obtained within the same context. In particular: the solution of the minimum output entropy conjecture [34], the determination of the communication capacity of quantum optical channels [20] and the multimode generalization of the majorization result [2].*

A large part of quantum communication theory is devoted to the transmission of electromagnetic radiation via bosonic Gaussian channels [6, 8–10]. The latter are formally defined as completely positive and trace preserving operations mapping Gaussian input states into Gaussian output states. The most relevant channels are also invariant under phase space rotations and are called phase-insensitive. For example, the transmission of optical quantum states through realistic physical devices [5] (like *e.g.* optical fibers, free space communication lines, dielectric media, *etc.*) can be described by phase-insensitive Gaussian channels.

In the spirit of classical communication theory [3], one may ask what is the minimum amount of "disorder" achievable at the output of a Gaussian channel. For quantum systems there are two main figures of merit which can be used to quantify the idea of disorder [11–14]: the von Neumann entropy and the concept of majorization. The entropy of a state ρ is defined as $S(\rho) = -\mathrm{Tr}[\rho \log(\rho)]$ and one can say that a state ρ_1 is more disordered than ρ_2 if $S(\rho_1) > S(\rho_2)$. A different (and stronger) way of saying that ρ_1 is more disordered than ρ_2 is the following:

$$\sum_{j=1}^{k} \lambda_j^{\rho_1} \leq \sum_{j=1}^{k} \lambda_j^{\rho_2}, \quad \forall k \geq 1, \tag{1}$$

where the vectors $\boldsymbol{\lambda}^{\rho_1}$ and $\boldsymbol{\lambda}^{\rho_2}$ consist of the eigenvalues of the respective states arranged in decreasing order. If the condition (1) is satisfied then one says that ρ_2 majorizes ρ_1 and this is usually indicated by the expression $\rho_2 \succ \rho_1$. The previous definition has a very intuitive operational interpretation since it can be shown that $\rho_2 \succ \rho_1$ if and only if ρ_1 can be obtained from ρ_2 by a proper convex combination of unitary operations [11–14]. These considerations extend also to the infinite dimensional case [15] relevant for the quantum description of electromagnetic modes.

According to the previous ideas of disorder, for a single-mode phase insensitive bosonic Gaussian channel it was conjectured [16] that:

(i) the minimum output entropy is achieved by coherent input states,

and

(ii) the output states resulting from coherent input states majorize all other output states.

A graphical representation of the last property is given in Fig. 1. Both conjectures have broad implications in many research areas like classical and quantum optics, telecommunication engineering, mathematical and statistical physics and for this reason they attracted the attention of many scientists. In particular, the validity of conjecture (i) and (ii) has a number of important corollaries and relations ranging from entanglement theory [17–20], channel capacities [8, 16, 20–24], entropic inequalities [21, 22, 25, 26] to quantum discord [27, 28].

In the last decade, many analytical and numerical evidences supporting both conjectures were presented [16, 21–26, 29–33] but a general proof was missing. Only very recently the first one was finally proved [20, 34] under the assumption of a finite mean energy. In this work we prove the second conjecture (ii) and highlight some of its implications. Moreover it is easy to show that $\rho_2 \succ \rho_1$ implies $S(\rho_1) \geq S(\rho_2)$, therefore the statement (ii) is stronger than the conjecture (i) and the result presented in this work can also be seen as a proof of the minimum output entropy conjecture, without any energy constraint. Thus both gaps in the theory are now definitely closed.

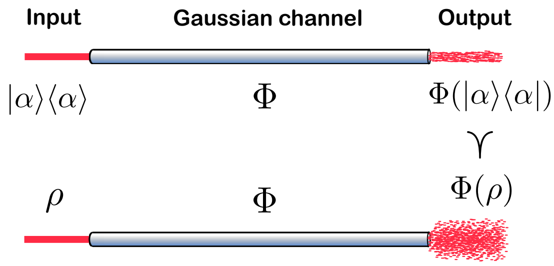

FIG. 1: Graphical representation of the majorization conjecture (2). A coherent state $|\alpha\rangle\langle\alpha|$ and an arbitrary state ρ are both transmitted through the same phase-insensitive Gaussian channel Φ. The respective output states always satisfy the majorization relation $\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho)$. This means that coherent input states produce less "noise" at the output of the communication channel.

Formally the core of our work are the following two results:

Result 1 (Minimization of strictly concave functionals): Let Φ be a phase-insensitive bosonic channel. Then, for every nonnegative unitary invariant and strictly concave functional F and for every quantum state ρ, we have

$$F(\Phi(\rho)) \geq F(\Phi(|\alpha\rangle\langle\alpha|)), \qquad \forall \alpha \in \mathbb{C}, \tag{2}$$

where $|\alpha\rangle$ is any coherent state. Moreover the equality is achieved only if ρ is a coherent state.

This result can be proved with a rather simple argument based on the following three ingredients: a particular decomposition of phase-insensitive channels in terms of quantum limited attenuators and amplifiers, the notion of complementary channel and a particular property of the quantum optical beam splitter. For the details see 1.

Notice that, since the von Neumann entropy is a strictly concave functional [13], the previous result constitutes an alternative proof (with respect to the one given in [34]) of the minimal output entropy conjecture. Actually applying with the choice $F(\rho) = -\mathrm{Tr}[\rho \log(\rho)]$, we get a slightly stronger version of the conjecture (i): the minimum output entropy of a phase-insensitive channel is achieved *only* by coherent input states. Moreover, choosing $F(x) = x - x^p$, $p > 1$, leads to the proof of the similar statement for the minimal output Renyi entropies of all orders $p > 1$.

We can finally state our main result which proves the validity of the majorization conjecture (ii):

Result 2 (Majorization at the output of the channel): Let Φ be a phase-insensitive bosonic channel. Then, for every input state ρ,

$$\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho), \qquad \forall \alpha \in \mathbb{C}, \tag{3}$$

where $|\alpha\rangle$ is any coherent state.

This fact follows almost straightforwardly from the previous minimization problem (Results 1), up to some non-trivial subtleties that we had to face. Such subtleties are related to the assumption of *strict* concavity and to the infinite dimension of the Hilbert space. Nonetheless we have been able to prove (for the details see 1) that Result 1 implies Result 2, therefore establishing the validity of the majorization conjecture (ii).

Our work, while closing two longstanding open problems in quantum communication theory, has a large variety of implications and consequences. For example, by using Result 1 and Result 2 one can: compute the entanglement of formation of non-symmetric Gaussian states (see the last section of [20]), evaluate the classical capacity of Gaussian channels [20] and compute the exact quantum discord [27] for a large class of channels [28]. Moreover, from Result 1, we conclude that coherent input states minimize every Schur-concave output function like Renyi entropies of arbitrary order [21, 22, 25, 26]. Finally, it is a simple implication that the pure entangled state $|\Psi_{out}\rangle$ obtained from a unitary dilation of a phase-insensitive Gaussian channel is more entangled than the output state $|\Psi_{out}\rangle'$ obtained with a coherent input. What is more, from the well known relationship between entanglement and majorization [11], we also know that $|\Psi_{out}\rangle'$ can be obtained from $|\Psi_{out}\rangle$ with local operations and classical communication. Finally, our result has been recently applied in the derivation of a strong converse theorem for the capacity of Gaussian channels [39].

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