Einstein-Podolsky-Rosen steering provides the advantage in entanglement-assisted subchannel discrimination with one-way measurements

Marco Piani[1,2] and John Watrous[3]

[1]Institute for Quantum Computing & Department of Physics and Astronomy, University of Waterloo, 200 University Avenue West, Waterloo, Ontario N2L 3G1, Canada
[2]SUPA and Department of Physics, University of Strathclyde, Glasgow G4 0NG, UK
[3]Institute for Quantum Computing & School of Computer Science, University of Waterloo, 200 University Avenue West, Waterloo, Ontario N2L 3G1, Canada

Steering is the entanglement-based quantum effect that embodies the "spooky action at a distance" disliked by Einstein and scrutinized by Einstein, Podolsky, and Rosen. Here we provide a necessary and sufficient characterization of steering, based on a quantum information processing task: the discrimination of branches in a quantum evolution, which we dub *subchannel discrimination*. We prove that, for any bipartite steerable state, there are instances of the quantum subchannel discrimination problem for which this state allows a correct discrimination with strictly higher probability than in absence of entanglement, even when measurements are restricted to local measurements aided by one-way communication. On the other hand, unsteerable states are useless in such conditions, even when entangled. We also prove that the above steering advantage can be exactly quantified in terms of the *steering robustness*, which is a natural measure of the steerability exhibited by the state.

Technical account. See [1] for a complete account of the results.

Introduction. The strongest feature exhibited by entangled systems is non-locality [2]. A weaker feature related to entanglement is *steering*: roughly speaking, it corresponds to the fact that one party can induce very different ensembles for the local state of the other party, beyond what is possible based only on a conceivable classical knowledge about the other party's "hidden state" [3, 4]. Steering embodies the "spooky action at a distance"—in the words of Einstein [5]—identified by Schroedinger [6], scrutinized by Einstein, Podolsky, and Rosen [7], and formally put on sound ground in [3, 4]. Not all entangled states are steerable, and not all steerable states exhibit nonlocality [3, 4], but states that exhibit steering allow for the verification of their entanglement in a *semi*-device independent way: there is no need to trust the devices used by the steering party, and the ability to determine the conditional states of the steered party is sufficient [3, 4, 8]. In general, besides its foundational interest, steering is interesting in practice in bipartite tasks, like quantum key distribution (QKD) [9], where it is convenient and/or appropriate to trust the devices of one of two parties, but not necessarily of the other party. For example, by exploiting steering it is possible to obtain key rates unachievable in a full device-independent approach [10], but still assuming less about the devices than in a standard QKD approach [11]. For these reasons, steering has recently attracted significant interest, both theoretically and experimentally [12–29], mostly directed to the verification of steering. On the other hand, an answer to the question "What is steering useful for?" that applies to states that exhibit steering can arguably be considered limited [8, 11]. Furthermore, the quantification of steering has just started to be addressed [23].

Summary of results. In this paper we fully characterize and quantify steering in an operational way that mirrors the asymmetric features of steering, and that breaks new ground in the investigation of the usefulness of steering. We prove that every steerable state is a resource in a quantum information task that we dub *subchannel discrimination*, a generalization of channel discrimination, in a practically relevant scenario where measurements can only be performed locally. Subchannel discrimination is the identification of which branch of a quantum evolution a quantum system undergoes. It is well known that entanglement between a probe and an ancilla can help in discriminating different channels [30–40]. In [41] it was proven that *every* entangled state is useful in some instance of the subchannel discrimination problem. Ref. [42] raised and analyzed the question of whether such an advantage is preserved when joint measurements on the output probe and the ancilla are not possible. Here we prove that, when only local measurements coordinated by forward classical communication are possible, *every* steerable state remains useful, while non-steerable entangled states become useless. We further prove that this usefulness, optimized over all instances of the subchannel discrimination problem, is exactly equal to the *robustness of steering*—a natural way of quantifying steering using techniques similar to the ones used in [23], but based on the notion of robustness [43–46]. We argue that the resulting quantification of steering, besides having operational interpretations both in terms of resilience to noise and usefulness, is quantitatively more detailed.

Entanglement and steering. In the following we will denote by a ˆ (hat) mathematical entities that are "normalized." So, for example, a positive semidefinite operator with unit trace is a (normalized) state $\hat{\rho}$. An *ensemble* $\mathcal{E} = \{\rho_a\}_a$ for a state $\hat{\rho}$ is a collection of *sub*states $\rho_a \leq \hat{\rho}$ such that $\sum_a \rho_a = \hat{\rho}$. Each substate ρ_a can

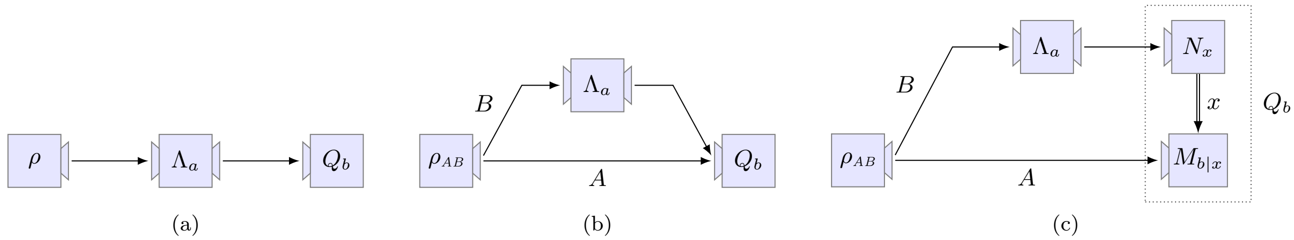

FIG. 1: Different strategies for subchannel discrimination. (a) No entanglement is used: a probe, initially in the state ρ, undergoes the quantum evolution $\hat{\Lambda}$, with branches Λ_a, and is later measured, with an outcome b for the measurement described by the POVM $\{Q_b\}_b$, which is the guess for which branch of the evolution actually took place. (b) The probe B is potentially entangled with an ancilla A; the output probe and the ancilla are jointly measured. (c) The probe is still potentially entangled with an ancilla, but the final measurement $\{Q_b\}_b$ is restricted to local measurements on the output probe and the ancilla, coordinated by one-way classical communication (single lines represent quantum systems, double lines classical information): the outcome x of the measurement performed on the output probe is used to decide which measurement to perform on the ancilla.

be seen as being proportional to a normalized state $\hat{\rho}_a$, $\rho_a = p_a\hat{\rho}_a$, with $p_a = \mathrm{Tr}(\rho_a)$ being the probability of $\hat{\rho}_a$ in the ensemble. An *assemblage* $\mathcal{A} = \{\mathcal{E}_x\}_x = \{\rho_{a|x}\}_{a,x}$ is a collection of ensembles $\mathcal{E}_x$ for the same state $\hat{\rho}$, one for each x, i.e., $\sum_a \rho_{a|x} = \hat{\rho}$, for all x. Along similar lines, a *measurement assemblage* $\mathcal{MA} = \{M_{a|x}\}_{a,x}$ is a collection of positive operators $M_{a|x} \geq 0$ satisfying $\sum_a M_{a|x} = \mathbb{1}$ for each x. i.e., a POVM for each x. For a fixed bipartite state $\hat{\rho}_{AB}$, every measurement assemblage on Alice gives rise to an assemblage on Bob via $\rho^B_{a|x} = \mathrm{Tr}_A\big(M^A_{a|x}\hat{\rho}_{AB}\big)$. On the other hand, every assemblage on Bob $\{\sigma_{a|x}\}_{a,x}$ has a quantum realization for some $\hat{\rho}_{AB}$ satisfying $\hat{\rho}_B = \mathrm{Tr}_A(\hat{\rho}_{AB}) = \sum_x \sigma_{a|x} =: \hat{\sigma}_B$ and for some measurement assemblage $\{M_{a|x}\}_{a,x}$ [47]. An assemblage $\mathcal{A} = \{\rho_{a|x}\}_{a,x}$ is *unsteerable* if $\rho^{\mathrm{US}}_{a|x} = \sum_\lambda p(\lambda)p(a|x,\lambda)\hat{\sigma}(\lambda) = \sum_\lambda p(a|x,\lambda)\sigma(\lambda)$, for all a, x, for some probability distribution $p(\lambda)$, conditional probability distributions $p(a|x,\lambda)$, and states $\hat{\sigma}(\lambda)$. Here λ indicates a (hidden) classical random variable, and we introduced also subnormalized states $\sigma(\lambda) = p(\lambda)\hat{\sigma}(\lambda)$. We say that an assemblage $\{\rho_{a|x}\}_{a,x}$ is *steerable* if it is not unsteerable. A *separable* (or *unentangled*) state is one that admits a decomposition $\hat{\sigma}^{\mathrm{sep}}_{AB} = \sum_\lambda p(\lambda)\hat{\sigma}_A(\lambda) \otimes \hat{\sigma}_B(\lambda)$, for $\hat{\sigma}_A(\lambda)$, $\hat{\sigma}_B(\lambda)$ local states, λ a classical label, and $p(\lambda)$ a probability distribution [48]. A state is *entangled* if it is not separable. An unsteerable assemblage can always be thought as the result of local measurements on some separable state, and a separable state can only give raise to unsteerable assemblages. It follows that entanglement is a necessary condition for steerability, and, in turn, a steerable assemblage is a clear signature of entanglement. Interestingly, not all entangled states lead to steerable assemblages by the action of appropriate local measurement assemblages [3, 4]; we call *steerable states* those that do, and *unsteerable states* those that do not. There exist entangled states that are steerable by one party but not the other (see, e.g., [21]). In this paper, when we refer to a state being steerable or unsteerable, it is always to be assumed that Alice is the steering party.

Channel and subchannel identification. A *subchannel* Λ is a linear completely positive map that is trace non-increasing: $\mathrm{Tr}(\Lambda[\rho]) \leq \mathrm{Tr}(\rho)$, for all states ρ. If a subchannel Λ is trace-preserving, $\mathrm{Tr}(\Lambda[\rho]) = \mathrm{Tr}(\rho)$, for all ρ, we use the $\hat{}$ notation and say that $\hat{\Lambda}$ is a *channel*. An *instrument* $\mathcal{I} = \{\Lambda_a\}_a$ for a channel $\hat{\Lambda}$ is a collection of subchannels Λ_a such that $\hat{\Lambda} = \sum_a \Lambda_a$. Every instrument has (in principle) a physical realization, where the (classical) index a can be considered available to some party [49–51]. Fix an instrument $\{\Lambda_a\}_a$ for a channel $\hat{\Lambda}$, and consider a measurement $\{Q_b\}_b$ on the output space of $\hat{\Lambda}$. The joint probability of Λ_a and Q_b for input ρ is $p(a,b) := \mathrm{Tr}(Q_b\Lambda_a[\rho]) = p(b|a)p(a)$, where $p(a) = \mathrm{Tr}(\Lambda_a[\rho])$ is the probability of the subchannel Λ_a for the given input ρ and $p(b|a) = p(a,b)/p(a)$ is the conditional probability of the outcome b given that the subchannel Λ_a took place (see Figure 1(a)). The probability of correctly identifying which subchannel was realized is $p_{\mathrm{corr}}(\{\Lambda_a\}_a, \{Q_b\}_b, \rho) = \sum_a \mathrm{Tr}(Q_a\Lambda_a[\rho])$. The best success probability in identifying subchannels $\{\Lambda_a\}_a$ optimizing over input and final measurement is $p^{\mathrm{NE}}_{\mathrm{corr}}(\{\Lambda_a\}_a) := \max_\rho \max_{\{Q_b\}_b} p_{\mathrm{corr}}(\{\Lambda_a\}_a, \{Q_b\}_b, \rho)$, where the superscript NE stands for "no entanglement" (see Fig. 1(a)). One may try to improve the success probability by using an entangled input state ρ_{AB} of an input probe B and an ancilla A (see Fig. 1(b)). This leads to the consideration of the optimal probability of success for a scheme that uses input entanglement and global measurements: $p^{\mathrm{E}}_{\mathrm{corr}}(\{\Lambda_a\}_a) := \max_{\rho_{AB}} \max_{\{Q^{AB}_b\}_b} p_{\mathrm{corr}}(\{\Lambda^B_a\}_a, \{Q^{AB}_b\}_b, \rho_{AB})$. We say that entanglement is useful in discriminating subchannels $\{\Lambda_a\}_a$ if $p^{\mathrm{E}}_{\mathrm{corr}}(\{\Lambda_a\}_a) > p^{\mathrm{NE}}_{\mathrm{corr}}(\{\Lambda_a\}_a)$. It is known that there are instances of subchannel discrimination, already in the simple setting $\{\Lambda_a\}_a = \{\frac{1}{2}\hat{\Lambda}_0, \frac{1}{2}\hat{\Lambda}_0\}$, where $p^{\mathrm{E}}_{\mathrm{corr}} \approx 1 \gg p^{\mathrm{NE}}_{\mathrm{corr}} \approx 0$ (see [42] and references therein).

In [41] it was proven that, for any entangled state ρ_{AB}, there exists a choice $\{\frac{1}{2}\hat{\Lambda}_0, \frac{1}{2}\hat{\Lambda}_1\}$ such that $p_{\mathrm{corr}}\left(\left\{\frac{1}{2}\hat{\Lambda}_0, \frac{1}{2}\hat{\Lambda}_1\right\}, \rho_{AB}\right) > p^{\mathrm{NE}}_{\mathrm{corr}}\left(\left\{\frac{1}{2}\hat{\Lambda}_0, \frac{1}{2}\hat{\Lambda}_1\right\}\right)$, i.e., that every entangled state is useful for the task of (sub)channel discrimination. In this sense, every entangled state, independently of how weakly entangled it is, is a resource. Nonetheless, exploiting such a resource may require arbitrary joint measurements on the output probe and ancilla [42]. From a conceptual perspective, one may want to limit measurements to those that can be performed by local operations and classical communication (LOCC), as this makes the input entangled state the only non-local resource. This limitation can be justified also from a practical perspective: LOCC measurements are arguably easier to implement, and might be the only feasible kind of measurements, especially in a scenario where only weakly entangled states can be produced. We do not know whether every entangled state stays useful for subchannel discrimination when measurements are restricted to be LOCC. In the following, though, *we prove that, if measurements are limited to local operations and forward communication (one-way LOCC), then only steerable states can and do remain useful.*

Steerability and subchannel identification by means of restricted measurements. We indicate a Bob-to-Alice one-way LOCC measurement, i.e., a POVM, by $\mathcal{M}^{B\to A} = \{Q_a^{B\to A}\}_a$. We define $p_{\mathrm{corr}}^{B\to A}(\mathcal{I},\rho_{AB}) := \max_{\mathcal{M}^{B\to A}} p_{\mathrm{corr}}(\mathcal{I}^B,\mathcal{M}^{B\to A},\rho_{AB})$ as the optimal probability of success in the discrimination of the instrument $\mathcal{I}^B = \{\Lambda_a^B\}_a$ by means of the input state ρ_{AB} and one-way LOCC measurements from B to A (see Fig. 1(c)). We say that ρ_{AB} is useful in this restricted-measurement scenario if $p_{\mathrm{corr}}^{B\to A}(\mathcal{I},\rho_{AB}) > p_{\mathrm{corr}}^{\mathrm{NE}}(\mathcal{I})$ for some instrument $\mathcal{I}$. One checks that no bipartite state ρ_{AB} is useful in one-way subchannel identification when the communication goes from the ancilla to the output probe. Furthermore, if the assemblage resulting from the measurement assemblage $\{N_{a|x}\}_{a,x}$ (see Fig. 1(c)) on the ancilla is unsteerable, then one can still achieve an equal or better performance with an uncorrelated probe. Thus, if ρ_{AB} is unsteerable, it is useless for subchannel discrimination with one-way measurements. This applies also to entangled states that are unsteerable, which are nonetheless useful in channel discrimination with arbitrary measurements [41].

Main result. We prove that every steerable state *is* useful in subchannel discrimination with one-way-LOCC measurements. To state our result in full detail we need to introduce the *steering robustness of* ρ_{AB},

$$R_{\mathrm{steer}}^{A\to B}(\rho_{AB}) := \sup_{\mathcal{MA}} R(\mathcal{A}), \quad R(\mathcal{A}) := \min\left\{ t\geq 0 \;\middle|\; \left\{\frac{\rho_{a|x}+t\,\tau_{a|x}}{1+t}\right\}_{a,x} \text{unsteerable}, \{\tau_{a|x}\}_{a,x} \text{ an assemblage}\right\}, \tag{1}$$

where the supremum is over all measurement assemblages $\mathcal{MA} = \{M_{a|x}\}_{a,x}$ on A, and $\mathcal{A}$ is obtained from ρ_{AB} with the measurement assemblage $\mathcal{MA}$ on A. The steering robustness $R(\mathcal{A})$ of $\mathcal{A}$ is a measure of the minimal "noise" needed to destroy the steerability of the assemblage $\mathcal{A}$, where such noise is in terms of the mixing with an arbitrary assemblage $\{\tau_{a|x}\}_{a,x}$. With the notation set, we have the following theorem.

Theorem 1. *Every steerable state is useful in one-way subchannel discrimination. More precisely, it holds*

$$\sup_{\mathcal{I}} \frac{p_{corr}^{B\to A}(\mathcal{I},\rho_{AB})}{p_{corr}^{\mathrm{NE}}(\mathcal{I})} = R_{steer}^{A\to B}(\rho_{AB})+1, \tag{2}$$

where the supremum is over all instruments $\mathcal{I}$, i.e., over all subchannel discrimination problems.

Idea of the proof: Using the definitions it is immediate to verify $p_{\mathrm{corr}}(\mathcal{I}^B,\mathcal{M}^{B\to A},\rho_{AB}) \leq (1+R_{\mathrm{steer}}^{A\to B}(\rho_{AB}))p_{\mathrm{corr}}^{\mathrm{NE}}(\mathcal{I})$, for any $\mathcal{M}^{B\to A}$ and any $\mathcal{I}$. On the other hand, we prove and use the fact that the steering robustness $R(\mathcal{A})$ of any assemblage $\mathcal{A} = \{\rho_{a|x}\}_{a,x}$ can be calculated via semidefinite programming (SDP) [52]. The dual of the SDP optimization problem provides then information that allows us to construct appropriate instances of the subchannel discrimination problem, and prove that the bound above can be approximated arbitrarily well. More in detail, the construction in the proof of Theorem 1 shows that, for any measurement assemblage $\mathcal{MA}$ on A such that the corresponding $\mathcal{A}$ exhibit steering with robustness $R(\mathcal{A})$, there exist instances of the subchannel discrimination problem with restricted measurements where the use of the steerable state ensures a probability of success approximately $(1+R(\mathcal{A}))$-fold higher than in the case where no entanglement is used.

Remarks. Theorem 1 implies that the robustnesses $R(\mathcal{A})$ and $R_{\mathrm{steer}}^{A\to B}(\rho_{AB})$ have operational meanings not only in terms of the resilience of steerability versus noise, but in applicative terms. Also, they constitute semi-device-independent lower bounds on the generalized robustness of entanglement $R_g(\rho_{AB})$ [44, 45], which is an entanglement measure with operational interpretations itself [53, 54]. Besides these observations, in [1] we argue that the way to quantify steerability that we have introduced is finer-grained than the approach of [23], while preserving the computational efficiency deriving from the use of semidefinite programming. Many questions remain open for further investigation: a closed formula for the steerability robustness of pure (maximally entangled) states; whether the result of Theorem 1 can be strengthened to prove that every steerable state is useful for channel—rather than general subchannel—discrimination with restricted measurements; whether general LOCC (rather than one-way LOCC) measurements can restore the usefulness of all entangled states for (sub)channel discrimination.

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