The information theoretic interpretation of the length of a curve

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I. SUMMARY

The anti-deSitter / Conformal Field Theory (AdS/CFT) correspondence relates the physics of quantum gravity in $d+2$ "bulk" dimensions to the physics of gravityless conformal field theories with degrees of freedom organized on the $d+1$-dimensional boundary of the bulk spacetime [1]. Even though the correspondence is close to twenty years old, the emergence of the additional spatial dimension from the boundary theory remains poorly understood. While the emergent dimension is known to be associated with scale in the boundary theory, it is difficult to make precise statements to that effect [2]. The purpose of this contribution is to provide a precise interpretation of a bulk geometrical quantity, namely the length of a curve, as a boundary information theoretic quantity, specifically the minimal amount of entanglement required to succeed at a certain boundary communication task determined by the geometry of the bulk curve. Our specific results, of which we will elaborate on 1, 2 and 3 below, are:

1. The exhibition of a protocol for merging the state of a boundary interval from Alice to Bob at an entanglement cost equal to the length of a bulk curve starting and ending at the endpoints of the interval, to leading order in the CFT central charge. In each step of the protocol, Alice and Bob act only in sub-intervals of the boundary determined by the geometry of the bulk curve.

2. A proof that, subject to appropriate locality constraints on Alice and Bob's actions, the entanglement cost is optimal: no procedure meeting the locality constraints can use less entanglement. The minimal constrained merging cost is, therefore, the length of the curve.

3. A demonstration that the smooth conditional min-entropy in a 1+1 dimensional CFT with large central charge is well approximated by the conditional von Neumann entropy. This is the limit in which the bulk is described by general relativity, in which case it can be understood as a classical geometry.

4. An analogous protocol for closed bulk curves, in which case Alice and Bob are required to swap their boundary intervals. There is at the moment, however, no matching optimality proof.

5. A refutation of the pre-existing conjecture that the length of the bulk curve is the maximum entropy among all boundary states matching certain consistency criteria. Detailed CFT calculations combined with the structure theory of quantum Markov chains reveals that the conjecture is not even approximately correct [3].

II. DIFFERENTIAL ENTROPY

For simplicity, we will focus on three-dimensional anti-de Sitter space (AdS_3), though our results are more general. We start with the metric on the Poincaré patch of AdS_3,

$$ds^2 = -\frac{R^2}{L^2}\, dT^2 + \frac{L^2}{R^2}\, dR^2 + R^2 d\tilde{x}^2. \tag{1}$$

We assume that this geometry arises as the dual description of a CFT living on its asymptotic boundary – that is, on an infinite line cross time. We denote the transversal coordinate in the bulk as $\tilde{x}$, in contrast to x, which we reserve for the spatial coordinate on the boundary. The starting point for our discussion is the seminal relation between entanglement entropies in the boundary theory and areas of minimal surfaces in the bulk, discovered by Ryu and Takayanagi [4, 5]. In the present context of pure AdS_3, which is dual to the vacuum of a 1+1-dimensional CFT, this relation takes the form

$$S(I) \equiv S(a) = \frac{c}{3}\log\frac{a}{\mu} = \frac{\text{length of geodesic connecting } \tilde{x} = \pm a/2 \text{ at } R = L^2/\mu}{4G}. \tag{2}$$

We use $S(I)$ to denote the entanglement entropy in the vacuum of an interval $I \equiv (-a/2, a/2)$ of total length a. The quantity μ, which is a UV cutoff in the CFT, also defines an IR cutoff L^2/μ on the dual gravity side, which regulates the otherwise infinite length of the geodesic. This is in accordance with the general holographic rule of thumb, which relates UV physics in the field theory to IR physics in the bulk. Eq. (2) relies on the Brown-Henneaux relation $c = 3L/2G$, which fixes the central charge of the 1+1-dimensional CFT in terms of the curvature scale L of the dual AdS_3 in Planck units (G is Newton's constant) [6].

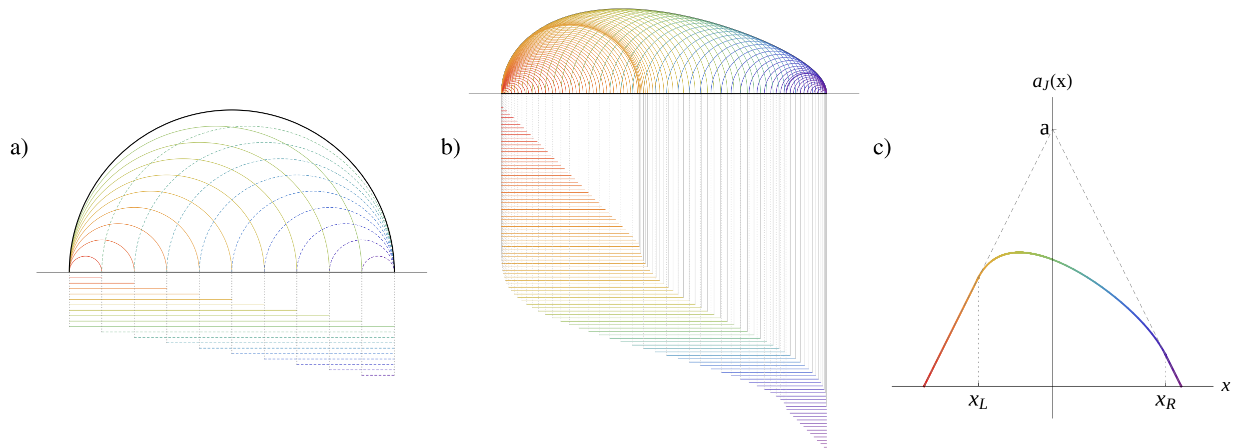

FIG. 1. (a) Geodesic g_I, which subtends a boundary interval I (black) and geodesics, which are tangent to g_I on the boundary along with the corresponding boundary intervals $I(x)$ (color). The dashed geodesics contribute zero to integral (3). (b) A curve h_J, which asymptotes to the geodesic g_I. We have again marked in color the geodesics tangent to the curve and the boundary intervals $J(x)$, which they subtend. (c) Graph of the function $a_J(x)$, the size of the interval $J(x)$ centered at x for the geodesic g_J. The dashed triangle corresponds to $a_I(x)$, the corresponding function for g_I. Because g_J asymptotes to g_I, the functions coincide for $x < x_L$ and $x > x_R$.

Ref. [7] (see also [8–10]) showed how to use relation (2) to give a boundary computation of the length of an arbitrary differentiable curve on a constant time slice in geometry (1). Given a convex [19] curve $R = R(\tilde{x})$, for every point $\tilde{x}$ one finds the geodesic that is tangent to the curve at $\tilde{x}$. The endpoints of the geodesic lie on the asymptotic boundary, so they select a boundary interval. We shall refer to this interval as $I(x)$, where x is the midpoint of the interval. Note that x depends on $\tilde{x}$ but is not equal to it. The construction is illustrated for the case of both a geodesic and a nongeodesic in Fig. 1. The length of the curve is then given by the formula

$$\frac{\text{length}}{4G} = \int \Big(S\big(I(x)\big) - S\big(I(x)\cap I(x-dx)\big)\Big) = \int S\big(I(x) - I(x-dx)\,|\,I(x)\cap I(x-dx)\big) \equiv S_{\text{diff}}. \tag{3}$$

Note that the integrand in (3), or rather the first nonvanishing term in its Taylor expansion, is a one-form, so the integral is well-defined. The right hand side was called "differential entropy" in [7] because the integrand can be rewritten in terms $\frac{dS}{dI}$. The form above is more relevant in the present context, however, because the presence of the conditional entropy suggests immediately that the length of the curve should have an interpretation in terms of state merging [11, 12]. Indeed, setting $x_j = -a/2 + j \cdot a/N$, (3) can be rewritten suggestively in terms of $A_j = I(x_j) - I(x_{j-1})$ and $B_j = I(x_j) \cap I(x_{j-1})$ as

$$\frac{\text{length}}{4G} = \lim_{N\to\infty} \sum_{j=1}^{N} \left[S\left(I(x_j)\right) - S\left(I(x_j)\cap I(x_{j-1})\right)\right] = \lim_{N\to\infty}\sum_{j=1}^{N} S(A_j|B_j). \tag{4}$$

III. CONSTRAINED MERGING

Now imagine that Alice has access to the CFT from outside. The CFT, for example, might describe a one-dimensional spin lattice system at a quantum phase transition that is sitting in Alice's laboratory. Bob has access to an isomorphic system initially factorizing with Alice's. Now Alice's goal is going to be to merge the state of interval I of her CFT to Bob. That is, using only Bell pairs and classical communication plus local operations inside their respective intervals I, Alice and Bob will prepare a state in Bob's lab isomorphic to Alice's original state on I *and* purifying I^c, the complement of I. Intuitively, Alice wishes to teleport the state of her I to Bob [13]. However, Alice and Bob are further constrained: they generally cannot act on all of their respective intervals I at once. The procedure will take place in N discrete steps and at the jth step, Alice and Bob are allowed to act *only* in their respective intervals $I(x_j)$. (N will ultimately be allowed to go to infinity as the UV cutoff μ goes to zero and the central charge to infinity.)

Consider Fig. 1a. For early values of j, corresponding to red and orange intervals, Alice and Bob are only permitted to act on the left side of I, while for blue and purple intervals they can only act on the right. For $x_j = 0$, however, corresponding to the full length green interval, they have access to all of I. So Alice could simply compress the state of I [14] in the $x_j = 0$ step then teleport it to Bob, who could decompress on his end. At all other steps, Alice and Bob would do nothing. The entanglement cost, postponing until the next section issues of single-shot versus von Neumann entropies and approximation, would be $S(I)$.

In contrast, for the nongeodesic curve of Fig. 1b, none of the intervals $I(x_j)$ span all of I. The simple-minded strategy of the previous paragraph therefore cannot succeed. Instead, Alice and Bob will act non-trivially in each interval $I(x_j)$. Specifically, in the jth step, Alice will merge A_j to Bob. Since $A_j \subseteq I(x_j)$, Alice's actions are consistent with the constraint. Moreover, by the jth step, Bob will already have reconstructed the entire interval $\cup_{i=1}^{j-1} I(x_i)$, of which the rules give him access only to the portion intersecting $I(x_j)$, namely $I(x_{j-1}) \cap I(x_j) = B_j$. The number of Bell pairs required to merge I subject to the locality constraints will therefore be $S(A_j|B_j)$ [11, 12]. By (4), the length of the bulk curve will therefore be approximated by $4G$ times the number of Bell pairs required to perform the merging protocol. The following theorem can be used to ensure that the entanglement cost of any N-step procedure conforming to the constraints cannot be significantly better.

Theorem 1 (Informal statement) *Let $|\psi\rangle \in \mathcal{H} \equiv \otimes_{x\in\mathbb{Z}} \mathcal{H}_x$, with $\dim \mathcal{H}_x < \infty$. Let $I \subseteq \mathbb{Z}$ be a finite interval and $I(x) \subseteq I$ itself be an interval for each x, such that the left and right endpoints of the intervals are non-decreasing with x. The initial configuration will be Alice holding $|\psi\rangle$ and Bob holding the state $\otimes_x |0\rangle$, in addition to E shared ebits. Suppose that Alice and Bob execute a multistep LOCC protocol, with one step for each $x \in I$, in increasing order. At each step, Alice and Bob can act only in their respective $I(x)$ subsystems and on the shared entanglement. Let $|\psi'\rangle \in (\otimes_{x\in I} \mathcal{H}_x^B) \otimes (\otimes_{x\notin I} \mathcal{H}_x^A)$ be the Alice-Bob state isomorphic to $|\psi\rangle$ but with the Hilbert spaces corresponding to $x \in I$ held by Bob. If, at the end of the procedure, Alice and Bob produce a state ρ such that $\langle\psi'|\rho|\psi'\rangle \geq 1-\delta$, then $E \geq \sum_{x\in I} \left[S\left(I(x)\right)_\psi - S\left(I(x) \cap I(x-1)\right)_\psi \right] - f(\delta) \sum_{x\in I} \dim \mathcal{H}_x$, where $f(\delta)$ vanishes as $\delta \to 0$.*

IV. SINGLE-SHOT VERSUS VON NEUMANN ENTROPIES

In the preceding discussion, the conditional von Neumann entropy was identified as the entanglement cost in each of the N merging steps, but the cost in the single-shot setting appropriate for use here is in fact the smooth conditional max-entropy $H^\epsilon_{\max}(A_j|B_j)$ [15, 16]. Since $H^\epsilon_{\max}(A_j|B_j) \leq H^{\epsilon/2}_{\max}(A_jB_j) - H^{\epsilon/4}_{\min}(B_j) + \text{const}$ [17], it suffices to bound the smooth $H_{\max}(AB)$ from above and $H_{\min}(B)$ from below. To do so, we use a formula found by Calabrese and Lefevre for the eigenvalue distribution of an interval's reduced density operator in either a vacuum or thermal state of a 1+1 dimensional CFT [18]:

$$P(\lambda) = \sum_i \delta(\lambda - \lambda_i) = \delta(\lambda - \lambda_{\max}) + \frac{b\Theta(\lambda_{\max} - \lambda)}{\lambda\sqrt{b\log(\lambda_{\max}/\lambda)}} I_1(2\sqrt{b\log(\lambda_{\max}/\lambda)}) \tag{5}$$

where $b = H_{\min}(I) = -\log(\lambda_{\max})$ and I_1 is a modified Bessel function of the first kind. Investigation of the limit $b \to \infty$ reveals the distribution to be well-approximated by a tightly peaked Gaussian after a suitable change of variables, yielding

$$H^\epsilon_{\min}(I) = S(I)\left(1 - O\left(\sqrt{2\log(1/\epsilon)/S(I)}\right)\right) \quad \text{and} \quad H^\epsilon_{\max}(I) = S(I)\left(1 + O\left(\sqrt{2\log(1/\epsilon)/S(I)}\right)\right) \tag{6}$$

To achieve a cumulative error of at most δ in the N-step protocol, with N chosen to be I/μ, each step can incur an error at most $\delta\mu/I$, which requires fixing $\epsilon = \sqrt{\delta\mu/I}$ [15]. Plugging into (6) shows that in the limit $\mu \to 0$, the subleading terms in the multiplicative correction to $S(I)$ are $O(1/\sqrt{c})$, vanishing in the limit of large central charge.

V. CONCLUSIONS

The Ryu-Takayanagi formula states that the entropy of a boundary interval I is the length of the shortest bulk curve starting and ending at the endpoints of I. (More generally, the shortest bulk curve homologous to I.) Entropy, however, also has an interpretation as the minimal entanglement cost required to teleport a given state. In this submission, we have demonstrated that those two minimizations, over bulk curves and boundary teleportation procedures, are effectively equivalent. Non-minimal length convex curves define constrained boundary state merging tasks whose optimal entanglement costs are the lengths of the curves themselves. This is a powerful insight into the relationship between bulk geometry and boundary entanglement structure.

While we have presented our results in the context of empty AdS_3 and the vacuum state of CFT_2, they extend immediately to AdS-Schwarschild black holes and the corresponding thermal CFT states. In higher dimensions, a more complicated geometrical construction [8, 9] can be used to extend our results to arbitrary convex codimension one bulk spatial surfaces homotopic to a convex boundary region. However, in the higher dimensional case, we do not have a proof that the conditional smooth max-entropy converges to the conditional von Neumann entropy, so Alice and Bob must act in parallel on many copies of the boundary system in order to instead make contact with asymptotic Shannon theory. Nonetheless, we strongly believe that the smooth max-entropy does converge in the correct limit, known as 't Hooft scaling. With luck, we may even have a demonstration in time for

QIP!

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