# Matrix Product Operators: Local Equivalences and Topological Order in 2D*

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Recent years have witnessed an enormously growing understanding of quantum many-body systems, based on the language and tools of entanglement theory. Along these lines tensor network methods have proven instrumental to our current understanding of gapped local Hamiltonians at low energies, in particular in 1D [2, 3, 4]. Often one is interested in the low-energy properties of an entire gapped phase, i.e. a collection of Hamiltonians which can be smoothly connected with one another by adiabatic paths (which in turn translate to local unitary (LU) circuits of constant depth). Ideally, one would like to identify stable properties which uniquely characterize a gapped phase, and enumerate those phases exhaustively. In 2D this goal has largely been elusive so far, not the least because of novel emergent phenomena like (intrinsic) topological order.

Here we focus on gapped phases in 2D with an emergent topological gauge theory [5], i.e. with topological order given by a (finite) gauge group $G$ and certain complex weights $\omega$. We introduce a framework of virtual symmetry matrix product operators (MPO) for projected entangled pair states (PEPS) and show how this MPO symmetry determines the emergent, topological properties of the PEPS [1]. In particular, we identify several kinds of *local* equivalences between symmetry MPOs all of which describe PEPS in the same gapped phase. In a wider context, this implies that many topological gauge theories are in fact equivalent and our results based on entanglement theory provide a microscopic explanation for this redundancy.

Generalizing $G$-injectivity [6], we require that local PEPS tensors are invariant under certain symmetry MPOs (Fig. 1). From this idea we can rigorously determine the emergent topological order of the PEPS as that of a Dijkgraaf-Witten topological quantum field theory (TQFT) [5]. This TQFT is defined via a discrete path integral $\mathcal{Z}$ over (2+1)D spacetime: one obtains $\mathcal{Z}$ by triangulating spacetime and assigning elements of $G$ and weights $\omega$ to the resulting tetrahedra in a triangulation invariant way. The MPO symmetry of local PEPS tensors can now be understood as a particular example of triangulation invariance in the emergent TQFT (Fig. 1). Other properties of the MPOs have a similar interpretation. However, while a TQFT has vanishing correlation length by definition, our framework of virtual MPO symmetry provides a rigorous extension of Dijkgraaf-Witten TQFTs to *non-zero* correlation length.

How are these MPOs constructed in detail? For simplicity we will again use the emergent TQFT for illustration, however, the construction works regardless. The triangulation invariance of $\mathcal{Z}$ implies that the tetrahedral weights are given by a so called 3-cocycle $\omega\colon G \times G \times G \to \mathbb{C}^\times$ of $G$. As sketched in Fig. 1, the

*Partially based on [1].

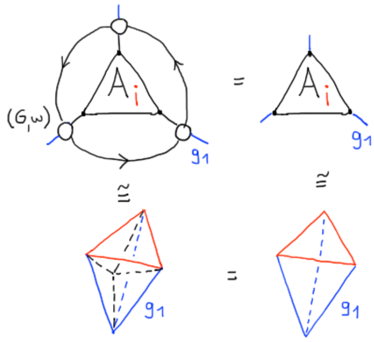


Figure 1: Virtual MPO symmetry vs. triangulation invariance in TQFTs.

MPO is defined by the same data $(G, \omega)$ as $\mathcal{Z}$. Microscopically, this ensures that our virtual MPO symmetry (and injectivity) correctly extends to any local region of the PEPS. It is easy to show that any nontrivial $\omega$ implies an MPO bond dimension $D > 1$. For trivial $\omega$ the MPO factorizes into a product of local operators and defines ordinary $G$-injectivity [6]. As is proven rigorously in [1], MPOs constructed from the data $(G, \omega)$ indeed lead to an emergent TQFT specified by the same data, even for non-zero correlation length. Parent Hamiltonians associated with the PEPS are frustration-free, but their interaction terms do not commute in general.

To what extent are these MPOs a unique property of the emergent gapped phase? Given $G$, a trivial observation shows that there is a continuous freedom in the choice of $\omega$. Group cohomology provides a convenient means to eliminate it: infinitely many 3-cocycles $\omega$ are replaced by a finite number of equivalence classes $[\omega]$ which form the third cohomology group $H^3(G, \mathbb{C}^\times)$ of $G$. This equivalence merely rescales the MPO and can be absorbed into an LU circuit of depth 1 at the physical level. In other words, the gapped phase determined by the MPO actually depends on $H^3(G, \mathbb{C}^\times)$ only. This very object has been suggested to classify topological gauge theories [7].

Let us briefly mention some simple examples. Since $H^3(\mathbb{Z}_2, \mathbb{C}^\times) = \{[1], [\omega]\}$ there are two topological gauge theories with $G = \mathbb{Z}_2$, and these are realized by the toric code [8] and the doubled semion model [9]. Both models have the same fusion rules for their emergent anyons and the same topological entanglement entropy [10, 11], yet can be distinguished by how their degenerate ground states on a torus respond to modular transformations.[1] Accordingly, PEPS for the toric code are described in our framework by MPOs with data $(\mathbb{Z}_2, [1])$, i.e. by ordinary $\mathbb{Z}_2$-injectivity, while the doubled semion model requires MPOs with data $(\mathbb{Z}_2, [\omega])$ and $D = 2$. Secondly, MPOs constructed from an *Abelian* group $G$ may yield PEPS with *non*-Abelian anyons, e.g. for certain data $(\mathbb{Z}_2^3, [\omega])$. This is impossible for ordinary $G$-injectivity unless $G$ itself is non-Abelian. The data $(\mathbb{Z}_2^3, [\omega])$ is also interesting as a source of non-Abelian anyons which can alternatively be described within the XS-stabilizer formalism [12]. More generally, emergent topological gauge theories are realized e.g. in quantum double models [8] and deformations thereof [13].

Perhaps surprisingly, group cohomology does *not* eliminate all freedom in

[1] This response is encoded in the topological $S$- and $T$-matrices.

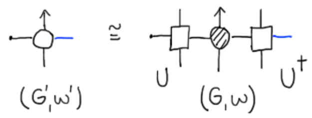


Figure 2: Local equivalence of virtual symmetry MPOs.

parametrizing our MPO symmetry. In order to see this, consider an automorphism $\sigma$ of $G$. While it cannot affect any structure solely determined by group multiplication, $\sigma$ does affect the weights via $(\sigma^{-1} \triangleright \omega)(g_1, g_2, g_3) \coloneqq \omega\big(\sigma(g_1), \sigma(g_2), \sigma(g_3)\big)$. So $\mathrm{Aut}(G)$ naturally induces a permutation action on equivalence classes $[\omega]$. We write $[\omega'] \sim [\omega]$ whenever two classes belong to the same orbit $[\![\omega]\!]$. Microscopically, an MPO with data $(G, [\omega])$ is then conjugated by a product of local operators to yield another MPO with data $(G, [\omega'])$. This can again be absorbed into an LU circuit of depth 1 at the physical level. Thus, gapped phases in $(2+1)$D with emergent topological gauge theories depend on $H^3(G, \mathbb{C}^\times)/\sim$ at most, which significantly improves the earlier result [7].

Often, the orbits $H^3(G, \mathbb{C}^\times)/\sim$ provide a provably complete classification for fixed $G$. We would like to illustrate this with a simple example. For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ one has $H^3(G, \mathbb{C}^\times) = \{\omega_{k_1 k_2 k_3}\}$, where $k_i \in \{0, 1\}$. The permutation action yields 4 disjoint orbits $[\![1]\!] = [\![\omega_{000}]\!]$, $[\![\omega_{001}]\!]$, $[\![\omega_{010}]\!]$, $[\![\omega_{111}]\!]$. It is now instructive to look at the topological $S$- and $T$-matrices which in this case are given by:

$$S_{(g,\mu)(h,\nu)} = \frac{1}{4}\,(-1)^{\mu_1 h_2 + \mu_2 h_1 + \nu_1 g_2 + \nu_2 g_1 + k_1 g_1 h_1 + k_2 g_2 h_2}\,(-\mathrm{i})^{k_3 (g_1 h_2 + g_2 h_1)}, \tag{1}$$

$$T_{(g,\mu)(h,\nu)} = \delta_{g,h}\,\delta_{\mu,\nu}\,\mathrm{i}^{k_1 g_1^2 + k_2 g_2^2 + k_3 g_1 g_2}\,(-1)^{g_1 \mu_2 + g_2 \mu_1}. \tag{2}$$

Here degenerate ground states are labelled by $(g, \mu) \in \{00, 01, 10, 11\}^2$ where $g_1$ ($g_2$) denotes the high (low) bit of $g$ (similarly for $\mu$). A simple calculation confirms that these matrices are constant on the above orbits (once ground state labels are suitably permuted), as expected from their invariant property [14]. Since the $T$-matrices are distinct for each orbit, there are exactly 4 topological gauge theories with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Furthermore, we show how two MPOs with data $(G, [\![\omega]\!])$ and $(G', [\![\omega']\!])$ respectively may determine the *same* gapped phase, although $G$ and $G'$ are *not* isomorphic. This is because symmetry MPOs may locally transform into each other under conjugation by an auxiliary MPO with $D > 1$ (Fig. 2). Perhaps surprisingly, we can reuse the above example for illustration: MPOs with the data $(\mathbb{Z}_2 \times \mathbb{Z}_2, [\![\omega_{001}]\!])$ and $(\mathbb{Z}_4, [\![1]\!])$ are in fact locally equivalent, as are the gapped phases they determine.

In this work we have introduced virtual symmetry MPOs and showed how they characterize PEPS in 2D with topological order given by a Dijkgraaf-Witten TQFT. We have further uncovered *local* equivalences of these symmetry MPOs in terms of auxiliary MPOs. While they arise at several, conceptually different stages, all these equivalences translate to LU circuits of constant depth at the physical level, so that the corresponding PEPS lie in the same gapped phase. What is more, these equivalences are stable since the virtual MPO symmetry is unaffected by physical operations which may cause a non-zero correlation length.

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