Self-correcting stabilizer quantum memories in 3 dimensions or (slightly) less

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Introduction.

The ability to store classical information robustly for long periods of time is one we often take for granted. We can encode classical bits onto magnetic hard drives and expect them to remain there undecayed into the foreseeable future. Intuitively, this is because the 3D Ising ferromagnet (our toy model for a magnetic hard drive bit) is a self-correcting classical memory. In comparison, current protocols for storage of quantum information on long timescales seem quite arduous, typically requiring constant active intervention to correct for errors that may have occurred. As such, there is significant interest from both an abstract and a practical perspective as to if and how self-correcting quantum memories might be realised. A practical self-correcting quantum memory would allow for arbitrarily long storage of quantum information at finite temperature without the need for active error-correction techniques.

The 4D toric code is a simple, exactly solvable example of a spin system with local interactions in 4 spatial dimensions that is known to have self-correcting properties [1, 2]. In this system, quantum information may be stored at finite temperature for an exponentially long time in the system size without decaying. This model also has a number of other desirable properties, such as being stable against Hamiltonian perturbations and having an efficient decoder for the stored qubits. It is unfortunate that it would require 4 spatial dimensions to realise such a model, and it is of interest to determine whether similar quantum self-correction can be engineered in a more physical setting.

In 2D, the toric code [3] is known to be unstable at finite temperature [4], and there are numerous no-go theorems that rule out broad classes of models for self-correction [5–7]. Despite this, some attempts have been made to engineer quantum memories in 2D systems with some partially self-correcting behaviour [8–10]. Many approaches towards realising some aspects of self-correction in 3D have also been found, notably including the Haah code [11–14], which gives a power-law relation between memory lifetime and system size - but only up to a certain (temperature-dependent) critical value of system size. The thermodynamic limit of such a system does not act as a reliable quantum memory at finite temperature. Other approaches to quantum self-correction in 3D are numerous and varied [15–21], including making use of long-range interactions, dissipative dynamics, and bosonic modes rather than finite dimensional spins. No previously known local Hamiltonian spin models in 2D or 3D are fully self-correcting in the sense of an asymptotically exponential memory lifetime. There are also several no-go results restricting possible self-correcting models in 3D [22–24].

Main result.

We propose a family of local CSS stabilizer codes in 3D as candidates for self-correcting quantum memories. Our approach is based on fractal geometries, and inspired by the classical self-correcting behavior of an Ising model on a Sierpinski carpet graph. The Sierpinski carpets [25] are a family of fractal subsets of $\mathbb{R}^2$ with Hausdorff dimension between 1 and 2. We propose a family of quantum CSS codes that can be regarded as 4D toric codes on the product of two Sierpinski carpets (with appropriate boundary conditions). Concretely, these codes are defined through the homological

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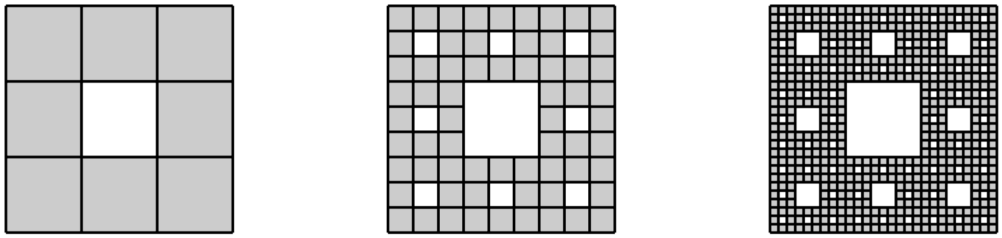

FIG. 1: The first few iterations of the Sierpinski carpet graphs (black) for the simplest Sierpinski carpet, overlaid on the carpet itself at the same iteration (gray)

product [26, 27] of two toric codes on 2D Sierpinski carpet graphs, yielding a code with extensive degeneracy. Though this model naturally embeds in $\mathbb{R}^4$, by choosing the Hausdorff dimension of the Sierpinski carpet small enough we can ensure that the resulting system may be embedded in $\mathbb{R}^3$ with finite distortion. We call these codes *embeddable fractal product codes* (EFPCs).

We show that the local stabilizer spin models corresponding to these codes have two phase transitions at finite temperature, one associated with each sector (X or Z) of the code. The tools we use to show this are generalized duality transformations and correlation inequalities. Given these phase transitions, we argue that it is likely the EFPC system acts as a self-correcting quantum memory at temperatures below the (lowest) critical temperature.

Embeddable fractal product codes.

Classically, the 2D Ising model is a self-correcting memory at finite temperature, while the 1D Ising model is not. Quantumly, the 4D toric code is a self-correcting quantum memory at finite temperature, while the 2D toric code is not. In this context, quantum looks like the "square" of the classical. Since we are interested in the possibility of quantum self-correction in a 3D system, a natural question is whether or not classical systems in 1.5 dimensions are able to act as self-correcting memories.

In order to talk sensibly about such fractional dimension objects, the natural context is fractal geometries. Fractal objects have dimension that interpolates between the familiar integral topological dimensions, and which can be quantified in several useful ways, such as the Hausdorff dimension or the box-counting dimension. The Sierpinski carpets [25] are a family of fractals with dimension between 1 and 2, and are naturally defined as self-similar subsets of $\mathbb{R}^2$. These Sierpinski carpets are usually obtained by subdividing the unit square into a number of smaller squares, deleting a subset of them, and iterating this procedure. Associated with each stage of this iteration is a Sierpinski carpet graph, which is a subgraph of the square lattice whose holes reflect the deleted regions of the Sierpinski carpet (for example, the first few graphs for a particular Sierpinski carpet are shown in Fig. 1).

It is possible to define a classical ferromagnetic Ising model on a Sierpinski carpet graph, and study the thermodynamic properties of such a model. These systems have 2-fold degenerate ground spaces, and thus can be considered as classical codes. General arguments suggest [28, 29], many numerical studies demonstrate (e.g. [30, 31]), and it can be rigorously proved [32–34], that such a family of Ising models has a phase transition at non-zero temperature. Intuitively, this is due to the fact that the Sierpinski carpet graphs have infinite ramification order (i.e. in the thermodynamic limit, an infinite number of bonds must be cut to separate the graph into two infinite pieces). Below the critical temperature, these systems act as a self-correcting classical memory.

The idea that "quantum is the square of the classical" suggests that a suitable self-correcting quantum memory may thus be defined on the product of two Sierpinski carpet graphs. In order to explicitly construct such codes, we make use of the homological product of two CSS codes [26, 27].

The homological product is a construction for building new CSS codes from existing ones, making use of the tools of algebraic topology. In this picture, each code is represented by a chain complex of three spaces corresponding to the X-type stabilizers, Z-type stabilizers, and physical qubits. The relationships between the stabilizers and qubits are represented by maps between these spaces.

The homological product gives a mechanism for combining two such chain complexes to give a new one. Many properties of the CSS code obtained in this way, for example its degeneracy, can also be found directly from properties of the component codes. We define the embeddable fractal product codes through the homological product of a toric code on a Sierpinski carpet graph (with punctures for each hole in the Sierpinski carpet) and its dual. The resulting code is intuitively the 4D toric code defined on the product of a Sierpinski carpet graph with its planar dual (again with appropriate punctures).

The logical qubits of this code come in two types. There are an extensive number of encoded qubits associated with punctures in the 4D lattice. These have logical operators that may be localized to the regions around each of these punctures, and so are local degeneracies in this sense (though the punctures appear at all length scales). There is also a single encoded qubit associated with the outer boundaries of the 4D lattice, and the corresponding logical operators are fractal membranes whose support lies on 2D cross-sections of the lattice. We anticipate that the phase transitions we identify in this system correspond to the appearance of thermal stability for this global encoded qubit.

The 4D fractal lattice just described is a discretization of the product of two Sierpinski carpets. The parameters of the Sierpinski carpets may be chosen such that the Hausdorff dimension of such a fractal object is less than 3, but it is naturally embedded in $\mathbb{R}^4$. Though it is generally difficult to determine whether or not a fractal may be embedded in another space, we can use embedding theorems for a special class of self-similar fractals satisfying the so-called strong separation condition [35] that guarantee low distortion (or bilipschitz) embeddings exist. This allows us to show that the embeddable fractal product codes are indeed realizable with local interactions in $\mathbb{R}^3$.

We study the thermodynamic properties of these codes, and relate each of the sectors (X or Z) of the quantum code to a classical generalized Ising model. Using duality transformations due to Merlini and Gruber [36], and the GKS correlation inequalities [37, 38], we are able to relate the thermodynamic properties of these classical Ising models to those of the Sierpinski carpet Ising models mentioned previously. Since it is known that the Sierpinski carpet Ising models have a finite temperature phase transition [32–34], we conclude that the EFPC Hamiltonian has two finite temperature phase transitions (one associated with each of the sectors). This strongly indicates that, below the critical temperatures, there will be an exponential memory lifetime for the encoded quantum information. Based on this evidence, it seems likely that the EFPC Hamiltonian may act as a self-correcting quantum memory in 3D.

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