No-Signalling Assisted Zero-Error Capacity of Quantum Channels and an Information Theoretic Interpretation of the Lovász Number

Full version at: arXiv:1409.3426

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Abstract: We study the one-shot zero-error classical capacity of quantum channels assisted by quantum no-signalling correlations, and the reverse problem of simulation. Both lead to simple semi-definite programmings whose solutions can be given in terms of conditional min-entropies. We show that the asymptotic simulation cost is precisely the conditional min-entropy of the Choi-Jamiołkowski matrix of the given channel. For classical-quantum channels, the asymptotic capacity is reduced to a quantum fractional packing number suggested by Harrow, which leads to the first information-theoretic operational interpretation of the celebrated Lovász ϑ function as the zero-error classical capacity of a graph assisted by quantum no-signalling correlations.

Keywords: Zero-error classical communication, Lovász ϑ function, Non-commutative bipartite graph, Quantum no-signalling correlations

When a communication channel $\mathcal{N}$ from Alice (A) to Bob (B) can be used to simulate another channel $\mathcal{M}$ that is also from A to B? We can abstractly represent the simulation process as the FIG.1. This problem has many variants according to the resources available to A and B. In

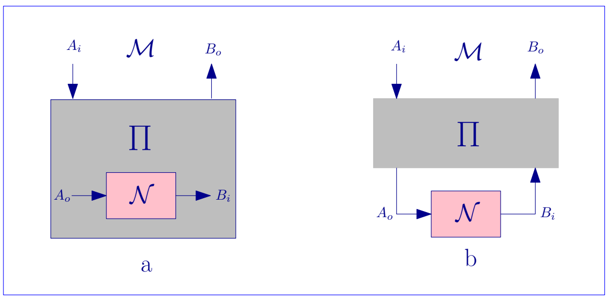

FIG. 1. A general simulation network: a) The general simulation procedure for implementing a channel $\mathcal{M}$ using another channel $\mathcal{N}$ just once, and the correlations between A and B; b) An equivalent way to redraw a), representing all correlations between A an B, and their pre- and post- processing as Π, a quantum no-signalling correlation.

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particular, the case when A and B can access unlimited amount of shared entanglement has been completely solved. Let $C_E(\mathcal{N})$ denote the entanglement-assisted classical capacity of $\mathcal{N}$ [1]. It was shown that, in the asymptotic setting, to optimally simulate $\mathcal{M}$, we need to apply $C_E(\mathcal{M})/C_E(\mathcal{N})$ times of $\mathcal{N}$ [2]. In other words, the entanglement-assisted classical capacity uniquely determines the property of the channel in the simulation process.

We are interested in the zero-error case first studied by Shannon in 1956 [3]. It is well known that determining the zero-error classical capacity is generally extremely difficult even for classical channels. Remarkably, by allowing a feedback link from the receiver to the sender, Shannon proved that the zero-error classical capacity is given by an interesting quantity which was later called the fractional packing number. This number only depends on the bipartite graph induced by the classical channel under consideration, and has a simple linear programming characterization. Recently Cubitt *et al* introduced classical no-signalling correlations into the zero-error simulation problems for classical channels, and proved that the well-known fractional packing number gives precisely the zero-error classical capacity of the channel [4].

Another major motivation for this work is to further explore the connection between quantum information theory and the so-called non-commutative graph theory suggested in [12]. Now it is well known that any classical channel induces a bipartite graph as well as a confusability graph, while a quantum channel induces a non-commutative bipartite graph and a non-commutative graph. The new insight is that we can simply regard a non-commutative (bipartite) graph as a high-level abstraction of all underling quantum channels, and study its information-theoretic properties. This leads us to a very general viewpoint: graphs as communication channels. It remains a great challenge to find feasible forms of various capacities for non-commutative (bipartite) graphs.

A class of quantum no-signalling correlations has been introduced as a natural generalization of classical non-signalling correlations [5–7]. Any such correlation is described by a two-input and two-output quantum channel with no-signalling constraints between A and B (refer to $\Pi : \mathcal{L}(A_i \otimes B_i) \to \mathcal{L}(A_o \otimes B_o)$ in FIG.1). We imitate the approach in [4] to study the zero-error classical capacity of a general noisy quantum channels and the reverse problem of simulation, both assisted by this more general class of quantum no-signalling correlations. We show below that both problems can be completely solved in the one-shot scenarios, and the solutions are given by semi-definite programmings (SDPs). Let $\mathcal{N}$ be a quantum channel with a Kraus operator sum representation $\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger$, where $\sum_k E_k^\dagger E_k = \mathbb{1}$. Let $K = \mathrm{span}\{E_k\}$ denote the Kraus operator space of $\mathcal{N}$, also referred as the non-commutative bipartite graph associated with $\mathcal{N}$. The Choi-Jamiołkowski matrix of $\mathcal{N}$ is given by $J_{AB} = (\mathrm{id}_A \otimes \mathcal{N})\Phi_{AA'}$ with $\Phi_{AA'}$ the unnormalized maximally entangled state. Let P_{AB} denote the projection on the support of J_{AB}.

The one-shot zero-error classical capacity of $\mathcal{N}$ assisted by quantum no-signalling correlations only depends on the Kraus operator space K, and is given by the integer part of following SDP

$$\Upsilon(K) = \max \mathrm{Tr}\, S_A \;\text{ s.t. }\; 0 \le U_{AB} \le S_A \otimes \mathbb{1}_B, \mathrm{Tr}_A\, U_{AB} = \mathbb{1}_B, \mathrm{Tr}\, P_{AB}(S_A \otimes \mathbb{1}_B - U_{AB}) = 0.$$

Similarly, the exact simulation problem has a SDP formulation. The one-shot zero-error classical cost of simulating a quantum channel $\mathcal{N}$ with Choi-Jamiołkowski matrix J_{AB} is given by $\lceil 2^{-H_{\min}(A|B)_J} \rceil$ messages per channel realization, where $H_{\min}(A|B)_J$ is the conditional min-entropy defined as follows [8]:

$$2^{-H_{\min}(A|B)_J} = \min \mathrm{Tr}\, T_B, \text{ s.t.}, J_{AB} \le \mathbb{1}_A \otimes T_B.$$

Since the conditional min-entropy is additive, it follows immediately that the asymptotic simulation cost of a channel is given by $-H_{\min}(A|B)_J$ bits per channel realization. As a direct consequence, the asymptotic zero-error classical simulation cost of the cq-channel $0 \to \rho_0$ and $1 \to \rho_1$, is given by $\log(1 + D(\rho_0, \rho_1))$, where $D(\rho_0, \rho_1) = \frac{1}{2}\|\rho_0 - \rho_1\|_1$ is the trace distance between ρ_0 and ρ_1. This provides a new operational interpretation of the trace distance between ρ_0 and ρ_1 as the asymptotic exact simulation cost for the above cq-channel.

The exact simulation cost $\Sigma(K)$ of the cheapest channel $\mathcal{N}$ such that $K(\mathcal{N}) < K$ (supporting on P_{AB}), is given by the integer part of

$$\Sigma(K) = \min \operatorname{Tr} T_B \text{ s.t. } 0 \le V_{AB} \le \mathbb{1}_A \otimes T_B, \operatorname{Tr}_B V_{AB} = \mathbb{1}_A, \operatorname{Tr}(\mathbb{1} - P)_{AB} V_{AB} = 0.$$

Let us now introduce the asymptotic zero-error channel capacity and simulation cost of K as follows,

$$C_{0,\text{NS}}(K) = \sup_{n \ge 1} \frac{1}{n} \log \Upsilon(K^{\otimes n}), \ G_{0,\text{NS}}(K) = \inf_{n \ge 1} \frac{1}{n} \log \Sigma(K^{\otimes n}).$$

In general, one-shot solutions do not give the asymptotic results (for instance, a c-q channel with two non-orthogonal pure states), and feasible formulas for the asymptotic capacity and simulation cost remain unknown.

Interestingly, for the case K corresponding to a cq-channel $\mathcal{N} : i \to \rho_i$, both quantities can be determined completely. Indeed, the asymptotic simulation cost is given by the one-shot simulation cost, i.e., $G_{0,\text{NS}}(K) = \log \Sigma(K)$, which immediately implies that $G_{0,\text{NS}}(K)$ is additive under tensor product.

The zero-error classical capacity exhibits more complexity and is given by the solution of the following simplified SDP

$$\mathsf{A}(K) = \max \sum_i s_i, \text{ s.t. } 0 \le s_i, \sum_i s_i P_i \le \mathbb{1},$$

and P_i is the projection on the support of ρ_i. $\mathsf{A}(K)$ was introduced by A. Harrow as a natural generalization of the Shannon's classical fractional packing number [9], and can be named as *semidefinite (fractional) packing number* associated with a set of projections $\{P_i\}$. Then we have $C_{0,\text{NS}}(K) = \log \mathsf{A}(K)$. More precisely, we have

$$\frac{1}{\text{poly}(n)} \mathsf{A}(K)^n \le \Upsilon(K^{\otimes n}) \le \mathsf{A}(K)^n.$$

The above capacity formula naturally generalizes the result in [4], and has two interesting corollaries. First, it implies that the zero-error classical capacity of cq-channels assisted by quantum no-signalling correlations is additive, i.e., $C_{0,\text{NS}}(K_1 \otimes K_2) = C_{0,\text{NS}}(K_1) + C_{0,\text{NS}}(K_2)$, for any two Kraus operator spaces K_1 and K_2 corresponding to cq-channels.

Second, and more importantly, we show that for any undirected classical graph $G = (V, E)$ with vertices $V = \{1, ..., n\}$ and edges $E \subset V \times V$, the Lovász ϑ function [10], is an achievable lower bound of the zero-error classical capacity assisted by quantum no-signalling correlations of any quantum channel $\mathcal{N}$ that has G as its non-commutative graph in the sense of [12]. For simplicity, we denote the non-commutative graph generated by the graph G as

$$G = \text{span}\{|i\rangle\langle j| : (i, j) \in E \text{ or } i = j, i, j \in V\}.$$

The zero-error classical capacity of a graph G assisted by quantum non-signalling correlations is defined as

$$C_{0,\text{NS}}(G) = \min\{C_{0,\text{NS}}(K) : K^\dagger K = G\}.$$

Then we have $C_{0,\text{NS}}(G) = \log \vartheta(G)$. Thus the Lovász ϑ function of a graph G can be operationally interpreted as the zero-error classical capacity of the graph assisted by quantum no-signalling correlations. To the best of our knowledge, this is the first complete information-theoretic interpretation of the Lovász ϑ function since 1979. Previously it was shown that the Lovász ϑ function is an upper bound for the zero-error entanglement-assisted classical capacity of a graph [11, 12].

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