Entanglement and swap of quantum states

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Suppose that two distant parties Alice and Bob share an entangled state ρ_{AB}, and they want to exchange the subsystems of ρ_{AB} by local operations and classical communication (LOCC). In general, this LOCC task (i.e. the LOCC transformation of $\rho_{AB} \to V\rho_{AB}V$ with V being a swap operator) is impossible deterministically, but becomes possible probabilistically. We study how the optimal probability is related to the amount of entanglement in the framework of positive partial transposed (PPT) operations, and show a remarkable class of states whose probability is the smallest among every state in two quantum bits.

Suppose that two distant parties Alice and Bob share an entangled state ρ_{AB}, but they need the other state such that the subsystems of ρ_{AB} are exchanged (i.e. $V\rho_{AB}V$ with V being a swap operator). Is it possible for them to swap a given ρ_{AB} by local operations and classical communication (LOCC)? This problem, first considered by Horodecki *et. al.* [1], is quite interesting, because it is closely related to the asymmetry of quantum entanglement. Indeed, if the entanglement contained in ρ_{AB} is asymmetric and the amounts of the entanglement of ρ_{AB} and $V\rho_{AB}V$ are not equal (in some entanglement measure), we can immediately conclude that ρ_{AB} cannot be swapped by LOCC, because the amount of entanglement cannot be increased by LOCC [1]. In Ref [1], it has also been shown that the entangled states which can be swapped by LOCC can be swapped by a local unitary transformation. This implies that the entangled state ρ_{AB} whose reduced density matrices ρ_A and ρ_B have different eigenvalues cannot be swapped by LOCC, and therefore almost all entangled states cannnot be swapped by LOCC. However, this is the case where a deterministic LOCC transformation with a unit probability is considered. As shown later, such states can also be generally swapped by stochastic LOCC (SLOCC), and therefore by examining the success probability, it becomes possible to quantitatively discuss the degree of difficulty of swapping. For example, it can be said that the state having the smallest swapping probability is the most difficult state to swap, and hopefully the entanglement of the state is the most asymmetric.

We numerically calculate the swapping probability under positive partial transposed (PPT) operations [2], which are known to contain LOCC. Concretely, we obtain the maximal p for given ρ_{AB} among trace-nonincreasing PPT operations Λ such that $\Lambda(\rho_{AB}) = pV\rho_{AB}V$. As a result, we show the special class of states whose swapping probability is the smallest among every state in two quantum bits (qubits). The class of states is

$$\xi = x\,|\psi\rangle\,\langle\psi| + (1-x)\,|01\rangle\,\langle 01|\,, \tag{1}$$

where $|\psi\rangle = \sqrt{y}\,|00\rangle + \sqrt{1-y}\,|11\rangle$, $0 \leq x \leq 1$, and $0 \leq y \leq 1$. The optimal (maximal) swapping probability p_ξ of ξ by SLOCC is as follows:

(i) For $x = 0, 1$ or $y = 0, 1$, $p_\xi = 1$ because ξ is pure or not entangled.

(ii) For $x \neq 0, 1$ and $0 < y \leq \frac{1}{2}$, $p_\xi = \frac{y}{1-y}$. The swap is realized by an SLOCC Λ as $\Lambda(\rho) = (A \otimes B)\rho(A \otimes B)^\dagger$ because $\Lambda(\xi) = \frac{y}{1-y} V \xi V$, where

$$A = B = |1\rangle\,\langle 0| + \sqrt{\frac{y}{1-y}}\,|0\rangle\,\langle 1|\,. \tag{2}$$

(iii) For $x \neq 0, 1$ and $\frac{1}{2} \leq y < 1$, it is clear that $p_\xi = \frac{1-y}{y}$ from the symmetry.

From these, p_ξ changes discontinuously at $x = 0, 1$ and $y = 0, 1$. In this way, the state ξ can be swapped with some finite probability even when it cannot be swapped by deterministic LOCC. It is quite interesting that p_ξ is independent of x for $0 < x < 1$.

Figure 1 shows the numerical results of relationship between the optimum swapping probability p and the concurrence C. The results for 20,000 random density matrices are plotted in each case of rank 2, 3, and 4. Our numerical results indicate that, in the limit $x \to 1$, p_ξ is a lower bound for the swapping probability of *any* state of same concurrence.

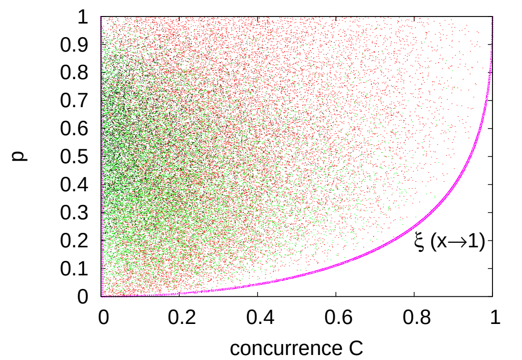

FIG. 1. The relationship between the swapping probability p and the concurrence C. Red, black, and green dots represent the result for rank 2, 3 and 4, respectively.

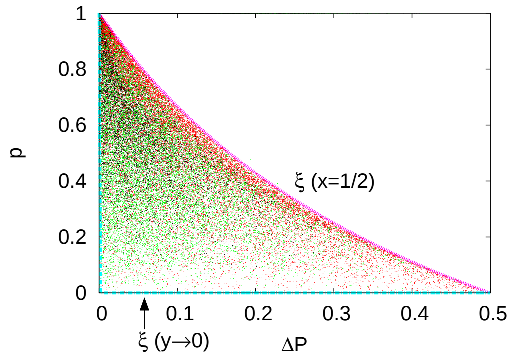

FIG. 2. The relationship of swapping probability p and ΔP.

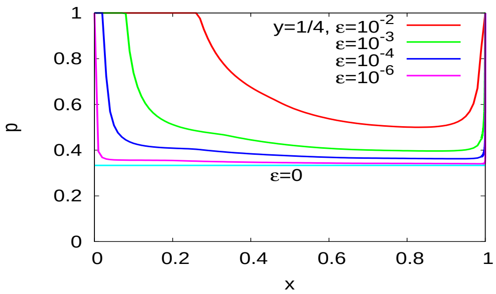

FIG. 3. The optimal swapping probability of ξ_{AB} with y=1/4 for various values of ϵ.

Figure 2 shows the relationship between the swapping probability p and the absolute value of the difference of the purity of the reduced density matrices (ΔP). Only when $\Delta P = 0$, the optimum swapping probability reaches 1, which reproduces the result by Horodecki et. al. [1].

In conclusion, we showed that almost all states in two qubits can be swapped with nonzero probability, and the lower bound of the optimal swapping probability increases for increasing the entanglement. In particular, we showed that the lower bound approaches to zero in the limit of vanishing entanglement. This is somewhat surprising because unentangled states can always be swapped with a unit probability. Moreover, we showed that the lower bond corresponds to the swapping probability of the state ξ in the limit of $x \to 1$. Namely, this state is the most difficult state to swap among the states which have the same amount of entanglement in two qubits.

Why is the state ξ extremely difficult to swap? The reason is not so clear but this state, the mixture of the pure entangled states $|\psi\rangle$ and $|01\rangle$, has relatively high entanglement. In order to swap this state, however, we are subject to the strong restriction that $|01\rangle$ must be transformed to $|10\rangle$, even if the portion of $|01\rangle$ is infinitesimally small. As a result, ξ has the special property that the optimal swapping probability is fully discontinuous around $x = y = 1$, i. e. the probability drops from 1 to 0 by the infinitesimal change of x.

Obviously, this discontinuity originates from the fact that we considered the exact transformation. If we consider the approximate transformation where the target state ρ'_{AB} is allowed to have a small margin ϵ such that $D(\rho'_{AB}, V\rho_{AB}V) \leq \epsilon$ (D is the trace distance), the discontinuity is smeared out. Indeed, the optimal swapping probability of ξ is smeared out as shown in Figure 3.

We also studied how the swapping probability is related to the mixedness, and showed that the lower bound ($p = 0$ for a purity larger than 1/2) is again explained by the state ξ (in the limit of $y \to 0$). Therefore, ξ is the most difficult state to swap in this sense also.

We hope that our results will shed some light on unknown properties of entanglement, in particular concerning the asymmetry of quantum correlation.

[1] K. Horodecki *et. al.*, Quant. Inf. & Comp. 10, (2010), 901-910.
[2] E. M. Rains, Phys. Rev. A **60**, 173 (1999).