Quantum-information Division and Optimal Uncorrelated Channel

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Abstract

We consider *quantum-information division*, which is characterized by a channel whose outputs have no correlation and are not completely randomized. We show that the quantum-information division is possible in a probabilistic manner by optimizing the average fidelity in the channel with M outputs in both deterministic and probabilistic cases. Moreover, we show that the optimal fidelities drastically change depending on the condition imposed on the outputs (symmetric and asymmetric), which is quite contrast to the case of imperfect cloning.

No-cloning theorem [2, 3], which states that an unknown quantum state cannot be perfectly copied, is a cornerstone of quantum physics. In spite of the no-go theorem, however, it was shown by Bužek and Hillery [4] that imperfect cloning, where average fidelity between the original unknown state (input) and the copied state (output) do not reach 1, is possible. After their insight, the Bužek-Hillery imperfect quantum cloning machine was proved to be optimal in the sense of average fidelity [5, 6, 7]. For the intensive review of quantum cloning, see Ref. [8].

At first glance, imperfect cloning seems to divide unknown quantum information of the input into the outputs. As we can see in the Bužek-Hillery imperfect quantum cloning machine for instance, however, there remains correlation among the output states in general. Namely, the output states are no longer independent to each other, and thus quantum information is regarded to be distributed rather than divided among the outputs in imperfect cloning. In oder to say that "quantum information is divided," it would be at least necessary that there exists no correlation including not only quantum correlation such as entanglement but also classical one among the outputs. Namely, the output state in the $1 \to M$ cloning for each input state $|\psi\rangle$ should have the form of $\rho_1(\psi) \otimes \rho_2(\psi) \otimes \cdots \otimes \rho_M(\psi)$, where $\rho_i(\psi)$ is the i-th output as a function of $|\psi\rangle$. Now we have a question: Is imperfect cloning without correlation among outputs possible, or is *uncorrelated cloning* possible? This is not a trivial issue. For instance, let us consider the cloning strategy called *the measurement-based procedure* [8], where an input state is measured in the $\{|0\rangle, |1\rangle\}$ basis, and either $|0\rangle|0\rangle$ or $|1\rangle|1\rangle$ is prepared depending on the measurement outcomes. This seems to achieve the desired uncorrelated cloning, but for the input state of $|\psi\rangle = \sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle$, the procedure results in the output state of $p|00\rangle\langle 00| + (1-p)|11\rangle\langle 11|$. Clearly, the two outputs are classically correlated unless $p = 0, 1$, and hence this procedure does not achieve the uncorrelated cloning.

Recently D'Ariano *et al.* studied this issue from the viewpoint of *quantum-state decorrelation* [9]. To be surprised the answer is negative, which means that no matter how small, multiple outputs cannot depend on the input state $|\psi\rangle$ simultaneously if no correlation among outputs is allowed; only one output can depend on $|\psi\rangle$. However, the quantitative evaluation by means of average fidelity is crucial to see whether unknown quantum information can be divided or not. So we study this issue by optimizing average fidelity in the information division into M outputs in d-dimensional systems in both deterministic and probabilistic cases. In each case, the optimization is performed by imposing symmetric or asymmetric condition on the output states [1].

In this poster we deal with the issue of dividing an unknown pure state in general. So we consider a non-trace-preserving (trace-decreasing) channel, which takes any d-dimensional pure state $|\psi\rangle$ as an input, and outputs M states each with the same dimension d. Moreover, when the output states always do not have any correlation, we call it an uncorrelated channel. (Note that an uncorrelated channel excludes even classical correlation among outputs). Thus a map of the uncorrelated channel from an input to the i-th output can be written as

$$|\psi\rangle \xrightarrow{\Lambda_{s_i}} p_{\psi}\rho_{s_i}(\psi), \tag{1}$$

where p_{ψ} is probability to output a state $\rho(\psi)$ and the suffix S_i stands for the ith-output system. In this channel, the output state $\rho_{s_i}(\psi)$ is realized with the probability p_{ψ} for input state $|\psi\rangle$. So the average fidelity of a trace-decreasing channel Λ_{s_i} is defined with the weight of probability p_{ψ} as

$$F_{s_i} = \frac{\displaystyle\int p_{\psi}\langle\psi|\rho_{s_i}(\psi)|\psi\rangle d\psi}{\displaystyle\int p_{\psi} d\psi}, \tag{2}$$

where the integral is over the uniform measure $d\psi$ on pure input states. This definition is the natural generalization of the average fidelity of a trace-preserving channel [10]. When the average fidelity is $1/d$, the input and output are independent each other. Conversely, if average fidelity is not $1/d$, the output has some sort of information of the input (i.e. not-randomized) . Therefore, it can be defined such that *quantum-information division* is possible if and only if $F_{s_i} \neq 1/d$ for all outputs S_i in an uncorrelated channel. With the fidelities F_{s_i}, we consider the average of F_{s_i} for all outputs S_i $(i = 1, 2 \cdots M)$

$$\bar{F} = \frac{1}{M}\sum_{i=1}^{M} F_{s_i}, \tag{3}$$

and optimize this $\bar{F}$ under the uncorrelated condition.

The derived optimal fidelities are summarized in TABLE 1. We notice that the optimal fidelities in the uncorrelated channels remarkably vary according to the conditions imposed on the channels. This result seems to present the strikingly contrast to the imperfect cloning, where the optimal fidelity does not change at all and is $\bar{F} = 5/6$ for every condition in $d = 2$, for instance (since the value $\bar{F} = 5/6$ is known to coincide with the boundary of the no-signaling condition [7], such invariance is expected).

The optimal deterministic and asymmetric uncorrelated channel can be realized by attaching randomized states to the intact input state, where the optimal average fidelity is thus $1/M + (M-1)/dM$ (this optimal channel is the same as the one called *trivial amplification* in Ref. [8]), and the optimal deterministic and symmetric channel can be realized

	Symmetric	Asymmetric
Deterministic	$\frac{1}{d}$ $(F_{S_k} = F_{S_i} = \frac{1}{d})$	$\frac{1}{M} + \frac{M-1}{dM}$ $(F_{S_k} = 1, F_{S_i} = \frac{1}{d})$
Probabilistic	$\frac{2}{d+1}(F_{S_k} = F_{S_i} = \frac{2}{d+1})$	$\frac{3M+d-1+D(d,M)}{2M(d+1)}$ $(F_{S_k} = \frac{(2-d)\xi+d+2\sqrt{\xi(1-\xi)(d-1)}}{d+1}, F_{S_i} = \frac{\xi+1}{d+1})$

Table 1: The optimal average fidelities in the d-dimensional $1 \to M$ uncorrelated channels ($D(d,M) \equiv \sqrt{(M+d-1)^2 - 4(M-1)(d-1)}$, $\xi \equiv 1/[1 + (d-1)/(d\lambda_m - d + 1)^2]$, $\lambda_m \equiv (M + d - 1 + D(d,M))/2d$. Note: The index S_i stands for all output systems except for a certain system S_k.)

by randomizing all output states, where the optimal average fidelity is $1/d$. In these cases (i.e. in the deterministic ones), the fidelities at multiple outputs cannot exceed $1/d$, that is, *quantum-information division* is impossible. This impossibility is also expected from [9].

On the other hand, in the optimal probabilistic uncorrelated channels, all the average fidelities at output can exceed $1/d$ (even in the symmetric channel). It is interesting that, even in this case, the optimal asymmetric channel is realized by attaching randomized states to an input-dependent state, and the optimal symmetric one is realized by randomizing all output states. With this similarity to the deterministic case, however, in the probabilistic channels whether the outputs exist or not can contain the input information, through which the randomized states can indirectly depend on the input states. This is why all the average fidelities at output can exceed $1/d$. Therefore, we can conclude that *quantum-information division* is possible probabilistically.

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