

# Generalized Schmidt rank and entanglement for composite systems of indistinguishable particles <sup>1</sup>

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## 1. Introduction

The possibility of identifying subsystems states in a given total state of a composite quantum system goes under the name of *separability*. In the case of pure states such a possibility is guaranteed if the composite state takes the form of the tensor product of subsystems states.

On the other hand, with the advent of Quantum Field Theory, we have identified elementary particles which are either bosons or fermions. As a matter of fact, according to the spin-statistics theorem all particles are either bosons or fermions. The difference is that a state is unchanged by the interchange of two identical bosons, whereas it changes the sign under the interchange of two identical fermions. The characterization of fermionic states contains already the lack of the factorization of the total state of the composite system. According to usual wisdom, this would always imply the presence of an entanglement. In our opinion this state of affairs cannot be maintained, so there is a need of a refinement of the notion of entanglement that describes better the situation when we are dealing with bosons and fermions or even with ‘parabosons’ or ‘parafermions’ arising from potentially meaningful parastatistics [1, 2].

In [3] we analyzed a concept of entanglement for a multipartite system with bosonic and fermionic constituents in purely algebraic way using the representation theory of the underlying symmetry groups. Correlation properties of indistinguishable particles become relevant when subsystems are no longer separated by macroscopic distances, like e.g. in quantum gates based on quantum dots, where they are confined to the same spatial regions [5]. In our approach to bosons and fermions we adopted the concept of entanglement put forward in [5, 6] for fermionic systems and extended in [7, 8] in a natural way to bosonic ones.

Our approach appeared to be sufficiently general to define entanglement also for systems with an arbitrary *parastatistics* in a consistent and unified way. For pure states we defined the S-rank, generalizing the notion of the Schmidt rank for distinguishable particles and playing an analogous role in the characterization of the degree of entanglement among particles with arbitrary exchange symmetry (parastatistics).

In the algebraic geometry, a canonical embedding of the product  $\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$  of complex projective spaces into  $\mathbb{C}P^{nm-1}$  is known under the name the *Segre embedding* (or the *Segre map*). In the quantum mechanical context, the complex projective space  $\mathbb{C}P^{n-1}$  represents pure states in the Hilbert space  $\mathbb{C}^n$ , and  $\mathbb{C}P^{nm-1}$  represent pure states in  $\mathbb{C}^n \otimes \mathbb{C}^m$ , so that the Segre embedding gives us a geometrical description of separable pure states and, as shown in [9, 10], this description can be extended also to mixed states.

In [4] we gave a geometric description of the entanglement for systems with arbitrary symmetry (with respect to exchanging of subsystems) in terms of *generalized Segre embeddings* associated with particular parastatistics. This description is complementary to the one presented in [3] in terms of the S-rank. For systems with arbitrary exchange symmetries, unlike for the systems of distinguishable particles, the spaces of states are not, in general, projectivizations of the full tensor products of the underlying Hilbert spaces of subsystems, but rather some parts of them. We show in the following how to extend properly the concept of the Segre embedding to achieve a geometric description analogous to that for distinguishable particles. This approach uses a unifying mathematical framework based on the representation theory and strongly suggesting certain concepts of the separability, thus of the entanglement, in the case of indistinguishable particles.

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## 2. Generalized Schmidt rank

Let  $\mathcal{H}$  be a finite-dimensional Hilbert space with a Hermitian product  $\langle \cdot | \cdot \rangle$ . In the tensor power  $\mathcal{H}^{\otimes k} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{k\text{-times}}$ , we distinguish the subspaces:  $\mathcal{H}^{\vee k} = \underbrace{\mathcal{H} \vee \cdots \vee \mathcal{H}}_{k\text{-times}}$  of totally symmetric tensors and  $\mathcal{H}^{\wedge k} = \underbrace{\mathcal{H} \wedge \cdots \wedge \mathcal{H}}_{k\text{-times}}$  of totally antisymmetric ones.

There are many concepts of a rank of a tensor used in describing its complexity. One of the simplest and most natural is the one based on the inner product operators defined in the previous section. This rank, called in [3] the *S-rank* and used there to define the entanglement for systems of indistinguishable particles, is a natural generalization of the *Schmidt rank* of 2-tensors.

**Definition 1.** Let  $u \in \mathcal{H}^{\otimes k}$ . By the *S-rank* of  $u$ , we understand the maximum of dimensions of the linear spaces  $\iota_{\mathcal{H}}^{k-1} \sigma(u)$ , for  $\sigma \in S_k$ , which are the images of the contraction maps

$$\mathcal{H}^{\otimes(k-1)} \ni \nu \mapsto \iota_{\nu} \sigma(u) \in \mathcal{H}. \quad (1)$$

Non-zero tensors of minimal S-rank in  $\mathcal{H}^{\otimes k}$  (resp.,  $\mathcal{H}^{\vee k}$ ,  $\mathcal{H}^{\wedge k}$ ) we will call *simple* (resp., *simple symmetric*, *simple antisymmetric*).

Note that the above definition has its natural counterpart for distinguishable particles, so tensors from  $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ . We just do the contractions with tensors from  $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}$  and the corresponding permutations. If particles are identical,  $\mathcal{H}_i = \mathcal{H}$ , and indistinguishable, e.g. the tensors are symmetric or skew-symmetric, we can skip using permutations. In other words, for  $u \in \mathcal{H}^{\vee k}$  (resp.,  $u \in \mathcal{H}^{\wedge k}$ ), the S-rank of  $u$  equals the dimension of the linear space which is the image of the contraction map,

$$\mathcal{H}^{\vee(k-1)} \ni \nu \mapsto \iota_{\nu} u \in \mathcal{H}, \quad (2)$$

(resp.,

$$\mathcal{H}^{\wedge(k-1)} \ni \nu \mapsto \iota_{\nu} u \in \mathcal{H}). \quad (3)$$

**Theorem 1.** ([3])

- (a) The minimal possible S-rank of a non-zero tensor  $u \in \mathcal{H}^{\otimes k}$  equals 1. A tensor  $u \in \mathcal{H}^{\otimes k}$  is of S-rank 1 if and only if  $u$  is decomposable, i.e., it can be written in the form

$$u = f_1 \otimes \cdots \otimes f_k, \quad f_i \in \mathcal{H}, \quad f_i \neq 0. \quad (4)$$

Such tensors span  $\mathcal{H}^{\otimes k}$ .

- (b) The minimal possible S-rank of a non-zero tensor  $v \in \mathcal{H}^{\vee k}$  equals 1. A tensor  $v \in \mathcal{H}^{\vee k}$  is of S-rank 1 if and only if  $v$  can be written in the form

$$v = f \vee \cdots \vee f, \quad f \in \mathcal{H}, \quad f \neq 0. \quad (5)$$

Such tensors span  $\mathcal{H}^{\vee k}$ .

- (c) The minimal possible S-rank of a non-zero tensor  $w \in \mathcal{H}^{\wedge k}$  equals  $k$ . A tensor  $w \in \mathcal{H}^{\wedge k}$  is of S-rank  $k$  if and only if  $w$  can be written in the form

$$w = f_1 \wedge \cdots \wedge f_k, \quad (6)$$

where  $f_1, \dots, f_k \in \mathcal{H}$  are linearly independent. Such tensors span  $\mathcal{H}^{\wedge k}$ .

In particular, the S-rank is 1 for simple and simple symmetric tensors and it is  $k$  for simple antisymmetric tensors from  $\mathcal{H}^{\wedge k}$ . Simple tensors have the form (4), simple symmetric tensors have the form (5), and simple antisymmetric tensors have the form (6).

### 3. Entanglement

Using the concept of simple tensors we can define simple (non-entangled or separable) and entangled pure states for multipartite systems of bosons and fermions.

#### Definition 2.

- (a) A pure state  $\rho_x$  on  $\mathcal{H}^{\vee k}$  (resp., on  $\mathcal{H}^{\wedge k}$ ),  $\rho_x = \frac{|x\rangle\langle x|}{\|x\|^2}$ , with  $x \in \mathcal{H}^{\vee k}$  (resp.,  $x \in \mathcal{H}^{\wedge k}$ ),  $x \neq 0$ , is called a *bosonic* (resp., *fermionic*) *simple* (or *non-entangled*) *pure state* if  $x$  is a simple symmetric (resp., antisymmetric) tensor. If  $x$  is not simple symmetric (resp., antisymmetric), we call  $\rho_x$  a *bosonic* (resp., *fermionic*) *entangled state*.
- (b) A mixed state  $\rho$  on  $\mathcal{H}^{\vee k}$  (resp., on  $\mathcal{H}^{\wedge k}$ ) we call *bosonic* (resp., *fermionic*) *simple* (or *non-entangled*) *mixed state* if it can be written as a convex combination of bosonic (resp., fermionic) simple pure states. In the other case,  $\rho$  is called *bosonic* (resp., *fermionic*) *entangled mixed state*.

According to Theorem 0.1, bosonic simple pure  $k$ -states are of the form

$$|e \vee \cdots \vee e\rangle\langle e \vee \cdots \vee e|$$

for unit vectors  $e \in \mathcal{H}$ , and fermionic simple pure  $k$ -states are of the form

$$k!|e_1 \wedge \cdots \wedge e_k\rangle\langle e_1 \wedge \cdots \wedge e_k|$$

for orthonormal systems  $e_1, \dots, e_k$  in  $\mathcal{H}$ .

Fixing a base in  $\mathcal{H}$  results in defining coefficients  $[u^{i_1 \dots i_k}]$  of  $u \in \mathcal{H}^{\otimes k}$ . Formulae characterizing simple tensors, thus simple pure states, can be written in terms of quadratic equations with respect to these coefficients as follows. The corresponding characterization of entangled pure states are obtained by negation of the latter.

#### Theorem 2. ([3])

- (a) The pure state  $\rho_u$ , associated with a tensor  $u = [u^{i_1 \dots i_k}] \in \mathcal{H}^{\otimes k}$ , is entangled if and only if there exist  $i_1, \dots, i_k, j_1, \dots, j_k$ , and  $s = 1, \dots, k$  such that

$$u^{i_1 \dots i_s \dots i_k} u^{j_1 \dots j_s \dots j_k} \neq u^{i_1 \dots j_s \dots i_k} u^{j_1 \dots i_s \dots j_k}. \quad (7)$$

- (b) The bosonic pure state  $\rho_v$ , associated with a symmetric tensor  $v = [v^{i_1 \dots i_k}] \in \mathcal{H}^{\vee k}$ , is bosonic entangled if and only if there exist  $i_1, \dots, i_k, j_1, \dots, j_k$ , such that

$$v^{i_1 \dots i_{k-1} i_k} v^{j_1 \dots j_{k-1} j_k} \neq v^{i_1 \dots i_{k-1} j_k} v^{j_1 \dots j_{k-1} i_k}. \quad (8)$$

- (c) The fermionic pure state  $\rho_w$ , associated with an antisymmetric tensor  $w = [w^{i_1 \dots i_k}] \in \mathcal{H}^{\wedge k}$ , is fermionic entangled if and only if there exist  $i_1, \dots, i_{k+1}, j_1, \dots, j_{k-1}$  such that

$$w^{[i_1 \dots i_k w^{i_{k+1}}] j_1 \dots j_{k-1}} \neq 0, \quad (9)$$

where the left-hand side is the antisymmetrization of  $w^{i_1 \dots i_k} w^{i_{k+1} j_1 \dots j_{k-1}}$  with respect to the indices  $i_1, \dots, i_{k+1}$ .

Note that the opposite to (9),  $w^{[i_1 \dots i_k w^{i_{k+1}}] j_1 \dots j_{k-1}} = 0$ , are sometimes called the *Plücker relations*.

## References

- [1] H. S. Green. A generalized method of field quantization. *Phys. Rev.*, 90:270–273, 1953.
- [2] Y. Ohnuki and S. Kamefuchi. *Quantum Field Theory and Parastatistics*. Springer, Berlin (1982).
- [3] J. Grabowski, M. Kuś, and G. Marmo. Entanglement for multipartite systems of indistinguishable particles. *J. Phys. A: Math. Gen.* 44:175302 (21pp), 2011.
- [4] J. Grabowski, M. Kuś, and G. Marmo. Segre maps and entanglement for multipartite systems of indistinguishable particles. *J. Phys. A: Math. Gen.* 45:105301 (17pp), 2012.
- [5] J. Schliemann, D. Loss, and A. H. MacDonald. Double-occupancy errors, adiabaticity, and entanglement of spin qubits in quantum dots. *Phys. Rev. B*, 63(8):085311, 2001.
- [6] J. Schliemann, J. I. Cirac, M. Kuś, M. Lewenstein, and D. Loss. Quantum correlations in two-fermion systems. *Phys. Rev. A*, 64:022303, 2001.
- [7] K. Eckert, J. Schliemann, D. Bruß, and M. Lewenstein. Quantum correlations in systems of identical particles. *Ann. Phys.*, 299:88–127, 2002.
- [8] R. Paškauskas and L. You. Quantum correlations in two-boson wave functions. *Phys. Rev. A*, 64(4):042310, 2001.
- [9] J. Grabowski, M. Kuś, and G. Marmo. Geometry of quantum systems: density states and entanglement. *J. Phys. A: Math. Gen.* 38:10217–10244, 2005.
- [10] J. Grabowski, M. Kuś, and G. Marmo. Symmetries, group actions, and entanglement. *Open Sys. Information Dyn.*, 13:343–362, 2006.