

# Understanding protected gates in topological quantum codes via anyons

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## Abstract

We provide restrictions on the logical gates that can be performed by constant-depth local circuits (as a form of protected gates) on topological quantum error-correcting codes (TQECC). In particular, our results apply to codes that can be described as topological quantum field theories, which provide a concise description in terms of anyons. These include the most promising TQECCs [16, 17, 6, 4, 18, 5, 15]. We show that such gates generate a finite group in any such code, and hence are not universal for quantum computation. The group of admissible gates is related to the symmetries of the underlying anyon model. For abelian anyon models, we find that these gates form a proper subgroup of the generalized Clifford group strengthening the result of Ref. [7]. For non-abelian models, such gates are even further restricted. For example, for Ising anyons, they must be elements of the Pauli group whereas for Fibonacci anyons there are no non-trivial gates. These results highlight the necessity for non-local information processing of some form.

A fully operational quantum computer should be capable of both storing and processing quantum information, while shielding it from noise that would otherwise render the computation useless. Kitaev's toric code [16] is arguably one of the most promising platforms for the realization of fault-tolerant quantum storage: its 2D geometry is attractive from an experimental viewpoint (allowing implementation with present-day technology [1, 10]). It only requires local operations to identify and correct errors, it has a macroscopic code distance (growing with the number of physical qubits), as well as a constant threshold for storage and computation [11, 21]. Similar properties are found in the wider class of topological quantum error-correcting codes (TQECCs), which are thus natural candidates for fault-tolerant quantum hardware. Examples of such systems include commuting-projector codes (of non-stabilizer type, such as Kitaev's quantum double models [16] or the Levin-Wen models [17]), fractional quantum Hall systems [18] and topological insulators with topologically nontrivial surfaces [5, 15]. The general framework of topological quantum field theories (TQFTs) encompasses all such systems and provides a simple mathematical description in terms of anyons without reference to microscopic details of the system (see e.g., the lecture notes of [20]).

**Braiding vs. constant-depth local circuits.-** There are two main methods to achieve protected processing of the quantum information stored in topological codes. The first is to braid anyons around one-another to implement a logical unitary. During the execution of a braid, the information stays encoded in a code with macroscopic distance, thereby protecting it from noise. The second method is to use a constant-depth local circuit (CDLC) to implement a logical unitary, which by design approximately preserves locality properties of preexisting errors. This offers protection in topological codes, which can correct local errors. A simple case is provided by the subset of transverse gates which act independently on each qubit. For example, in the toric code, applying physical  $X$  operators on all the qubits along a homologically non-trivial loop implements a logical  $\bar{X}$ . Of both methods, CDLCs provide a faster implementation since braiding must, in general, be carried out in a time proportional to the code distance. Furthermore, a gate implemented by braiding of abelian anyons can be realized by a CDLC whereas generally not for non-abelian anyons [2].

The computational power of braiding of anyons is well-studied and understood [14]: for example, braiding of Ising anyons generates the Clifford group, whereas braiding of Fibonacci anyons produces a dense subgroup of the set of unitaries on a suitable subspace, hence providing universality. In contrast,

the power of CDLCs in the context of general TQFTs has not been previously considered. The favorable properties of CDLCs motivate our study of the unresolved question of whether such unitaries can be found more generally in systems exhibiting topological order and what gates they can perform. We consider the more general class of *locality-preserving* unitaries, which include CDLCs as well as constant-time local-Hamiltonian evolution. Our main question is therefore:

**What is the computational power of locality-preserving unitaries preserving the ground space of a topologically ordered system?**

We aim to characterize unitaries  $U$  which (i) preserve the code space defined by the TQFT, and (ii) preserve the locality of operators (for any operator  $A$ , the operator  $UAU^\dagger$  is supported on a constant-size neighborhood of the support of  $A$ ). We call such unitaries *locality-preserving logical unitaries* (of the code), or simply *protected gates*.

**Results: Characterization of locality-preserving gates in TQFTs.-** The following three results illustrate that locality-preserving gates are severely limited.

- (i) the set of locality-preserving logical unitaries generates only a finite group of unitaries for any TQFT. This means that they cannot provide universality on their own. This result can be seen as a “topological” analogue of the Eastin-Knill theorem characterizing transversal gates in quantum codes [12].
- (ii) for systems with abelian anyons, locality-preserving logical unitaries belong to a proper subgroup of the Clifford group, which we call homology-preserving Cliffords. Thus, our work strengthens the result of [7] when specialized to stabilizer Hamiltonians, which can be seen as multiple copies of the toric code [23].
- (iii) for systems of non-abelian anyons, locality-preserving logical unitaries are generically very limited: we derive a number of general constraints applicable to general TQFTs. As explained below, these constraints relate locality-preserving logical unitaries to symmetries of the underlying anyon model. Using these constraints, we obtain the following statements for two paradigmatic examples. **Ising:** For systems with Ising anyons, locality-preserving logical unitaries belong (after a suitable choice of basis of the Hilbert space) to the Pauli group. Note that braiding, in contrast, generates the whole Clifford group. **Fibonacci:** There is no non-trivial locality-preserving logical unitary in a system with Fibonacci anyons. However, in this case, braiding is known to be universal.

Unfortunately, our results exclude the possibility of achieving universal quantum computation through the exclusive use of locality-preserving unitaries. However, by supplementing these gates with additional resources, universality can be recovered. In particular, this may be achieved by relying on non-local processing, be it by braiding of non-abelian anyons or by the non-local classical syndrome processing involved in gauge-fixing of gauge-color codes [19, 3] in magic state distillation [8], as well as for *cutting-out* and *re-gluing* macroscopic regions on the lattice [13].

Interestingly, we may conclude that only a finite number of ground states of a Hamiltonian with a topologically ordered ground space are connected by a locality preserving unitary. This indicates that there is a finer grained notion of phase induced by states than the one induced by ground spaces [9].

**Methods.-** Our work proceeds by studying the action of locality-preserving logical unitaries (automorphisms) on logical operators which take the form of ‘strings’ or ‘ribbons’ in the case of TQFTs. It is illustrative to consider the case of Kitaev’s toric code ( $D(\mathbb{Z}_2)$ -code on a torus). There is a pair of commuting logical operators  $\bar{X}(C)$ ,  $\bar{Z}(C)$  associated with each non-trivial cycle  $C \in \{C_1, C_2\}$  on the torus. Here  $\bar{X}(C)$  consists of a tensor product of Pauli- $X$ -operators along  $C$  on the lattice, whereas  $\bar{Z}(C)$  is a tensor product of Pauli- $Z$ -operators along  $C$  on the dual lattice<sup>1</sup>. In the language of anyons, the operators  $F_1(C) = \text{id}$ ,  $F_e(C) = \bar{X}(C)$ ,  $F_m(C) = \bar{Z}(C)$ ,  $F_c(C) = \bar{X}(C)\bar{Z}(C)$  carry the interpretation of creating

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<sup>1</sup>In the stabilizer language, one commonly talks about the first and second logical qubits, setting  $(\bar{X}_1, \bar{Z}_1) = (\bar{X}(C_1), \bar{Z}(C_2))$  and  $(\bar{X}_2, \bar{Z}_2) = (\bar{X}(C_2), \bar{Z}(C_1))$ , but this language is unsuitable for generalization to non-abelian anyons.

particle-antiparticle pairs of type  $a \in \{1, e, m, \epsilon\} = \mathbb{A}$ , dragging one of them around the cycle  $C$ , and then re-annihilating. The logical operators  $\{F_a(C)\}_{a \in \mathbb{A}}$  associated with a closed loop  $C$  form a commutative algebra. For non-abelian anyons  $a$ , the operators  $\{F_a(C)\}_{a \in \mathbb{A}}$  may not be unitary. However, they constitute a representation of the so-called Verlinde algebra [22], a commutative, associative  $C^*$ -algebra entirely determined by the fusion rules. This is the starting point of our considerations, which are based on the following key observation:

**Observation 1.** *The action of any locality-preserving automorphism  $U$  by conjugation on ribbon-operators  $\{F_a(C)\}_{a \in \mathbb{A}}$  associated to a particular loop  $C$  realizes an automorphism of the Verlinde algebra.* In particular, this means that  $U$  simply permutes the idempotents  $\{P_a(C)\}_{a \in \mathbb{A}}$  of the algebra. The latter are the projections onto certain ‘fluxes’ and are indexed by particle labels; for example,  $P_e(C) = \frac{1}{4}(\text{id} + F_e(C))(\text{id} + F_m(C))$  in the toric code. A general expression for  $P_a(C)$  (in terms of  $\{F_a(C)\}_{a \in \mathbb{A}}$ ) can be given in terms of the  $S$ -matrix, based on the central statement that  $S$  diagonalizes the fusion rules (a result called the Verlinde formula [22]).

Observation 1 immediately implies result (i) since we can give an operator basis for the code space consisting of ribbon operators associated with a finite set  $\mathcal{C} = \{C_j\}_{j=1}^m$  of closed loops; specifying a permutation  $\pi^C$  of  $P_a(C)_{a \in \mathbb{A}}$  for each  $C \in \mathcal{C}$  completely determines the logical action of the locality-preserving automorphism  $U$ . As there is a finite number of such permutations for each loop  $C$  the allowed unitaries are contained in a finite group. Observation 1 is also sufficient to imply result (ii) for the  $D(\mathbb{Z}_2)$  model underlying Kitaev’s toric code<sup>2</sup>.

To go beyond observation 1, we examine ‘global’ constraints on the collection of permutations  $\{\pi^C\}_{C \in \mathcal{C}}$ . Here we use the fact that the string-operators  $\{F_a(C)\}_{a \in \mathbb{A}}$  and  $\{F_a(C')\}_{a \in \mathbb{A}}$  associated with different (but non-intersecting) loops  $C$  and  $C'$  are not independent (for example, if  $C$  and  $C'$  are homologically equivalent, then  $F_a(C) = F_a(C')$ ). These dependencies can be expressed in terms of fusion rules, and determine the structure of the code space; one may enumerate basis states by consistently labeled fusion trees (i.e., trivalent graphs) with labels belonging to a certain subset of  $\mathbb{A} \times \mathbb{A} \times \mathbb{A}$  at each vertex. A fusion-space is spanned by the set of diagrams obtained by fixing labels at certain boundaries. For a locality-preserving automorphism  $U$ , this means the following.

**Observation 2.** *Restricting  $U$  to fusion-spaces results in isomorphisms.* In particular, for a maximal set of non-intersecting non-contractible loops  $\mathcal{C}$  (referred to as a DAP-decomposition), the collection  $\{\pi^C\}_{C \in \mathcal{C}}$  must permute fusion-consistent labelings of the underlying fusion tree.

Observation 2 can be applied directly by computing dimensions of fusion spaces. Generically, this severely restricts the set of allowed permutations in non-abelian models, hence constraining the unitary  $U$ . This is a key ingredient in establishing result (iii): for example, it implies that for Fibonacci, any locality-preserving automorphism must act as a diagonal unitary in any fusion-tree basis.

While observations 1 and 2 constrain the action of  $U$  by conjugation on string-operators, an expression in terms of basis states is often more informative (e.g., to assess the computational power), and needed to complete the proof of (iii). Basis changes between fusion-tree diagrams can be obtained by local ‘moves’ determined by the so-called  $F$ -matrix. A key fact here is

**Observation 3.** A locality-preserving automorphism  $U$  must be compatible with basis changes, i.e., commute with them.

We show how to use observation 3 to extract information about phases. Combining observations 1-3, we obtain a gate characterization for Fibonacci and Ising anyon models. Our reasoning can be applied more generally to any system described by a TQFT, if the corresponding data (particles labels  $\mathbb{A}$ , fusion rules, as well as  $S$ - and  $F$ -matrices) are known.

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<sup>2</sup>Result (ii) for general abelian anyons is derived by considering different constraints of  $U$ : the argument is similar to cleaning-type arguments used in [7] for stabilizer codes.

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