Many-body localisation implies that eigenvectors are matrix-product states

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The phenomenon of many-body localisation received a lot of attention recently in a variety of physical settings. In condensed-matter physics such systems can be insulators even at non-zero temperature. In the context of the foundations of quantum statistical mechanics, they provide examples that fail to thermalise under out-of-equilibrium dynamics. We show how quantum information ideas can be used to connect dynamical properties – the absence of a group velocity and transport – with the entanglement of individual eigenvectors. Specifically, we prove that if a system is localising, in the sense of a zero velocity Lieb-Robinson bound, then eigenstates of the model have exponentially clustering correlations. In one dimension, they fulfil an area law for entanglement entropies and can be efficiently approximated by matrix-product states. In contrast to previous work, such as the exponential clustering theorem, our results are not limited to the ground state of the model. Depending on the precise notion of dynamical localisation adopted, they can be applied to many or even all of its eigenstates. Our results significantly contribute to an ongoing debate in the condensed matter community about the different aspects of many-body localisation and possible rigorous definitions.

Interacting many-body systems continue to pose some of the most challenging and intriguing problems in the field of quantum information. Crucial to their understanding is the role of locality and its impact for the static and dynamic properties of Hamiltonian systems. Building on the seminal work by Lieb and Robinson [1], there is a large body of literature trying to capture the precise way in which a local decomposition of the Hamiltonian influences the dynamics. Often it can be characterised by a finite group velocity associated with a light-cone like spreading of information and excitations [2, 3].

These dynamical insights also provide powerful mathematical proof tools and in turn can be employed to understand the static properties of local systems. On the level of individual eigenstates, this lead to the exponential clustering theorem showing that models with a finite group velocity and a spectral gap have exponentially decaying correlations in their ground state [4]. Similarly, in one dimension, it can be shown that such ground states can efficiently be written in terms of tensor networks [5]. These results provide ideal examples of how tools from the quantum information community, such as entropies, tensor networks and Lieb-Robinson bounds can be used to shed light on the exceedingly complex behaviour of strongly correlated condensed matter systems. Despite the great importance of these results, there has been little progress since and accessing the properties of eigenstates above the ground state seems to be a very difficult task. Aside from translational invariant systems, where the first excited states can sometimes be characterised [6], the most promising systems to obtain stronger and more general results are local models, where random potentials are added. This random noise is expected to strongly suppress the propagation of information, thus leading to dynamical localisation. If this is the case, then exponential clustering correlations in the ground state can indeed be shown under weaker assumptions on the spectral gap [7, 8].

While these systems are surely already interesting from a purely mathematical point of view, they are also paradigmatic to the research on many-body localisation. This phenomenon received a lot of attention recently, as it allows systems to be an insulator even at non-zero temperature and provides examples of systems showing the absence of thermalisation following out-of-equilibrium dynamics [9, 10]. As such, its is of crucial importance for condensed-matter physics as well as in the context of the foundations of quantum statistical mechanics.

The basis for the understanding of the absence of transport and of diffusion of waves in disordered media, was laid in the seminal work by Anderson [11] which carefully and comprehensively discusses the single-particle case. Despite considerable efforts, the extension of these results to interacting many-body systems is still a heatedly debated topic and even the very definition of many-body localisation seems unclear to date. On the one hand probes involving real-time dynamics [12, 13] have been discussed, showing excitations "getting stuck", or seeing suitable signatures in density-auto-correlation functions [14], leading to a dynamic reading of the phenomena. On the other hand, statistics of energy levels has been considered as an indicator [15] as well as a lack of entanglement in the eigenbasis [10] and a resulting violation of the eigenstate thermalisation hypothesis (ETH) [16–18].

In our work [19], we are able to make fundamentally new connections between the dynamic and static properties of Hamiltonian models. We not only present new ways how quantum information tools can be applied to understand the structure of energy eigenstates, but with this also significantly help to clarify an emerging debate in the condensed matter community how many-body localisation should be defined in rigorous terms. The starting point for our results is a comparison of different notions of dynamical localisation and different ways to state zero-velocity Lieb-Robinson bounds. For this, we assume that we have a lattice system with N sites and some distance measure $d(.,,)$. We identify the following very strong form of dynamical localisation that implies that information cannot be send through the system by controlling the local Hamiltonians.
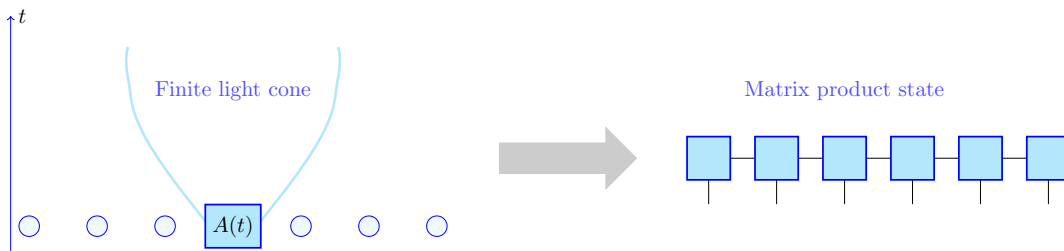
FIG. 1.   Schematic presentation of our main result. We show that a zero-velocity Lieb-Robinson bound of a Hamiltonian implies that eigenstates cluster exponentially and can, in one dimension, be efficiently written in terms of matrix-product states.

Definition 1 (Strong dynamical localisation). *A Hamiltonian H exhibits strong dynamical localisation iff its time evolution satisfies*

$$\|A(t) - e^{itH_A^l}Ae^{-itH_A^l}\| \leq c_{\mathrm{loc}}e^{-\mu l} \,, \tag{1}$$

for a suitable $\mu > 0$, where A is an arbitrary local observable, $c_{\mathrm{loc}} > 0$ a constant independent of the system size and H_A^l denotes a Hamiltonian which includes all interactions contained in a region of distance no more than l from the support of A.

This estimate relies on a very strong form of Lieb-Robinson bounds, namely one where the Hamiltonian itself rather than the observable is truncated in space. Moreover this property has to hold as a norm estimate and thus necessarily holds for all states. Therefore, we also provide a variant of our main theorem that relaxes this assumption considerably by making use of the weaker commutator version of zero velocity Lieb-Robinson bounds and moreover by restricting this property to hold only for expectation values with respect to individual eigenstates.

Definition 2 (Weak dynamical localisation). *A Hamiltonian H exhibits weak dynamical localisation for one of its eigenstates $|k\rangle$ iff its time evolution satisfies for all times t,*

$$|\langle k|\,[A(t), B]\,|k\rangle| \leq \min(t, 1)c_{\mathrm{mob}}e^{-\mu d(A,B)} \,, \tag{2}$$

with constants $c_{\mathrm{mob}}, \mu > 0$ independent of t and $d(A, B)$ refers to the distance between the support of A and B.

In the context of many-body localisation, it is for example often assumed that the low-energy subspace shows an absence of transport, while information might be able to spread once the state has support on eigenstates with an energy above a so-called mobility edge [7, 8]. Such a setting is covered by the second part of our main theorem, where we only rely on Eq. (2) for the particular eigenstate in question and we are thus able to make statements about the correlation structure for all states below such a mobility edge.

The other type of assumptions we will need refer to the eigenvalues of the Hamiltonian and will be generic in the sense that they will almost surely be fulfilled for random local models.

- *Non-degenerate energies (AI):* The energies of the full Hamiltonian are assumed to be non-degenerate. The smallest gap between these energies will be called γ.

- *Locally independent gaps (AII):* The energies of reduced Hamiltonians H_A and H_B which include all interactions inside rectangular regions A and B respectively, are assumed to be non-degenerate when viewing them as operators on their respective truncated Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The smallest gap for all possible rectangular sets will be called $\tilde{\gamma}$. Moreover, the gaps of these Hamiltonians need to be locally independent with respect to each other, in the sense that

$$E_a - E_{a'} = E_b - E_{b'} \Rightarrow a = a', b = b', \tag{3}$$

  where a, a' and b, b' label the eigenvalues of H_A and H_B respectively. The smallest difference of these gaps for all rectangular regions A and B will be called η.

- *Non-symmetric gaps (AIII):* For all eigenvalues E_k of the full Hamiltonian the spectrum is no-where symmetric in the sense that

$$\min_{r \neq s}\big||E_r - E_k| - |E_s - E_k|\big| \geq \zeta > 0 \ \forall k. \tag{4}$$

In order to state our results, we need to define one more quantity, namely the number of states up to energy E,

$$\Theta(E) := \sum_{\substack{l \\ E_l \leq E}} 1. \tag{5}$$

We now state our main theorem that shows that eigenstates of localising models have exponentially decaying correlations [19]. In stark contrast to previous results, we are no longer limited to the ground state and depending on the precise notion of localisation adapted, even all eigenstates can be accessible.

Theorem 1 (Clustering of correlations of eigenvectors). *The dynamical properties of a local Hamiltonian imply clustering correlations of its eigenvectors in the following way.*

> ***a*** *If the Hamiltonian shows strong dynamical localisation and its spectrum fulfils assumptions **AI** and **AII** then all its eigenvectors $|k\rangle$ have exponentially clustering correlations, i.e.*
>
> $$| \langle k| AB |k\rangle - \langle k| A |k\rangle\langle k| B |k\rangle | \leq 4c_{\mathrm{loc}}e^{-\mu \mathrm{d}(A,B)/2}. \tag{6}$$

> ***b*** *If the Hamiltonian shows weak dynamical localisation for an eigenstate $|k\rangle$ and its spectrum fulfils assumptions **AI** and **AIII**, then the eigenvector $|k\rangle$ has exponentially clustering correlations, in the sense that for all $\kappa > 0$*
>
> $$
> \begin{aligned}
> &|\langle k| AB |k\rangle - \langle k| A |k\rangle\langle k| B |k\rangle| \\
> &\leq \left(12\pi\Theta(E_k + \kappa)c_{\mathrm{mob}} + \ln\frac{\pi\mu \mathrm{d}(A,B)e^{4+2\pi}}{\kappa^2} \right) \frac{e^{-\mu \mathrm{d}(A,B)}}{2\pi} \,,
> \end{aligned}
> \tag{7}
> $$
>
> *where $\Theta(E)$ is the number of eigenstates up to energy E and κ can be chosen arbitrarily to optimize the bound.*

The second, more elaborate part of the theorem allows to freely choose an energy cut-off κ that serves as an artificial energy gap and can be used to optimize the bound. For typical local models, we expect the density of states to behave like a Gaussian, an intuition that can be rigorously proved in a weak sense [20]. In this case the number of states will behave like a low-order polynomial in the system size at energies close to the ground state energy $E_k + \kappa$. Thus fixing a $\kappa > 0$ independent of the system size will lead to a prefactor that scales like a low-order polynomial in the system size where the order of the polynomial will increase as one moves to higher energies. In the bulk of the spectrum, the number of states will grow exponentially with the system size, thus rendering our bounds useless as one moves to high energies.

Based on recent insights about the connection between exponential decay of correlations and bounds for the smooth max-entropies of truncated states [21], we proceed to provide a corollary [19] that in one dimension, eigenstates of localising models can efficiently be written in terms of matrix-product states (see Fig. 1).

Corollary 2 (Area laws and matrix-product states). *An eigenvector $|k\rangle$ of a localising Hamiltonian can be approximated as an MPS with fidelity $|\langle k|\mathrm{MPS}\rangle| \geq 1 - \epsilon$, where the bond dimension for a sufficiently large system size is given as follows.*

> ***a*** *If the Hamiltonian shows strong dynamical localisation and its spectrum fulfils assumptions **AI** and **AII**, then the approximation has a bond dimension*
>
> $$D = C \left(N/\epsilon\right)^{\frac{16}{\mu \log_2 e}} \,, \tag{8}$$
>
> *for some constant $C > 0$.*

> ***b*** *If the Hamiltonian shows weak dynamical localisation for an eigenvector $|k\rangle$ and its spectrum fulfils assumptions **AI** and **AIII**, then the approximation has a bond dimension*
>
> $$D = \mathrm{poly}\left(\Theta(E_k + \kappa), N\right), \tag{9}$$
>
> *for any fixed κ which enters in the precise form of the polynomial.*

It has been noted before that many-body localisation should be connected to eigenvectors fulfilling an area law (see Conjecture 1 in Ref. [10]), and eigenvectors being well approximated by matrix-product state vectors. This corollary rigorously proves this connection. The two main results above establish a clear link between dynamic and static properties of localising models. We have shown that dynamical localisation implies that eigenvectors cluster exponentially and can thus in one dimension be approximated by matrix-product states. These results are not only relevant as a general contribution to the understanding of the structure of eigenstates of local Hamiltonian models, but also relate to an ongoing debate about the nature of many-body localisation. Our work is a first significant step in clearing up this debate and eventually arrive at a comprehensive definition of the phenomenon. Moreover, we expect that our methods will be helpful in numerically describing such systems and to obtain a better understanding when and how states of localising models can be efficiently parametrised on a classical computer.

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