Common Resource State for Preparing Multipartite Quantum Systems via Local Operations and Classical Communication

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Given a set of multipartite entangled states, can we find a common state to prepare them by local operations and classical communication? Such a state, if exists, will be a common resource for this set of states. We completely solve this problem for bipartite pure states case by explicitly constructing a unique optimal common resource state for any given set of states. For multipartite setting the problem is quite complicated, so we are interested in finding nontrivial common resources for the whole multipartite state spaces. We show that $|GHZ_3\rangle = 1/\sqrt{3}(|000\rangle + |111\rangle + |222\rangle)$ is a nontrivial common resource for 3-qubit systems.

The problem of transforming one entangled state to another by local operations and classical communication (LOCC) is of central importance in quantum entanglement theory. This problem has been studied extensively in last two decades, and many interesting results have been reported. Notably, Nielsen pointed out that local transformations between bipartite pure states can be completely characterized by an algebraic relation of majorization between their Schmidt coefficient vectors [1]. The majorization characterization can be extended to a class of multipartite pure states having Schmidt decompositions [2]. Unfortunately, a Schmidt decomposition for a generic multipartite pure state doesn't exist. So it is still an open problem to determine whether a multipartite pure state can be transformed into another one by LOCC except for some special cases.

For some multipartite state spaces, we can actually find special entangled states that can be transformed into any other state in the same state space. These kind of states are called *maximally entangled states* [3]. Such state spaces have the property that the dimension of one subsystem is equal to or larger than the product of dimensions of all other subsystems. As a simple corollary, there is no maximally entangled state in three-qubit system. Even stronger, it has been shown that 3-qubit states can be entangled in two different ways even under a coarse version of LOCC, namely, stochastic local operations and classical communication (SLOCC): the so-called W-type states and Greenberge–Horn–Zeilinger (GHZ)-type states [4]. However, we can obtain a W-type state from any GHZ-type state via SLOCC with a rate asymptotically approaching unity if multi-copy transformation is allowed [5].

To the best of our knowledge, an interesting problem has never been researched even for bipartite case. This problem can be best described in the following scenario. Assume that Alice and Bob often do different quantum entanglement experiments together. Each time they will use different entangled state. They want to prepare a certain state, instead of preparing all required states, which can be transformed into any state they want in the experiment by LOCC. So the question is: when we have a given set of pure states, is there a certain state which can be locally transformed into all of them? Answers to this question for bipartite case are given explicitly in this paper. In addition, we study the case when the set of target states is whole system. If the dimension of one subsystem is not smaller than the product of all the other subsystems' dimensions, this kind of states always exist [3]. Bell states are just such examples, which can be transformed into any pure state in 2-qubit system. This kind of states can also exist in those quantum systems where the largest dimension of subsystems is less than the product of other dimensions. Interestingly, we find a non-trivial state $|GHZ_3\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$ which can be transformed into any 3-qubit pure state by LOCC.

Let us now formulate our problem. Let $S = \{|\psi_1\rangle, |\psi_2\rangle, \cdots\}$ be a set of (multipartite) states, possibly infinite. A common resource state $|\psi\rangle$ to S can be transformed into any state in S by LOCC. We say $|\psi\rangle$ an *optimal common resource* (OCR) if for any other common resource $|\phi\rangle$ we have either $|\phi\rangle$ can be transformed into $|\psi\rangle$ by LOCC, or $|\phi\rangle$ and $|\psi\rangle$ are not comparable under LOCC. In general, it is a hard problem to find OCR for a set of multipartite states. For bipartite pure states, majorization characterizes the LOCC transformation between two pure state [1]. Given a bipartite pure state $|\psi\rangle$, λ^ψ denotes a probability vector whose entries are in descending order of the Schmidt coefficients of $|\psi\rangle$. If $\lambda^\psi = (x_1, \cdots, x_d)$ and $\lambda^\phi = (y_1, \cdots, y_d)$ satisfy *i)* $\forall k, 1 \leq k \leq d, \sum\limits_{j=1}^{k} x_j \leq \sum\limits_{j=1}^{k} y_j$, *ii)* $\sum\limits_{j=1}^{d} x_j = \sum\limits_{j=1}^{d} y_j$, we say that λ^ψ is majorized by λ^ϕ and write $\lambda^\psi \prec \lambda^\phi$. Nielsen established the following fundamental result: A bipartite pure state $|\psi\rangle$ can be transformed to another pure state $|\phi\rangle$ by LOCC if and only if $\lambda^\psi \prec \lambda^\phi$ [1].

Nielsen's result together with the properties of majorization leads us to an explicit construction of the unique OCR of a set of bipartite pure states.

Theorem 1. *Let $S = \{|\phi_i\rangle, i \in I\}$ be a set of $d \otimes d$ pure states, where I is an index set (finite or infinite). Assume that the Schmidt coefficient vector of $|\phi_i\rangle$ is given by $\lambda_{\phi_i} = (x_1^{(i)}, \cdots, x_d^{(i)})$. Then if I is a finite set, the OCR for S always exists and is unique. The OCR state $|\psi\rangle$ is given* *by* $\lambda^\psi = (y_1, \cdots, y_d)$, *where*

$$y_k = \min_{i\in I} \sum_{j=1}^{k} x_j^{(i)} - \min_{i\in I} \sum_{j=1}^{k-1} x_j^{(i)}.$$

Furthermore, if I is infinite, the min *sign in above equations should be replaced with* inf*.*

Let us consider an example example to demonstrate the application of the above theorem. Let the $d-$dimensional bipartite target set be $S_a = \{|\phi\rangle | \lambda_1^\phi \geq a\}$, where $a \geq 1/d$. Then an OCR $|\psi\rangle$ for S_a can be chosen as $|\psi\rangle$ such that $\lambda^\psi = (a, \frac{1-a}{d-1}, \frac{1-a}{d-1}, \cdots, \frac{1-a}{d-1})$. The maximal entangled state $\frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |kk\rangle$ is always a common resource but usually not optimal.

We shall now move to multipartite setting, and consider the problem "what is the common resource of the whole system". For $n-$partite quantum system $d_1 \otimes \cdots \otimes d_n$ where $d_1 \geq d_2 \geq \cdots \geq d_n$, the maximal entangled state exists, in the sense that all other states in the system can be obtained from the state by LOCC, if and only if $d_1 \geq \prod_{i=2}^{n} d_i$ [3]. For instance, the state $\frac{1}{2}(|000\rangle + |101\rangle + |210\rangle + |311\rangle)$ is an OCR of tripartite $\mathcal{H}_4 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$ system. Interestingly, the OCR exists even if any sub-system's dimension is less than the product of other sub-system's dimensions.

Theorem 2. $|GHZ_3\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$ *is a common resource of* $3-$*qubit system.*

In conclusion, we introduce a notion of optimal common resource for a set of entangled states, and explicitly construct it for any bipartite pure state set. We also show that $|GHZ_3\rangle$ state is a nontrivial common resource for 3-qubit system, and conjecture its optimality. We hope this problem will stimulate further research interest in entanglement transformation theory.

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[1] M. Nielsen, *Phys. Rev. Lett.* **83**, 436 (1999).

[2] Y. Xin and R. Duan, *Phys. Rev. A* **76**, 044301 (2007).

[3] R. Duan and Y. Shi, *Quantum Information and Computation* **10** (11): 925–935 (2010).

[4] W. Dür, G. Vidal, and J. I. Cirac, *Phys. Rev. A* **62**, 062314 (2000).

[5] N. Yu, C. Guo and R. Duan, *Phys. Rev. Lett.* **112**, 160401 (2014).

[6] D. Li, X. Li, H. Huang and X. Li, *Phys. Lett. A* **359**, 428- 437 (2006).

[7] L. Chen, Y. Chen, and Y. Mei, *Phys. Rev. A* **74**, 052331 (2006).

[8] E. Chitmbar, C. Miller and Y. Shi, *J. Math. Phys.* **51**, 072205 (2010) .

[9] M. Nielsen and I. Chuang, Quantum Computation and Quantum Information (2000).

[10] E. Chitambar, R. Duan and Y. Shi, *Phys. Rev. Lett.* **101**, 140502 (2008).

[11] N. Yu, E. Chitambar, C. Guo and R. Duan, *Phys. Rev. A* **8**1, 014301 (2010).

[12] S. Kintas and S. Turgut, *J. Math. Phys.* **51**, 092202 (2010).

[13] S. Turgut, Y. Gul and N. K. Pak, *Phys. Rev. A* **81**, 012317 (2010).