Asymptotic Entanglement Preservability of LOCC Conversions

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Recently, entanglement concentration was shown to be irreversible. However, it is still not clear what kind of LOCC conversion is reversible. We derive the necessary and sufficient condition for the reversibility of LOCC conversions between two bipartite pure entangled states in the asymptotic setting. Moreover, conversion can be asymptotically achieved perfectly and reversibly with only local unitary operation under such condition. Astonishingly, our result implies that an error-free reversible conversion is asymptotically possible even between states whose copies can never be locally unitarily equivalent with any finite numbers of copies, although such a conversion is impossible in the finite setting.

Entangled states are used as resources for many quantum information processes. However, the most preferable entangled state depends on the type of the information processes to be applied. For example, measurement based quantum computation [1] and quantum channel estimation [2] require entangled states that are not necessarily maximally entangled while maximally entangled states are used as typical resource of entanglement. In such a situation, it is required to prepare the desired entangled state with high accuracy. When the initial entangled state is different from the desired form and we are not allowed to apply the global operation, we need to convert the given initial state by local operations and classical communications (LOCC). This type of conversion is called LOCC conversion. Bennett et. al. [3] studied the asymptotic conversion between the multiple-copy states of two distinct pure entangled states, which are not necessarily maximally entangled. The optimal conversion rates are given by the ratio between von Neumann entropies H_ψ and H_ϕ of the reduced density matrices of the initial state ψ and the target state ϕ. Since the opposite conversion rate is the inverse of the original conversion rate, this kind of conversion was seemed to be reversible, as pointed out in [4][5][6][7][8]. However, two of the authors [9] explicitly revealed that this kind of conversion is irreversible in the case of entanglement concentration, i.e., the case when the target entangled state is maximally entangled, although Hayden and Winter [10] and Harrow and Lo [11] implicitly suggested this fact. This problem was not discussed when the initial and target states are not maximally entangled. Recently, two of the authors [12] investigated the second order asymptotics and derived the second-order optimal LOCC conversion rate between general pure states, which clarifies the relation between the accuracy and the asymptotically optimal conversion rate up to the second order. However, the paper [12] did not consider the reviersibility. That is, it is still unsolved what kind of LOCC conversions are asymptotically successful and reversible or not, namely, entanglement preservability of LOCC conversion.

In this paper, we study entanglement preservability of LOCC conversions from copies of an arbitrary pure entangled state ψ to copies of another arbitrary pure entangled state ϕ on bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$. To investigate the preservability, we consider the minimum conversion-recovery error (MCRE) defined as

$$\begin{aligned}&\delta_n(\psi,\phi)\\ :=&\min_{m\in\mathbb{N},C,D:\mathrm{LOCC}} B(C(\psi^{\otimes n}),\phi^{\otimes m}) + B(\psi^{\otimes n}, D\circ C(\psi^{\otimes n})),\end{aligned} \tag{1}$$

where B is the Bures distance defined as $B(\psi,\phi)=\sqrt{1-F(\psi,\phi)}$, F denotes the fidelity, C and D are conversion and recovery LOCC operations respectively. The limit $\lim_{n\to\infty}\delta_n(\psi,\phi)$ represents the asymptotic compatibility of the two operations because its convergence to zero means that both operations can be perfectly accurately done. On the other hand, when it does not go to zero, we have to consider a trade-off between the errors of the convertibility and the reversibility even in the asymptotic setting. In the asymptotic analysis of LOCC conversion, it is shown that the quantity $C_{\psi,\phi} := \frac{H_\psi}{V_\psi}\left(\frac{H_\phi}{V_\phi}\right)^{-1}$, plays an important role where $V_\psi := \mathrm{Tr}\{(\mathrm{Tr}_B\psi)(-\log(\mathrm{Tr}_B\psi)-H_\psi)^2\}$ [12].

We show that

$$\lim_{n\to\infty} \delta_n(\psi,\phi) = 0, \tag{2}$$

if and only if $C_{\psi\phi} = 1$ [13]. This condition is the criterion of asymptotic entanglement preservability of LOCC conversions.

Conversion with Only Local Unitary Operation: Even if LOCC conversion is asymptotically reversible, it is not reversible for finite number of copies in general. If our operation is restricted to local unitary (LU) operations, reversibility is perfectly guaranteed even for the non-asymptotic setting. We define the error $\epsilon_n(\psi,\phi)$ of LU conversion from ψ to ϕ as

$$\begin{aligned}&\epsilon_n(\psi,\phi)\\ &:= \min\left\{ B((U_A\otimes U_B)\psi^{\otimes n}, \phi^{\otimes m}) \;\middle|\; \begin{array}{l} m\in\mathbb{N} \\ U_A : \mathcal{H}_A^{\otimes n} \to \mathcal{H}_A'^{\otimes m} \\ U_B : \mathcal{H}_B^{\otimes n} \to \mathcal{H}_B'^{\otimes m} \\ U_A, U_B : \text{unitary} \end{array} \right\},\end{aligned} \tag{3}$$

This definition of the error represents the achievability of conversion with the optimal LU operation and the optimal number of copies of the target state. The following gives the formula of the asymmmptotic minimum error.

$$\lim_{n\to\infty} \epsilon_n(\psi,\phi)^2 = 1 - \sqrt{\frac{2}{C_{\psi\phi}{}^{\frac{1}{2}} + C_{\psi\phi}{}^{-\frac{1}{2}}}}. \tag{4}$$

Moreover, the optimal number of copies of ϕ is $\frac{H_\psi}{H_\phi}n + o(\sqrt{n})$ [13]. When ϕ is a maximally entangled state and ψ is not, $C_{\psi\phi} = 0$ and $\lim_{n\to\infty}\epsilon_n(\psi,\phi) = 1$ hold, which means this LU conversion is totally impossible. If $C_{\psi\phi}$ is close to 1, $\lim_{n\to\infty}\epsilon_n(\psi,\phi)$ is close to 0, i.e., the copies of the target state can be precisely approximated by making $C_{\psi\phi}$ close to 1. Since LU conversion is a kind of LOCC, $\delta_n(\psi,\phi) \le \epsilon_n(\psi,\phi)$ holds by definition. Therefore, (4) implies that $\lim_{n\to\infty}\delta_n(\psi,\phi) = 0$ if $C_{\psi\phi} = 1$, which is a sufficient condition for (2). Notice that possible LU operations are restricted to changes of their respective Schmidt basis. Hence, the error is 0 only in a limited case, i.e., the case when the Schmidt coefficient P_ψ^n of ψ is equal to P_ϕ^m of ϕ up to reordering with a certain m. However, due to (4), in the limit that the number n goes to infinity, the error goes 0, i.e., these two states are asymptotically inter-convertible by LU conversion under the weaker condition $C_{\psi\phi} = 1$. In fact, such a non-trivial example exists.

Examples in Bipartite Two Qubit System: Now, we give examples in bipartite two qubit system $\mathcal{H}_A\otimes\mathcal{H}_B$, where $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2\otimes\mathbb{C}^2$. Let $\{|0\rangle, |1\rangle\}$ be an orthonormal basis of $\mathbb{C}^2$. At first, in order to see the asymptotic behavior of the error of the LU conversion given in (4), we define initial and target pure states $|\psi\rangle \in \mathcal{H}_A\otimes\mathcal{H}_B$ and $|\phi(x)\rangle \in \mathcal{H}_A\otimes\mathcal{H}_B$ for $0\le x\le 1$ as

$$\begin{aligned}|\psi\rangle &:= \sqrt{0.0048}\,|00\rangle_A\otimes|00\rangle_B + \sqrt{0.4752}\,|01\rangle_A\otimes|01\rangle_B + \sqrt{0.0052}\,|10\rangle_A\otimes|10\rangle_B \\ &\quad + \sqrt{0.5148}\,|11\rangle_A\otimes|11\rangle_B, \qquad (5)\\ |\phi(x)\rangle &:= \sqrt{(-ax+0.5)(-bx+0.5)}\,|00\rangle_A\otimes|00\rangle_B + \sqrt{(-ax+0.5)(bx+0.5)}\,|01\rangle_A\otimes|01\rangle_B \\ &\quad + \sqrt{(ax+0.5)(-bx+0.5)}\,|10\rangle_A\otimes|10\rangle_B + \sqrt{(ax+0.5)(bx+0.5)}\,|11\rangle_A\otimes|11\rangle_B, \qquad (6)\end{aligned}$$

where $a := 0.225$ and $b := 0.1996180626854719$. In FIG. 1, the solid line is the asymptotic error $\lim\epsilon_n(\psi,\phi(x))$ given in (4) as a function of x, where we see that $\lim\epsilon_n(\psi,\phi(x)) = 1$ with $x = 0$, and $\lim\epsilon_n(\psi,\phi(x)) \approx 0$ with $x = 1$ because $\phi(0)$ is a maximally entangled state, and $C_{\psi,\phi(1)} = 1 + 1.11\times10^{-15} \approx 1$. We can also see that the limit of the error can take various values in proportion to target states. As for the error for the non-asymptotic setting, the dots in FIG. 1 are the result of numerical calculation of $\epsilon_{3000}(\psi,\phi(x))$ for $x = 0.1j$ $(j = 0,1,\ldots,10)$. Indeed, we can see that the error for large $n = 3000$ is close to (4). In particular, ψ and $\phi(1)$ are obviously not locally unitarily equivalent, and satisfy $C_{\psi,\phi(1)} \approx 1$ as mentioned above. Hence, the pair $\psi, \phi(1)$ is

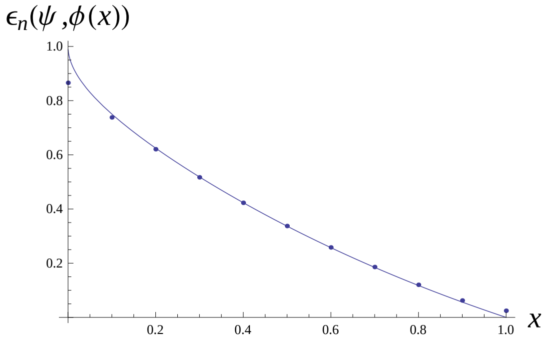

FIG. 1: The solid line is the graph of $[1 - \mathrm{sech}^{\frac{1}{2}} \frac{1}{2} \log C_{\psi,\phi(x)}]^{1/2}$, which equals to the asymptotic error $\lim \epsilon_n(\psi, \phi(x))$ of LU conversion according to (4). We see that $\lim \epsilon_n(\psi, \phi(x))$ is 1 with $x = 0$, and is almost equal to 0 with $x = 1$ because $C_{\psi,\phi(0)} = 0$ and $C_{\psi,\phi(1)} \approx 1$. $\lim \epsilon_n(\psi, \phi(x))$ takes various values in proportion to x. The dots are the result of numerical calculation of $\epsilon_{3000}(\psi, \phi(x))$ for $x = 0.1j$ $(j = 0, 1, \ldots, 10)$. Indeed, they are close to the asymptotic curve given as the solid line.

a non-trivial example of asymptotically precisely LU convertible pairs.

Conclusion.: We have addressed the asymptotic preservability of LOCC conversion between two arbitrary bipartite pure entangled states. We have introduced the MCRE in terms of the Bures distance in order to evaluate the errors of conversion and recovery operations simultaneously, and derived the necessary and sufficient condition for their asymptotic compatibleness. Consequently, we have found that LOCC conversion is asymptotically preservable if and only if $C_{\psi\phi} = 1$. Moreover, local unitary operation is enough to obtain the copies with the optimal rate of number of the target state when a case of $C_{\psi\phi} = 1$, and the asymptotic error is small if $C_{\psi\phi}$ is close enough to 1. This result suggests a new criterion H_ψ/V_ψ of a kind of asymptotic equivalence relation between pure states. It remains a future problem to exactly formulate the trade-off relation between the conversion and the recoverability for general LOCC conversions because we have solved it only for LU conversion. Moreover, it is important to analyze the LOCC conversion and its reversibility in a finite-length setting for utility, though only asymptotic analysis is treated in this paper. In fact, even if $C_{\psi\phi} = 1$, the minimum sum of both errors is not zero and should be less under general LOCC conversion than that of under LU conversion with finite n. Since the limit has been shown to be zero under both of them, we are interested in the convergence speed of the minimum sum. It is also an open problem to clarify the asymptotic behavior.

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