A closed-form necessary and sufficient condition for any two-qubit state to show hidden nonlocality w.r.t the Bell-CHSH inequality

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Nonlocality, other than being one of the most characteristic features of quantum mechanics has also been established as a resource for quantum information processing([1]). Particularly, in recent years *device independent quantum information processing* has emerged where quantum nonlocality is considered to be the main resource as opposed to entanglement[1]. The characterization and quantification of quantum non-locality is thus of prime importance from an information theoretic point of view.

Nonlocality of certain quantum states can be *revealed* by post-selection through local filters before performing a standard Bell-test. This phenomenon(called 'hidden nonlocality') has received widespread attention([1],[5]) in the study of quantum non-locality and its interrelation with entanglement ever since the first examples of it were produced in [11], [4].However in spite of the progress made so far it is not known for any Bell inequality(in a closed-form), what are the necessary and sufficient conditions for a quantum state to show hidden non-locality. In this work we fill this gap by providing a closed-form necessary-sufficient condition for any two-qubit state to show hidden nonlocality with a single copy w.r.t the Bell-CHSH inequality.

Defn.

Consider a local filtering transformation taking any two-qubit state ρ to another two-qubit state

$$\rho' = \frac{(A \otimes B)\rho(A^\dagger \otimes B^\dagger)}{Tr(A^\dagger A \otimes B^\dagger B \rho)} \tag{1}$$

Then, ρ is said to show hidden non-locality w.r.t the Bell-CHSH inequality iff ρ' violates the Bell-CHSH inequality [2] for at least one choice of A,B.

In [13], [12] it was shown by Verstraete et. al that *any* two-qubit state can be brought by means of local filtering operations(eq. 1) into either a Bell-diagonal form or some other canonical forms. It was further shown that the other canonical forms can be further 'quasi-distilled' [6] to a Bell-diagonal state , where 'quasi-distillation' refers to an asymptotic process where the filtered state can be brought arbitrarily close to a Bell-diagonal state at the expense of the probability of

getting the state out of the filter going to zero. In [14] it was further shown that the above Bell-diagonal or quasi-distilled Bell-diagonal state is the state showing maximum Bell violation among the states connected to a two-qubit state ρ by means of local filtering transformations of the form of eqn. (1). In this work we show that as a simple consequence of the above one can obtain a closed-form necessary sufficient condition for any two-qubit state to show hidden non-locality with respect to the Bell-CHSH inequality. As far as the Bell-CHSH inequality is concerned this provides the necessary and sufficient condition for any two qubit state to violate it under any SLOCC [15]. Together with the necessary-sufficient condition known for violation of the Bell–CHSH inequality by two-qubit states [7] our condition thus completes the study of Bell-CHSH inequality for the same.Our main result is summarized in Theorem 1 below.

Let R be the real 4×4 matrix with $R_{ij} = Tr(\rho \sigma_i \otimes \sigma_j), i, j = 0, 1, 2, 3$ (where $\sigma_0 = I_2$). Further let $C_\rho = MRMR^T$ with $M = \mathrm{diag}(1, -1, -1, -1)$.

Theorem 1: Let $\lambda_i(C_\rho), (i = 0, 1, 2, 3)$ denote the eigenvalues of C_ρ in descending order for an arbitrary two-qubit state ρ. Then, ρ shows hidden nonlocality w.r.t the Bell-CHSH inequality iff

$$\lambda_1(C_\rho) + \lambda_2(C_\rho) > \lambda_0(C_\rho). \tag{2}$$

The maximum Bell violation obtained from the optimal filtered (or quasi-distilled) Bell-diagonal state being $2\sqrt{\frac{(\lambda_1(C_\rho)+\lambda_2(C_\rho))}{\lambda_0(C_\rho)}}$.

Applications:

Using theorem 1 we have numerically computed the relative volume of states showing hidden Bell-CHSH non-locality , among all two-qubit states with one-sided reduction maximally mixed. The latter form a six parameter family isomorphic to the set of all qubit channels. The relative volumes of states which do *not* show hidden Bell-CHSH non-locality and separable states turn out to be about 0.39 and 0.24 respectively, while that of states which satisfy the Bell-CHSH inequality *without* post-selection through local filters is about 0.81 . Thus the post-selection restriction considerably reduces the difference between entangled and non-local states and it will be interesting to see how far more it is reduced as one considers more inequalities like I_{3322} [3].

Proof Sketch: As mentioned before, in [13] and [12] it was shown that a general two-qubit state can be brought through reversible local filtering transformations like eqn. (1) with full rank qubit filters A, B into either a Bell-diagonal form or four other canonical forms.Of the four canonical

forms three correspond to product states, while the other correspond to rank three or two states. It was further shown that the latter canonical form can be 'quasi-distilled' to a Bell-diagonal form. In [14](Theorem 3) it was shown that the Bell-diagonal state or the quasi-distilled Bell-diagonal state is the state having maximal Bell-violation among all the states connected to ρ by transformations of the form of eqn. (1).

Our proof rests on the fact that in the R picture transformations like eqn. (1) act as proper orthochronous Lorentz transformations [13]. This is used to show that for reversible transformations of the form of eqn. (1) with full rank filters A, B $\quad \lambda(C_\rho)/\lambda_{\max}(C_\rho)$ remains invariant. For the case where the states can be filtered into a Bell-diagonal form, Theorem 1 follows immediately. For the cases corresponding to rank three or two states we show that they are entangled and hence can only come from action of full rank filters on some entangled states. Then by using the invariance and by explicitly constructing the 'quasi-distillation' filters we show that Theorem 1 holds for these cases. We show that the Bell-CHSH inequality is *always* violated by the 'quasi-distilled' Bell-diagonal states and hence the states whose normal form corresponds to rank 3 or rank 2 states always show hidden non-locality.

For the case where the normal form corresponds to product states, the parent states must also be separable due to reversibility of the filtering operation. We also have $\lambda_i(C_\rho) = 0$ for all i for these cases, and hence Theorem 1 holds.

Our main contribution over the work previously done is to show that the eigenvalues of the C_ρ matrix correctly reflects if the state can show hidden non-locality or not even for the cases where the canonical form corresponding to a state is non-Bell diagonal.

It will be interesting to see the implications of Theorem 1 for Lemma 1 derived in [8], which also discusses a necessary-sufficient condition for CHSH-type inequalities although not in a closed form.

The proof of Theorem 1 appears in [10].

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[15] This can be proved for example by convexity arguments using a Krauss representation of the SLOCC and using Result 2 of [9] which says that if a Bell-inequality with two two-outcome measurement settings per site is violated by an N-party state then it can be transformed by an SLO to an N-qubit state which violates the same inequality by an equal or larger amount