Bell Inequalities for Temporal Order

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Introduction: Quantum mechanics forces us to question the notion that physical quantities (such as spin, positions or energy) have predefined values. Bell's theorem [1] shows that, if observable quantities were determined by some locally-defined variables, it would be impossible to accomplish certain tasks – such as the violation of Bell inequalities – whereas such tasks are possible in quantum mechanics [2–4]. However, the *time ordering* of events remains fixed in quantum mechanics: whether an event A is in the past, in the future, or space-like separated from another event B is pre-defined by the location of such events in space-time. In general relativity, space-time itself is dynamical [5]: the presence of massive objects affects local clocks and thus the time order of events defined with respect to them. Still, even if dynamical, the causal structure of classical general relativity is pre-defined: the causal relation between any pair of events is uniquely determined by the distribution of matter-energy degrees of freedom in the past light-cone of the events. Causal relations are always determined by local classical variables. The picture changes if we consider *quantum states* of gravitating degrees of freedom [6]. Here we show that a preparation of a massive system in a superposition of two distinct states, each yielding a different causal structure for future events, allows (in principle) for causal relations which display " quantum properties". In the spirit of Bell's theorem we formulate a task that cannot be accomplished if the time ordering between the events was pre-determined by some local variables, while the task becomes possible if the considered events are in a space-time region affected by the gravitational field of a massive object prepared in a specific quantum state.

Gravitational time dilation and superposition of temporal orderings: A space-time event can be meaningfully specified only in relation to some physical system, e.g. in terms of the time shown by a given clock. The presence of massive bodies can alter the relative rate at which different clocks tick [5]. For example, in a weak-field approximation, a clock sitting in a gravitational potential V will appear to "run slower" by a factor $1 - \frac{V}{c^2}$. Consider two observers, a and b, sitting with two initially synchronized clocks on two different pre-defined world-lines[1]. Some massive body is then brought in the vicinity of the two observers, so to induce time dilation between the two. The position of the massive body is decided by a third observer, which chooses between two options, K_1 and K_2. For the choice K_1, the massive body is positioned such that the event A, defined by observer a measuring time $t_a = t^*$ on her local clock, ends up in the past light cone of the event B, defined by observer b measuring time $t_b = t^*$. If K_2 is chosen instead, the mass is prepared in a different configuration (e.g. closer to a than to b), such that the event B ends up in the past of event A.

A possible way to realize configuration K_1 is to place an approximately point-like body of mass M closer to b than to a. If a and b are respectively at the fixed distances r_a and $r_b = r_a - h$ in Schwarzschild coordinates, the respective gravitational potentials are given by $V(r_a) = -\frac{GM}{r_a}$

[1] We can imagine that the world-lines are fixed with respect to some far-away system, and thus are not affected by the position of massive bodies in their vicinities. In general, this can imply that, depending on the local masses, different accelerations will have to be applied to the observers in order to keep them on the pre-defined world-lines. In this work we show that it is not necessary: one can consider mass distributions that produce gravitational time dilation between inertial observers.

and $V(r_b) \approx -\frac{GM}{r_a}(1+\frac{h}{r_a})$. Thus, the event defined by b's clock showing time t^*, when observed from a, will correspond to the time $\bar{t}_a = \left(1+\frac{GM}{r_a c^2}h\right)t^*$. If $t^* \leq \frac{r_a c^3}{GMh^2}$, there is enough time for a not-faster-than-light signal emitted at the event A, given by $t_a = t^*$, to cover the distance h and reach observer b at the event B, defined by $t_b = t^*$. Configuration K_2 can be similarly arranged by placing the mass closer to a than to b.

When A is in the past light-cone (causal past) of B, which is denoted $A \prec B$, a physical system can be transferred from A to B. We can consider a quantum system S initially prepared in the state $|\psi\rangle^S$ which undergoes the unitary U_A at event A and the unitary U_B at event B (we ignore for simplicity any additional time evolution between the two events), resulting in the final state

$$|\widetilde{\psi_1}\rangle^S = U_B U_A |\psi\rangle^S. \tag{1}$$

If $B \prec A$, and we start with the same initial state and apply the same unitaries at the events A and B, the final state will be

$$|\widetilde{\psi_2}\rangle^S = U_A U_B |\psi\rangle^S. \tag{2}$$

A situation can therefore be arranged such that state (1) is produced for configuration K_1 and (2) is produced for K_2. If quantum mechanics applies with no restriction to the massive system, we can assign the quantum states $|K_1\rangle^M$, $|K_2\rangle^M$ to the two configurations and it must be possible, at least in principle, to prepare the superposition state $\frac{1}{\sqrt{2}}\left(|K_1\rangle^M + |K_2\rangle^M\right)$. If the two configurations are macroscopically distinguishable, we can assume that the two states are orthogonal. By a straightforward application of the superposition principle, the final state of the mass M and of the system S will be given by

$$\frac{1}{\sqrt{2}}\left(|K_1\rangle^M U_B U_A |\psi\rangle^S + |K_2\rangle^M U_A U_B |\psi\rangle^S\right). \tag{3}$$

At the formal level, the process described corresponds to a *quantum control* of the order [7–10] between the unitaries performed at the events A and B. If the time order between the space-time events A and B was *classically* determined, it would not be possible to prepare such a superposition state by simply applying the unitaries U_A, U_B at the events A and B respectively.

Entanglement and Bell inequalities for temporal order: Based on the above observation, we propose an argument to rule out the existence of a pre-defined event order based only on the operations and measurement outcomes of local observers, similar in spirit to Bell's argument against local classical variables. We consider a protocol in which two space-like separated *pairs* of observers act on two parts of a bipartite quantum system. We prove a theorem which asserts that if the observers initially do not share any entangled state; all measurements/operations of each observer are faithfully described by quantum mechanics; actions on one subsystem are space-like separated with respect to actions on the other subsystem; and if operations are *classically ordered*[2], then the final state of the system is always separable. If the event order is "entangled", then by acting only with local operations on subsystems in a product state, it is nevertheless possible to create an entangled state and, via appropriate measurements, use it to violate Bell inequalities.

Violation of Bell inequalities for temporal order: The scenario involves a bipartite system, made of subsystems S_1 and S_2 and initially in the state $|\psi_1\rangle^{S_1}|\psi_2\rangle^{S_2}$. The system is sent to two different regions of space such that observers a_1, b_1, and c_1 only interact with S_1, while a_2, b_2,

[2] We define a set of events as *classically ordered* if for each pair of events A and B, there exists a space-like surface and a classical variable λ defined on it that determines the causal relation between A and B: for each given λ, either $A \prec B$ or $B \prec A$ or $A||B$ (A and B space-like separated).

and c_2 only interact with S_2. Observers a_1, a_2 perform respectively the unitaries U_{A_1}, U_{A_2} at the events A_1, A_2, while observers b_1, b_2, perform the unitaries U_{B_1}, U_{B_2} at the events B_1, B_2. c_1 and c_2 finally measure S_1 and S_2 at events C_1 and C_2, respectively. Assume that a massive system can be prepared in two possible configurations, K_1 and K_2, such that $A_1 \prec B_1 \prec C_1$ (event A_1 is in the causal past of B_1 which is in the causal past of C_1) and $A_2 \prec B_2 \prec C_2$ for K_1, while $B_1 \prec A_1 \prec C_1$ and $B_2 \prec A_2 \prec C_2$ for K_2; and such that the two groups of three events are always space-like separated from each other. If the mass is prepared in the superposition $\frac{1}{\sqrt{2}}\left(|K_1\rangle^M + |K_2\rangle^M\right)$, the joint mass-system state after the unitaries is

$$\frac{1}{\sqrt{2}}\left(|K_1\rangle^M U_{B_1}U_{A_1}|\psi_1\rangle^{S_1} U_{B_2}U_{A_2}|\psi_2\rangle^{S_2} + |K_2\rangle^M U_{A_1}U_{B_1}|\psi_1\rangle^{S_1} U_{A_2}U_{B_2}|\psi_2\rangle^{S_2}\right). \tag{4}$$

Now we introduce the observer d who, at the event D, measures the massive system in the superposition basis $|\pm\rangle^M = \frac{1}{\sqrt{2}}\left(|K_1\rangle^M \pm |K_2\rangle^M\right)$. The state of the system conditioned on the outcome of the measurement at D, is

$$\frac{1}{\sqrt{2}}\left(U_{B_1}U_{A_1}|\psi_1\rangle^{S_1} U_{B_2}U_{A_2}|\psi_2\rangle^{S_2} \pm U_{A_1}U_{B_1}|\psi_1\rangle^{S_1} U_{A_2}U_{B_2}|\psi_2\rangle^{S_2}\right). \tag{5}$$

If the states $U_{B_1}U_{A_1}|\psi_1\rangle^{S_1}$, $U_{B_2}U_{A_2}|\psi_2\rangle^{S_2}$ are orthogonal to the states $U_{A_1}U_{B_1}|\psi_1\rangle^{S_2}$, $U_{A_2}U_{B_2}|\psi_2\rangle^{S_2}$, respectively, then (5) is a maximally entangled state. Local measurements at C_1, C_2, can be performed on it to violate Bell inequalities, conditioned on the measurement outcome at D. Notice that the measurement settings to be used at C_1 and C_2 are independent of the outcome at D, thus the three measurements can be performed at space-like separation and the violation of the inequality will be recovered when all the data are compared.

The violation of Bell inequalities implies that at least one of the assumptions of the theorem does not hold. We propose a concrete scenario in which the violation of the inequalities leads to the conclusion that assumption of classical order of events is violated, i.e. in the proposed scenario space-time events are not classically ordered. The physical system on which the operations are performed is considered to be Fock space of a single photonic mode, more precisely, the two-level system spanned by the vacuum and a single-photon state. We also discuss the details of the mass distributions. Using spherical shells of different radii (and of the same mass) for the configurations K_1 and K_2, we can achieve a situation where only the rates of clocks are affected, but no differential force is exerted in the regions of space where the operations are performed (which could reveal the state of the mass and thus spoil the protocol).

Discussion: An effect considered in this work appears in a semi-classical, albeit non-perturbative, regime which does not require quantization of the gravitational field. Entirely new conceptual features appear in this regime from the combination of general relativity and quantum mechanics. No inconsistency or conceptual tensions arise when applying superposition principle to space-time, in contrast to [11–13]. It is possible to consistently describe superpositions of space-times with different time evolutions or even different causal structures: for each probability amplitude the time-dilation effects introduced by gravity can be treated completely classically. The considered process involves a simple superposition of such amplitudes and the final probability amplitude is given by the sum of the different amplitudes.

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