Fraction of Determinism Restricts Winning Chances of $2 \times n$ Input Cardinality Games

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Since Bell's paper [1] entanglement has been studied and explored in depth. Saying that quantum information branch emerged from extensive studies of entanglement wouldn't be an exaggeration. Entanglement has been used in many applications which either give advantage over classical counterpart or there exist no classical counterpart e.g. QKD[2] and DIQKD[3], teleportation, super dense coding[4]. Pseudo-Telepathy (PT) is another such application where entanglement supplies advantage over classical world. It was first explicitly introduced by Brassard et. al.[5]. Since then there has been a significant amount of work exploring Pseudo-telepathy games and it's features. See [6] for more results.

It was shown by Gisin, Scarani and Methot [7] that there exists no $2 \times n$ input cardinality Pseudo-telepathy game which wins over any classical strategy. In this work we tackle the same issue and give a quantitative result for such games which has not been studied before. To show bounds on winning probability for $2 \times n$ PT games, we introduce notion of "fraction of determinism" (FOD) and show that the winning probability is less than one by the factor corresponding to the FOD which is nonzero for $2 \times n$ input cardinality games. Moreover here we give bounds on $2 \times n$ input cardinality Bell type inequalities in terms of difference between β_{alg}^{max} and β_{det}^{max} making it more general result for such Bell type inequalities.

In finding lower bound for FOD, we have proven a fundamental property of quantum states which is interesting on its own. Namely, if ρ_1 and ρ_2 are far from σ then any convex mixture of them is also far from σ i.e. for $\epsilon \geq 0$ and if $||\rho_1 - \sigma|| \geq 2 - \epsilon$ and $||\rho_2 - \sigma|| \geq 2 - \epsilon$ then

$$||p\rho_1 + (1-p)\rho_2 - \sigma|| \geq 2 - \mathcal{O}(\sqrt{\epsilon}).$$

Moreover, we have numerical evidence to conjecture a stronger version of this fact which states that the mixture is also far away from σ by the order $\mathcal{O}(\epsilon)$. If it holds true, it would give significant gain to the bound. Also note that this conjecture is in a sense dual to the triangle inequality.

We exploit another geometrical property of quantum states which is related with steering. Using entangled states one can create ensembles at distant site by steering them [8]. If two different ensembles are created from same entangled states then surely they cannot be steered in a way that they will be perfectly distinguishable when no communication is allowed. Indeed, if the elements of one ensemble would be orthogonal to the elements of other, the person who steers could then signal by choosing appropriate measurements which create these ensembles. We use this property and bound the probability of distinguishing between the elements of ensembles. Characterizing distance between elements of these ensembles is an interesting topic which has not been explored.

Results:

Fraction of determinism (FOD): Consider a non-signalling box B. One can always express it as convex combination of $B = (1 - c)X + cD$, where X is any NS-box and D is a deterministic box. For NS-boxes B, X and D the *fraction of determinism* is defined as

$$FOD := \max_D \{c | B = (1-c)X + cD\} \tag{1}$$

Classical Fraction(CF): A non-signalling box B, can always be expressed as a convex combination of $B = (1 - \sum_i c_i)X + \sum_i c_i D_i$, where X is any NS-box and D_i are deterministic boxes. We call $\sum_i c_i$ the *Classical fraction*.

If a box has some fraction c of "determinism", then this fraction implies bound on maximal value of linear functions (in particular Bell type inequalities).

Proposition 1. *Consider a box $P = \{P(a,b|x,y)\}$ with inputs $x \in \{x_1, \ldots, x_n\}$ on Alice side and $y \in \{y_1, \ldots, y_m\}$ on Bob's side. Suppose that we can find numbers $a_0^{(1)}, \ldots, a_0^{(n)}, b_0^{(1)}, \ldots, b_0^{(m)}$ such that*

$$\forall i,j \quad P(a_0^{(i)}, b_0^{(j)} | x_i, y_j) \geq c \tag{2}$$

Then for any linear function β of the box, we have

$$\beta(P) \leq \beta_{alg}^{\max} - c(\beta_{alg}^{\max} - \beta_{det}^{\max}) \tag{3}$$

where $\beta_{alg}^{\max}$ is the maximum value over all boxes, while $\beta_{det}^{\max}$ is the maximum over all classical deterministic boxes. This follows from the fact that any such box can be expressed as convex combination of a deterministic box and any other box i.e. $P = cD + (1-c)X$ and simply taking maximum value of β.

We have shown two important results. First one gives a general bound on all $2 \times n$ input cardinality Bell type inequalities which is as follows

Theorem 1. *For input cardinality $2 \times n$, the fraction of determinism is bounded by the following quantity:*

$$c \geq \frac{1}{2kl} \frac{9^{-\theta}}{m_0} \left(\frac{m_0 - 2}{m_0 - 1}\right)^{\theta} \tag{4}$$

with $\theta = 2^{2\lceil \log l \rceil}$ and $m_0 = (3 + \theta + \sqrt{(3+\theta)^2 - 8})/2$. Here, $k = \max\{|x_1|, ...|x_n|\}$ and $l = \max\{|y|, |y'|\}$, where $\{x_1, ...x_n\}$ are inputs on Alice's side while $\{y, y'\}$ are inputs on Bob's side.

The second result is an important theorem which is stated as follows

Theorem 2. *Let $\epsilon \geq 0$ and suppose that $||\rho_1 - \sigma|| \geq 2 - \epsilon$ and $||\rho_2 - \sigma|| \geq 2 - \epsilon$. Then*

$$||\rho - \sigma|| \geq 2 - \phi(\epsilon) \tag{5}$$

where $\rho = p_1\rho_1 + p_2\rho_2$ with $p_1 + p_2 = 1$ and $\phi(\epsilon) = 3\sqrt{2\epsilon}$.

We have numerical evidence to believe that this bound can further be improved. We conjecture that

Conjecture 1. *Let, ρ_1, ρ_2 and σ be any density matrices. If $||\rho_1 - \sigma||_1 \geq 2 - \epsilon$ and $||\rho_2 - \sigma||_1 \geq 2 - \epsilon$ then*

$$||\rho - \sigma|| \geq 2 - \phi(\epsilon) \tag{6}$$

where $\rho = p_1\rho_1 + p_2\rho_2$, $p_1 + p_2 = 1$, $\phi(\epsilon) = 2\epsilon$(i.e. is a linear function of ϵ).

Example: Consider the case when Alice and Bob both have binary outcomes, $\theta = 2^{2\lceil \log 2 \rceil} = 4$ and $m_0 = (7 + \sqrt{41})/2$ which gives value of FOD as

$$c \geq 1.675 \times 10^{-5} \tag{7}$$

And since $\beta_{alg} = 4$ and $\beta_{det} = 2$ we have bound on linear functional for these boxes,

$$\beta(P) \leq \beta_{alg} - c(\beta_{alg} - \beta_{det}) = 3.9971. \tag{8}$$

Using conjecture instead of using theorem (2) and the above values of bounds can be improved. These values are, $c \geq 1.0723 \times 10^{-2}$ and

$$\beta(P) \leq \beta_{alg} - c(\beta_{alg} - \beta_{det}) = 3.9786. \tag{9}$$

which gives slight improvement.

Discussion: In classical theory, the constant c is not 1, but is $1/k_A k_B$ where k_A is number of Alice's outcomes, and k_B is number of bob's outcomes (in the case, when the numbers of outcomes are the same for all observables). Indeed, c says about fraction of determinism. Classical theory is less deterministic, if we have maximally mixed state, which gives the above value. Interesting thing is that we can use our estimate for β of (2) to evaluate maximal β in classical theory. We will get very rough result: since c is so small.

$$\beta_{cl} \leq \beta_{alg}^{\max} - c(\beta_{alg}^{\max} - \beta_{det}^{\max}) = 4 - \frac{1}{4}(4-2) = 3\frac{1}{2} \tag{10}$$

In classical case, we obtain bound taking maximal noise, as there is minimal fraction of determinism in such state.

In quantum case, the set of states is larger, hence we might in principle have states with zero fraction of determinism. However it is not the case as shown here. PR-boxes are noiseless and they do not have any fraction of determinism. The latter is equivalent to saying, that they provide a perfectly secure correlations. Indeed, a fraction of determinism

is at the same the fraction that can be known by third person. This result is interesting since here we can answer a fundamental question why does the bound $2\sqrt{2}$ for QM case??

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