# Fidelity of recovery and geometric squashed entanglement

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**Introduction.** The conditional quantum mutual information (CQMI) is a central information quantity that finds numerous applications in quantum information theory [6, 16], the theory of quantum correlations [11, 5], and quantum many-body physics [10, 1]. For a quantum state $\rho_{ABC}$ shared between three parties, say, Alice, Bob, and Charlie, the CQMI is defined as

$$I(A; B|C)_\rho \equiv H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho, \tag{1}$$

where $H(F)_\sigma \equiv -\mathrm{Tr}\{\sigma_F \log \sigma_F\}$ is the von Neumann entropy of a state $\sigma_F$ on system $F$ and we unambiguously let $\rho_C \equiv \mathrm{Tr}_{AB}\{\rho_{ABC}\}$ denote the reduced density operator on system $C$, for example. The CQMI captures the correlations present between Alice and Bob from the perspective of Charlie in the independent and identically distributed (i.i.d.) resource limit, where an asymptotically large number of copies of the state $\rho_{ABC}$ are shared between the three parties.

In an attempt to develop a version of the CQMI, which would be relevant for the "one-shot" or finite resource regimes, we along with Berta [3] recently proposed Rényi generalizations of the CQMI. We proved that these Rényi generalizations of the CQMI retain many of the properties of the original CQMI in (1). We used them to define a Rényi squashed entanglement and a Rényi quantum discord [12], which retain several properties of the respective, original, von Neumann entropy-based quantities.

One contribution of [3] was the conjecture that the proposed Rényi CQMIs are monotone increasing in the Rényi parameter, as is known to be the case for other Rényi entropic quantities. That is, for a tripartite state $\rho_{ABC}$, and for a Rényi conditional mutual information $\widetilde{I}_\alpha(A; B|C)_\rho$ defined as [3, Section 6]

$$\widetilde{I}_\alpha(A; B|C)_\rho \equiv \frac{1}{\alpha - 1} \log \left\| \rho_{ABC}^{1/2} \rho_{AC}^{(1-\alpha)/2\alpha} \rho_C^{(\alpha-1)/2\alpha} \rho_{BC}^{(1-\alpha)/2\alpha} \right\|_{2\alpha}^{2\alpha}, \tag{2}$$

[3, Section 8] conjectured that the following inequality holds for $0 \le \alpha \le \beta$:

$$\widetilde{I}_\alpha(A; B|C)_\rho \le \widetilde{I}_\beta(A; B|C)_\rho. \tag{3}$$

Proofs were given for this conjectured inequality when the Rényi parameter $\alpha$ is in a neighborhood of one and when $1/\alpha + 1/\beta = 2$ [3, Section 8]. We also pointed out implications of the conjectured inequality for understanding states with small conditional quantum mutual information [3, Section 8] (later stressed in [2]). In particular, we pointed out that the following lower bound on the conditional quantum mutual information holds as a consequence of the conjectured inequality in (3) by choosing $\alpha = 1/2$ and $\beta = 1$:

$$I(A; B|C)_\rho \ge -\log F\left(\rho_{ABC}, \mathcal{R}_{C\to AC}^P(\rho_{BC})\right) \tag{4}$$

$$\ge \frac{1}{4} \left\| \rho_{ABC} - \mathcal{R}_{C\to AC}^P(\rho_{BC}) \right\|_1^2, \tag{5}$$

where $\mathcal{R}_{C\to AC}^P$ is a quantum channel known as the Petz recovery map [8], defined as

$$\mathcal{R}_{C\to AC}^P(\cdot) \equiv \rho_{AC}^{1/2} \rho_C^{-1/2} (\cdot) \rho_C^{-1/2} \rho_{AC}^{1/2}. \tag{6}$$

The fidelity is a measure of how close two quantum states are and is defined for positive semidefinite operators $P$ and $Q$ as

$$F(P, Q) \equiv \left\| \sqrt{P} \sqrt{Q} \right\|_1^2. \tag{7}$$

The trace distance bound in (4) was conjectured previously in [9] and a related conjecture (with a different lower bound) was considered in [15].

The conjectured inequality in (4) revealed that (if it is true) it would be possible to understand tripartite states with small conditional mutual information in the following sense: *If one loses system A of a tripartite state $\rho_{AC}$ and is allowed to perform the Petz recovery map on system C alone, then the fidelity of recovery in doing so will be high.* The converse statement was already established in [3, Proposition 35] and independently in [7, Eq. (8)]. Indeed, suppose now that a tripartite state $\rho_{ABC}$ has large conditional mutual information. Then if one loses system $A$ and attempts to recover it by acting on system $C$ alone, then the fidelity of recovery will not be high no matter what scheme is employed (see [3, Proposition 35] for specific parameters). These statements are already known to be true for a classical system $C$, but the main question is whether the inequality in (4) holds for a quantum system $C$.

**Summary of results.** In [13], we observe that the RHS of the conjectured inequality in (4) can be lower bounded in terms of a quantity that we call the *surprisal of the fidelity of recovery*:

$$-\log F\left(\rho_{ABC}, \mathcal{R}^P_{C \to AC}(\rho_{BC})\right) \geq I_F(A; B|C)_\rho \equiv -\log F(A; B|C)_\rho, \tag{8}$$

where the *fidelity of recovery* is defined as

$$F(A; B|C)_\rho \equiv \sup_{\mathcal{R}} F\left(\rho_{ABC}, \mathcal{R}_{C \to AC}(\rho_{BC})\right). \tag{9}$$

That is, rather than considering the particular Petz recovery map, one could consider optimizing the fidelity with respect to all such recovery maps.[1] We show that the surprisal of the fidelity of recovery $F(A; B|C)_\rho$ obeys many of the same properties as the conditional mutual information $I(A; B|C)_\rho$. For example, it is non-negative, it is monotone under quantum operations on systems $A$ and $B$ in the sense that

$$I_F(A; B|C)_\rho \geq I_F(A'; B'|C)_\omega, \tag{10}$$

where $\omega_{ABC} \equiv (\mathcal{N}_{A \to A'} \otimes \mathcal{M}_{B \to B'})(\rho_{ABC})$ and $\mathcal{N}_{A \to A'}$ and $\mathcal{M}_{B \to B'}$ are quantum channels acting on systems $A$ and $B$, respectively, and it obeys a duality relation given by

$$I_F(A; B|C)_\psi = I_F(A; B|D)_\psi. \tag{11}$$

We also show that it obeys a dimension bound given by

$$I_F(A; B|C)_\psi \leq 2 \log |A|, \tag{12}$$

where $|A|$ is the dimension of the system $A$, and obeys a "weak" chain rule:

$$I_F(AC; B|D)_\rho \geq I_F(A; B|CD)_\rho. \tag{13}$$

Our other contribution in [13] is to define an entanglement measure of a bipartite state based on $I_F(A; B|C)_\rho$ of (8), which we call the *geometric squashed entanglement*. (The quantity can be easily extended to the multipartite case.) To motivate this quantity, recall that the squashed entanglement of a bipartite state $\rho_{AB}$ is defined as

$$E^{\text{sq}}(A; B)_\rho \equiv \frac{1}{2} \inf_{\omega_{ABE}} \{I(A; B|E)_\omega : \rho_{AB} = \text{Tr}_E\{\omega_{ABE}\}\}, \tag{14}$$

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[1]**Note:** After the completion of this work, we learned of the recent breakthrough result of [7], in which the inequality $I(A; B|C)_\rho \geq -\log F(A; B|C)_\rho$ was established for any tripartite state $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Thus, for states with small conditional mutual information (near to zero), the fidelity of recovery is high (near to one).

where the infimum is over all extensions $\omega_{ABE}$ of the state $\rho_{AB}$ [5]. The interpretation of $E^{\text{sq}}(A;B)_\rho$ is that it quantifies the correlations present between Alice and Bob after a third party (often associated to an environment or eavesdropper) attempts to "squash down" their correlations. In light of the above discussion, we define the geometric squashed entanglement simply by replacing the conditional mutual information with $I_F$:

$$E_F^{\text{sq}}(A;B)_\rho \equiv \frac{1}{2} \inf_{\omega_{ABE}} \{I_F(A;B|E)_\omega : \rho_{AB} = \text{Tr}_E\{\omega_{ABE}\}\}. \tag{15}$$

We also employ the related quantity throughout the paper:

$$F^{\text{sq}}(A;B)_\rho \equiv \sup_{\omega_{ABE}} \{F(A;B|E)_\rho : \rho_{AB} = \text{Tr}_E\{\omega_{ABE}\}\}, \tag{16}$$

with the two of them being related by

$$E_F^{\text{sq}}(A;B)_\rho = -\frac{1}{2} \log F^{\text{sq}}(A;B)_\rho. \tag{17}$$

We prove the following results for the geometric squashed entanglement, justifying it as an entanglement measure in its own right:

1. **(Entanglement Monotone)** The geometric squashed entanglement of $\rho_{AB}$ does not increase under local operations and classical communication. That is, the following inequality holds

$$E_F^{\text{sq}}(A;B)_\rho \geq E_F^{\text{sq}}(A';B')_\omega, \tag{18}$$

   where $\omega_{AB} \equiv \Lambda_{AB \to A'B'}(\rho_{AB})$ and $\Lambda_{AB \to A'B'}$ is a quantum channel realized by local operations and classical communication. The geometric squashed entanglement is also convex, i.e.,

$$\sum_x p_X(x) E_F^{\text{sq}}(A;B)_{\rho^x} \geq E_F^{\text{sq}}(A;B)_{\overline{\rho}}, \quad \text{where } \overline{\rho}_{AB} \equiv \sum_x p_X(x) \rho_{AB}^x. \tag{19}$$

2. **(Faithfulness)** The geometric squashed entanglement of $\rho_{AB}$ is equal to zero if and only if $\rho_{AB}$ is a separable (unentangled) state. In particular, we prove the following bound by appealing directly to the argument in [15]:

$$E_F^{\text{sq}}(A;B)_\rho \geq \frac{1}{512 |A|^4} \left\| \rho_{AB} - \text{SEP}(A:B) \right\|_1^4, \tag{20}$$

   where the trace distance to separable states is defined by

$$\left\| \rho_{AB} - \text{SEP}(A:B) \right\|_1 \equiv \min_{\sigma_{AB} \in \text{SEP}(A:B)} \left\| \rho_{AB} - \sigma_{AB} \right\|_1. \tag{21}$$

3. **(Reduction to geometric measure)** The geometric squashed entanglement of a pure state $|\phi\rangle_{AB}$ reduces to a variant of the well known geometric measure of entanglement [14] (see also [4] and references therein):

$$E_F^{\text{sq}}(A;B)_\psi = -\frac{1}{2} \log \max_{|\varphi\rangle_A} \langle\phi|_{AB} (\varphi_A \otimes \phi_B) |\phi\rangle_{AB} \tag{22}$$

4. **(Normalization)** The geometric squashed entanglement of a maximally entangled state $\Phi_{AB}$ is equal to $\log d$, where $d$ is the Schmidt rank of $\Phi_{AB}$.

5. **(Subadditivity)** The geometric squashed entanglement is subadditive for tensor-product states, i.e.,

$$E_F^{\text{sq}}(A_1A_2;B_1B_2)_\omega \leq E_F^{\text{sq}}(A_1;B_1)_\rho + E_F^{\text{sq}}(A_2;B_2)_\sigma, \tag{23}$$

   where $\omega_{A_1B_1A_2B_2} \equiv \rho_{A_1B_1} \otimes \sigma_{A_2B_2}$.

6. **(Continuity)** If two quantum states $\rho_{AB}$ and $\sigma_{AB}$ are close in trace distance, then their respective geometric squashed entanglements are close as well.

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