Error suppression in Hamiltonian based quantum computation using energy penalties

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A major problem on the road to building scalable quantum computers is the difficult task of protecting the system from errors, such as those due to unwanted environmental interactions. In the usual circuit model of quantum computation, the theory of quantum error correction has been well-developed, culminating in the threshold theorem which proves that, provided the error rate in a quantum computing system can be reduced to below a certain threshold, errors can be suppressed arbitrarily well using quantum error correcting codes. The situation for the Hamiltonian model of quantum computation as used in, for example, adiabatic quantum computing, continuous-time quantum walks, and Hamiltonian simulation problems, is less understood and no fault-tolerant theorem is known. Here we take steps towards establishing such a theorem.

In the Hamiltonian model, the computational system is described by a Hamiltonian, which is a (possibly time-dependent) Hermitian operator, H_{comp}, that governs the time-evolution of the system according to

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\phi(t)\rangle = H_{\text{comp}}(t)|\phi(t)\rangle\,,$$

where $|\phi(t)\rangle$ is the state of the computational system at time t. In this model, the goal is to evolve some initial state $|\phi(0)\rangle$ to a final state $|\phi(T)\rangle$, the measurement of which reveals some information about the problem to be solved. Note that no instantaneous unitary gates are applied, nor are any intermediate measurements performed. To consider the effects of unwanted environmental interaction, one must consider the Hamiltonian $H_{\text{comp}} + H_{\text{environment}} + H_{\text{interaction}}$ that governs the evolution of the entire system-environment supersystem. The goal of error suppression is to ensure that the state of the system at time T is approximately as though the evolution had been governed by just H_{comp} alone.

It is not clear how to adapt the successful error correcting code techniques of the circuit model to the Hamiltonian model. In a conventional quantum error correcting code [2], each qubit is encoded as a logical qubit, comprised of several physical qubits, so that the occurrence of any single-qubit error on any physical qubit can be detected. The use of such a code in the error correcting circuit model essentially consists of four steps: the state is encoded, the state is allowed to evolve, a measurement is made to determine what error has occurred (if any), and gates are applied to correct that error. In our Hamiltonian model, we do not allow intermediate measurements or the application of instantaneous gates, and therefore rule out any active determination and correction of errors; thus, a different strategy is required.

The error suppression strategy that we use is that of *energy penalties*, first suggested in [1], in which the system Hamiltonian is modified according to a quantum error detecting code and a constant (time-independent) term is added to the Hamiltonian. This extra term, the energy penalty, penalizes states that have been corrupted by, say, single-qubit errors. It is believed that such a penalty will suppress the occurrence of environmentally-induced errors, as it imposes an energy barrier that must be surmounted for an error to occur. We note that since we will not be performing active error correction, we do not need an error *correcting* code, which gives information about which error occurred; rather, it suffices to use an error *detecting* code, which only detects whether any error has occurred.

In this work, we prove that, in principle, this energy penalty method does indeed work; we show that it successfully suppresses errors arbitrarily well when the penalty is arbitrarily large. We focus on 1-local errors and use a 1-qubit error detecting code; however, we note that our result can be generalized to k-local errors when using a k-qubit error detecting code. We also explore (in the 1-local error case) how well the penalty terms work when the penalty is not infinite but of a reasonable size. We then show the results of small-system numerical simulations that suggest that the achieved protection is even better than our bounds can predict.

Specifically, we assume that we have a $[[\ell, 1]]$ quantum error detecting code, meaning that by encoding a single qubit as a logical qubit comprised of ℓ physical qubits, we can detect arbitrary 1-qubit errors. We use this code to protect our system of n qubits, meaning that each qubit of the original H_{comp} is encoded to be ℓ qubits, so that the full encoded system consists of $n_s = \ell n$ qubits. For each qubit register i, the original computational basis states $|0\rangle_i$ and $|1\rangle_i$ are encoded as the ℓ-qubit logical states $|0_L\rangle_i$ and $|1_L\rangle_i$. The codespace of the ith logical qubit is then the span of the logical states, $\{a|0_L\rangle_i + b|1_L\rangle_i : |a|^2 + |b|^2 = 1\}$. Associated with this codespace is the projection operator

$$P_i = |0_L\rangle\langle 0_L|_i + |1_L\rangle\langle 1_L|_i \ ,$$

where P_i acts as the identity on all physical qubits other than those associated with the logical qubit i. Note that states in the codespace are invariant under P_i, whereas P_i kills states that are orthogonal to the codespace of the ith qubit. The full codespace for the entire logical space (over all n logical qubits) corresponds to the projector

$$P = P_1 P_2 \cdots P_n \, .$$

Using our error detecting code we may also encode Pauli matrices, in that we may substitute any Pauli matrix σ_i on qubit i with an equivalent Hermitian operator σ_i^L that has the same effect on logical qubit i as σ_i had on qubit i. Since any Hermitian matrix can be written in terms as a sum of tensor products of Pauli matrices, we encode our desired n-qubit computational Hamiltonian, H_{comp}, to be the n_s-qubit Hamiltonian, $H^{\text{L}}_{\text{comp}}$.

Including environment interactions and our energy penalty, the total Hamiltonian is described by

$$H(t) = H_0(t) \, + \, \lambda V(t) \, + \, E_P \tilde{Q} \, ,$$

where

$$H_0(t) = H^{\text{L}}_{\text{comp}}(t) \otimes \mathbb{1}_{\text{env}} + \mathbb{1}_{\text{sys}} \otimes H_{\text{env}}(t)$$

governs the evolution of the encoded system in the absence of any system-environment interaction, V is the system-environment interaction term, λ is a constant indicating the strength of the interaction, and $E_P \tilde{Q}$ is the energy penalty, penalizing all states outside of the codespace by an amount of at least E_P. Explicitly,

$$\tilde{Q} = \sum_{i=1}^{n} (\mathbb{1} - P_i) \, . \tag{1}$$

Since we focus on 1-local errors, V acts 1-locally on the system. Because we are using a quantum code that can detect 1-qubit system errors, this enables us to penalize the states that arise from the action of V, and therefore have hope of suppressing V's effect.

We prove the following theorem, which states that in the limit of infinitely large energy penalty E_P, the system evolves as though there were no errors due to V. Our proof technique involves exact integration of the Schrodinger equation, making no use of master equations or their assumptions. Indeed, the proof only relies on the fact that, because V causes 1-local errors on the system and we are using a code that can detect arbitrary 1-qubit errors, the action of V is to take codewords out of the codespace, i.e.

$$PVP = 0\,.$$

Theorem. Suppose that the Hamiltonian of a system coupled to an environment is

$$H(t) = H^{\mathrm{L}}_{\mathrm{comp}}(t) + H_{\mathrm{env}}(t) + \lambda V(t) + E_P \tilde{Q},$$

where V acts 1-locally on the system, $H^{\mathrm{L}}_{\mathrm{comp}}$ is encoded in a quantum code that can detect single-qubit errors, and $\tilde{Q}$ is the operator defined in Eq. (1). Then, in the limit of an infinitely large energy penalty (positive or negative) E_P, the actual evolution in the codespace is as if there were no errors due to V. That is, for any time T we have

$$\boxed{\lim_{E_P \to \pm\infty} U(T)P = U_0(T)P\,,}$$

where U and U_0 are the actual and error-free evolution operators corresponding to H and H_0, respectively, and $P = \prod_{i=1}^{n} P_i$ is the full codespace projection operator.

We then derive bounds for the finite energy penalty scenario, under physical assumptions and locality constraints. Suppose the system/environment is initially in the pure state $|\psi\rangle$, and it evolves under U for time T. We show that the fidelity, $\mathcal{F}$, between the desired final state, $U_0\,|\psi\rangle$, and the actual final state, $U\,|\psi\rangle$, is bounded by

$$1 - \mathcal{F}^2(T) \leq \frac{\lambda^2 n}{E_P^2}\ell^2\big[\mathcal{O}(1) + \mathcal{O}(1)\,T\big]^2 + \mathcal{O}\big(\lambda^3\big)$$

when using a ℓ-qubit code to encode each logical qubit. We also explain the limited use of such a bound for large T.

Finally, we show the results of numerical simulations of a single logical qubit, encoded using four physical qubits, that is coupled to an environment of eight qubits, with a range of different energy penalty strengths. The results suggests that the energy penalty method achieves even greater protection than our bounds indicate. We also present evidence that may suggest especially good protection for the case of adiabatic quantum computing.

References

[1] S. P. Jordan, E. Farhi, and P. W. Shor. *Error-correcting codes for adiabatic quantum computation.* Phys. Rev. A, 74(5):052322. 2006; `http://link.aps.org/doi/10.1103/PhysRevA.74.052322`

[2] Nielsen, M. A. and Chuang, I. L. *Quantum computation and quantum information.* Cambridge University Press, Cambridge, UK. 2000.