# Quantum Circuit Design for Accurate Simulation of Qudit Channels

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*Background and Motivation*

Quantum simulation [1–3] has become a promising application of quantum computing since Feynman's seminal text [4]. One of the main techniques of quantum simulation algorithm is the Trotter-Suzuki product formula [5, 6], which has wide applications for Hamiltonian dynamics simulation in quantum chemistry, quantum control, and also computational condensed-matter physics. The Trotter-Suzuki formula can also be generalized to Liouvillian superoperator [7], and then employed for the quantum simulation of Markovian open-system dynamics.

Quantum evolution can be generally described by completely positive mappings, which, for instance, include non-Markovian dynamics and discrete-time processes such as quantum measurement. The formalism of completely positive mappings is more general than the description based on Hamiltonian. As a result, quantum simulation algorithms beyond the Trotter-Suzuki formula need to be developed for the study of much broader quantum phenomena.

*Problem*

Recently, quantum simulation of channels, which are completely positive, trace-preserving mappings, has seen some progresses [8–10]. In this work, we study the problem of quantum simulation of arbitrary qudit channels. We develop a classical algorithm for the approximate decomposition of a quantum channel into a convex combination of quasiextreme channels [11], and also design quantum circuits for efficient simulation of arbitrary qudit channels within a pre-specified error tolerance with respect to the diamond norm distance [12]. Our quantum channel simulation algorithm serves as a distinct approach from Hamiltonian simulation algorithms.

In detail, the problem is, given the mathematical description of a quantum channel $\mathcal{E}$ [13], one needs to design a quantum circuit to approximately simulate it such that the quantum circuit can generate output states within the error tolerance $\epsilon$ for all input states. To this end, there exists an approach based on Stinespring dilation theorem [14], which extends the channel to a unitary operator, followed by a partial trace over the ancilla. For a quantum channel $\mathcal{E}$ acting on a qudit, the dilated unitary operator $U$ is $d^3$-dimensional, which requires $O(d^6)$ primary gates (such as single-qubit rotation and controlled-NOT gate) [15, 16]. The circuit complexity is much worse than that for a unitary gate $O(d^2)$, which is also the lower bound. For instance, one would obtain a unitary operator acting on three qubits for a qubit channel, and the quantum circuit requires 20 controlled-NOT gates [16], which generally is beyond the range of current experiments.

*Results and Ideas*

We take a different approach from the dilation theorem to address the question that whether the circuit complexity to simulate a quantum channel can be reduced, and whether it can be reduced to the lower bound. Our previous result on single-qubit channel simulation [9] shows that a reduction of circuit complexity is indeed possible. The main idea is to decompose a qubit channel into a convex combination of two quasiextreme channels, each of which can be simulated by unitary operators acting on only two qubits.

The decomposition of an arbitrary quantum channel into a convex combination of quasiextreme channels is known as an open problem in quantum information [17]. From an algorithmic point of view, we consider an approximate decomposition using quasiextreme channels instead of looking for an analytical exact formalism. Our main method is as follows. We first construct a Kraus operator-sum representation [18] of an arbitrary quasiextreme channel $\mathcal{E}^{\rm e}$ and design a quantum circuit to simulate it. Then in the Choi-state representation [11], we employ an optimization algorithm to decompose the Choi state $\mathcal{C}$ for an input channel $\mathcal{E}$ into a convex sum of $d$ quasiextreme Choi states $\mathcal{C}_i^{\rm e}$, each corresponding to one $\mathcal{E}_i^{\rm e}$ $(i \in \mathbb{Z}_d)$.

A channel is extreme if and only if it cannot be written as a convex sum of other channels. Choi proved a crucial property of extreme channels that a channel, given by the Kraus operators $\{K_i\}$, is extreme if and only if the set $\{K_i^\dagger K_j\}$ is linearly independent [11]. Furthermore, Ruskai showed that the set of channels with rank at most $d$ is the closure of the set of extreme channels, and channels in the closure are called quasiextreme channels [17]. Based on Choi's and Ruskai's results, we construct the Kraus operators for an arbitrary extreme channel using the Heisenberg-Weyl basis $\{X_iZ_j; i,j \in \mathbb{Z}_d\}$ for $X_i = \sum_{\ell=0}^{d-1} |\ell\rangle\langle \ell + i|$ and $Z_j = \sum_{\ell=0}^{d-1} \mathrm{e}^{\mathrm{i}2\pi\ell j/d}|\ell\rangle\langle\ell|$. We prove that an extreme channel $\mathcal{E}^{\rm e}$ is represented by $d$ Kraus operators $K_i = WF_iV$, for any unitary operators $V, W \in SU(d)$, and $F_i := X_iE_i$, $E_i := \sum_{j=0}^{d-1} a_{ij}Z_j$, $i \in \mathbb{Z}_d$, provided that $\{a_{ij} \in \mathbb{C}\}$ is chosen such that the set $\{F_i^\dagger F_j\}$ is linearly independent and $\sum_{i=0}^{d-1} F_i^\dagger F_i = \mathbb{1}$ is satisfied. Moreover, the set $\{F_i^\dagger F_j\}$ is linearly independent for almost all values of $\{a_{ij}\}$, except for a (Lebesgue) measure zero subset. The existence of the unitary operators $V$ and $W$ follows from the property that composition with unitary channels does not change the extreme property of one extreme channel [19]. Note that the cases when $\{F_i^\dagger F_j\}$ becomes linearly dependent are for quasiextreme channels which are not truly extreme. That is, our construction can yield quasiextreme channels which are truly extreme or not.

From the Kraus operators $\{K_i\}$, we design a quantum circuit based on dilation method. The quantum circuit acts on two qudits, one for the qudit system, and the other for ancilla since there are $d$ Kraus operators. Concisely, the circuit unitary operator $U^{\mathrm{e}}$ for a quasiextreme channel $\mathcal{E}^{\mathrm{e}}$ is a product of controlled-$X_i$ (Heisenberg-Weyl) gates and controlled-Givens rotations [13] acting on the two qudits, and a pre and post- unitary operator, $V$ and $W$, acting on the system qudit. In the circuit, each Givens rotation contains a single variable, and there are totally $d^2 - d$ controlled-Givens rotations. From a straightforward analysis, we obtain that the circuit complexity of $U^{\mathrm{e}}$ is $O(d^2)$, achieving the circuit lower bound. Combined with Solovay-Kitaev type algorithms for gate decomposition using universal instruction set (such as $T$ gates and Hadamard gates) [20–22], the circuit complexity becomes $O\left(d^2 \log \frac{d^2}{\epsilon}\right)$.

With the extreme channel circuit developed above, we can use optimization algorithm to decompose an arbitrary input qudit channel, which can be simulated by random concatenations of $d$ quasiextreme-channel circuits. We performed numerical decomposition successfully for quantum channels of dimension up to four, and the results manifest that it is feasible to obtain the decomposition with high accuracy (around$\sim$ 0.01) and reasonable computation resources. However, the optimization method we employed is not convex, it is an open problem whether this can be converted to a convex optimization problem or not.

*Potential Impacts*

Our quantum channel simulation algorithm can be viewed as an approach for dissipative Solovay-Kitaev algorithm of decomposing non-unitary quantum gates, hence a dissipative Solovay-Kitaev theorem [23]. It is shown that the circuit complexity for simulating a quantum channel achieves the same circuit lower bound for simulating a unitary gate, by exploring classical resources. Quantum channel simulation is also a dissipative quantum computing [7, 24], which facilitates dissipative quantum-state engineering [25], the study of dissipative quantum many-body phenomena and quantum thermodynamics [26, 27], and also the design of algorithms for mathematical and computational problems. Our quantum channel simulators can be employed to study properties of well-known channels, such as unital channels, the entanglement-breaking channels for the study of entanglement and channel capacity [28], Davis channels for the thermalization of a system [29], as well as much more general mappings including trace non-preserving mappings and positive mappings.

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