LOCC Conversion via Entanglement Storage

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I. INTRODUCTION

When two distant parties would like to perform some quantum protocol using a specific suitable entangled state, those parties need to prepare the entangled state by LOCC with high accuracy. However, it can be thought that the parties can only generate an entangled state different from the desired one by their quantum channel. To overcome the problem of state preparation, they can perform LOCC conversion for the given entangled state and generate the target entangled state. In the conventional setting of LOCC conversion, LOCC operations is supposed to directly generate a target entangled state from a given initial entangled state by LOCC conversion, and the asymptotic behavior has been intensively studied [? ? ? ? ? ? ?]. In particular, although the second-order rate of typical quantum tasks are represented by the standard normal distribution, the authors showed in [?] that the second-order rate of LOCC conversion can not be represented by the standard normal distribution and is characterised by a new kind of probability distribution named the Rayleigh-normal distribution. In this sense, the asymptotics of LOCC conversion contains novel aspect which has not appeared in known typical quantum tasks.

Here, unlike the conventional setting of LOCC conversion, we suppose that LOCC conversion passes through a quantum system to store entangled states named *entanglement storage*. That is, an initial entangled state is once transformed into the entanglement storage with smaller dimension by LOCC and then transformed again to generate a target entangled state by LOCC. As a special case, when the target entangled state is the same as the initial entangled state, this conversion can be regarded as LOCC compression of entangled states into the given entanglement storage. Since the storage to keep the entangled states is implemented with a limited resources, the analysis for LOCC compression is expected to be useful to store entanglement in small quantum system. In the asymptotics of LOCC conversion via entanglement storage, a kind of extension of the Rayleigh-normal distribution play an important role.

II. LOCC CONVERSION VIA RESTRICTED STORAGE

We consider two-step LOCC conversions for entangled states. As stated above, an initial state is converted into the intermediate quantum system called storage by LOCC in the first step, and the converted state is converted again to a target state by LOCC in the second step. We call such an intermediate quantum system the entanglement storage. In the following, let the entanglement storage has the form of $\mathcal{H}_{qubit}^{\otimes N}$ where $\mathcal{H}_{qubit}$ is the two-qubit system $\mathbb{C}^2 \otimes \mathbb{C}^2$ and thus has the storage size of N ebits. Then we analyze the asymptotic behavior of LOCC conversion via entanglement storage when an initial state and a target state are i.i.d. and pure. Since there is the entanglement storage unlike the conventional LOCC conversion, the storage size restricts the number of copies of a target state which can be approximated by LOCC even when the number of copies of an initial state is large enough. In this section, we clarify the relation between the storage size and the number of copies of a target state under accuracy constraint. We adopt the fidelity $F(\cdot, \cdot)$ as a measure of closeness between two quantum states. Then the maximum accuracy of LOCC conversion via entanglement storage is described as follows

$$F(\psi \to \phi|N) := \sup_{\Gamma, \Gamma' : LOCC} F(\Gamma' \circ \Gamma(\psi), \phi) \tag{1}$$

where ψ and ϕ are quantum states on bipartite systems $\mathcal{H}$ and $\mathcal{H}'$ respectively, Γ and Γ' are LOCC conversions from $\mathcal{H}$ to the storage and from the storage to $\mathcal{H}'$, and the sup is taken over all pairs (Γ, Γ') of LOCC conversions. Here, we note that a converted state by LOCC in storage is not necessarily a pure state. However, in the optimal process, we can assume that the converted state by LOCC in storage is pure [?].

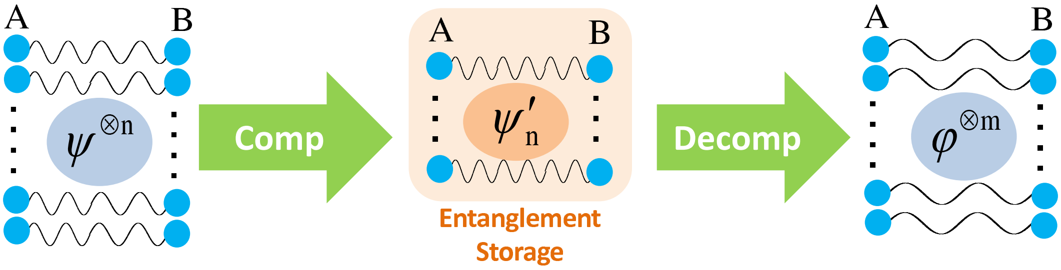

FIG. 1: LOCC conversion via the entanglement storage.

In the first-order asymptotic setting, the following rate region represents the relation between the storage size and the number of copies of a target state under accuracy constraint $\nu > 0$:

$$\mathcal{R}_1(\nu) := \left\{ (s_1, t_1) \in \mathbb{R}_+^2 \middle| \liminf_{n\to\infty} F\left(\psi^{\otimes n} \to \phi^{\otimes t_1 n} | s_1 n\right) \geq \nu \right\}.$$

Then we have the following characterization of the first-order rate region:

Theorem 1 *For $\nu \in (0, 1)$,*

$$\mathcal{R}_1(\nu) = \left\{ (s_1, t_1) \middle| 0 < s_1, 0 < t_1 \leq \frac{\min\{S_\psi, s_1\}}{S_\phi} \right\}, \tag{2}$$

where S_ψ and S_ϕ are the von Neumann entropy of ψ and ϕ, respectively.

The form of the first-order rate region is shown in the left hand side of Fig. **??**.

Then we proceed to the second-order asymptotics. In the following, we set a first-order rate pair (s_1, t_1) to the optimal one $(S_\psi, \frac{S_\psi}{S_\phi})$. Similarly to the first-order case, in the second-order asymptotic setting, the following rate region represents the set of achievable second-order rate pairs between the storage size and the number of copies of a target state under accuracy constraint $\nu > 0$:

$$\mathcal{R}_2(\nu) := \left\{ (s_2, t_2) \in \mathbb{R}^2 \middle| \liminf_{n\to\infty} F\left(\psi^{\otimes n} \to \phi^{\otimes \frac{S_\psi}{S_\phi} n + t_2\sqrt{n}} | S_\psi n + s_2\sqrt{n}\right) \geq \nu \right\}.$$

To derive the concrete form of the above second-order rate region, we introduce a new kind of probability distribution function below. For $\mu \in \mathbb{R}$ and $v \in \mathbb{R}_+$, let $\Phi_{\mu,v}$ be the cumulative distribution function of the normal distribution with the mean μ and the variance v. We denote $\Phi_{0,1}$ simply by Φ. We generalize the Rayleigh-normal distribution defined in [?] as follows.

Definition 2 *For $v > 0$ and $s \in \mathbb{R}$, a generalized Rayleigh-normal distribution function $Z_{v,s}$ on $\mathbb{R}$ is defined by*

$$Z_{v,s}(\mu) = 1 - \left(\sup_{A\in\mathcal{A}_s} \int_{\mathbb{R}} \sqrt{\frac{dA}{dx}} \sqrt{\frac{d\Phi_{\mu,v}}{dx}} dx \right)^2, \tag{3}$$

where the set $\mathcal{A}_s$ of functions $A : \mathbb{R} \to [0, 1]$ is defined by

$$\mathcal{A}_s = \left\{ A \middle| \text{ continuously differentiable monotone increasing}, \ A(s) = 1, \ \Phi \leq A \leq 1 \right\}.$$

The generalized Rayleigh-normal distribution function is a cumulative distribution function, and thus, it determines a probability distribution on $\mathbb{R}$. We note that this function is a new kind of probability distribution function, and its concrete computable form and properties are given in our paper [?] for the first time.

Then the second-order rate region is characterized by the generalised Rayleigh-normal distribution by using constants $V_\psi := \mathrm{Tr}\{(\mathrm{Tr}_B\,\psi)(\mathrm{Tr}_B\,\psi - S_\psi I_A)^2\}$, $C_{\psi,\phi} := \frac{S_\psi V_\phi}{V_\psi S_\phi}$ and $D_{\psi,\phi} := \frac{S_\phi}{\sqrt{V_\psi}}$ as follows:

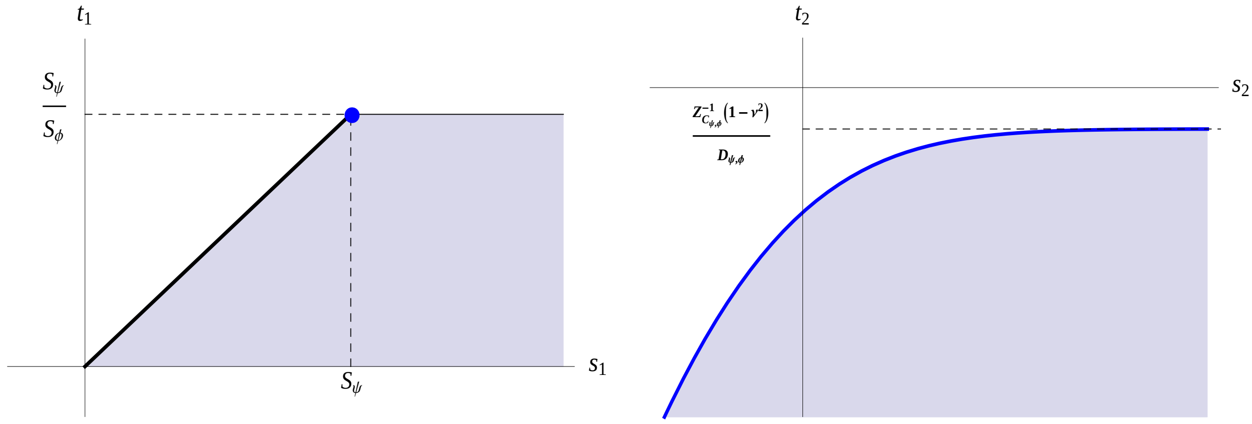

FIG. 2: The left figure is the first-order rate region $\mathcal{R}_1(\nu)$. The right figure is the second-order rate region $\mathcal{R}_2(s_1, t_1, \nu)$ when both ψ and ϕ are non-maximally entangled states with $C_{\psi,\phi} < 1$. The function $Z_v = \lim_{s\to\infty} Z_{v,s}$ is the Rayleigh-normal distribution in [?]. The figures of other cases (e.g. $C_{\psi,\phi} \geq 1$ or ϕ is maximally entangled) are shown in [?].

Theorem 3 *For* $0 < s_1 \leq S_\psi$, $s_2 \in \mathbb{R}$ *and* $\nu \in (0, 1)$,

$$\mathcal{R}_2(\nu) = \left\{ (s_2, t_2) \middle| t_2 \leq D^{-1}_{\psi,\phi} Z^{-1}_{C_{\psi,\phi}, \sqrt{V_\psi}^{-1} s_2} (1 - \nu^2) \right\}.$$

Thanks to Theorem **??**, the concrete form of the second-order rate region can be shown as in the right hand side of Fig. **??** although it can not be directly obtained from the definition.

III. ENTANGLED STATE COMPRESSION BY LOCC

We consider the case when an initial state ϕ equals a target state ψ. Then the LOCC conversion via entanglement storage is regarded as a compression process for entangled states. There already exist some studies about LOCC compression for entangled states. In particular, Schumacher [?] derived the optimal first-order rate of LOCC compression for entangled states in the framework of the first-order asymptotics. Here, we consider the LOCC compression in the framework of the second-order asymptotics and derive some observations which essentially can not be obtained from the first-order asymptotics. We focus on the following maximum number of copies of a target state under the constraint of accuracy and storage size:

$$L_n(\nu, N) \;=\; \max\{L \in \mathbb{N} | F(\psi^{\otimes n} \to \psi^{\otimes L} | N) \geq \nu\}.$$

When the storage size has the optimal first-order compression rate S_ψ and the second-order rate s_2, the difference between the numbers of copies of the initial and optimally recoverd states is given by Theorem **??** as

$$n - L_n(\nu, S_\psi n + s_2\sqrt{n}) \cong -D^{-1}_{\psi,\psi} Z^{-1}_{1, \sqrt{V_\psi}^{-1} s_2} (1 - \nu^2)\sqrt{n}. \tag{4}$$

The formula (**??**) relates with the irreversibility of entanglement concentration [?]. That is, when s_2 is smaller than $\sqrt{V_\psi}\Phi^{-1}(\nu^2)$ for a required accuracy ν, the right-hand side in (**??**) is positive as shown in [?] and represents the loss which inevitably occurs even in the optimal compression process. Moreover, from [?], the LOCC conversion in the optimal compression coincides with LOCC conversion used in the optimal entanglement concentration. In addition, (**??**) also relates with LOCC cloning [?]. That is, when s_2 is larger than $\sqrt{V_\psi}\Phi^{-1}(\nu^2)$, the right-hand side in (**??**) is negative and it represents that the number of copies of the recovered state after the compression process exceeds that of the initial state under the accuracy constraint. While we argued about approximate LOCC cloning without entanglement storage (or with infinite storage) in [?], the above fact says that approximate LOCC cloning can be realized even when there is entanglement storage with the tight first-order rate S_ψ as long as the second-order rate of the size of storage is large enough.

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