Perturbative gadgets without strong interactions

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Quantum many-body interactions are central to a broad area of quantum physics and chemistry. In quantum information and complexity theory we often emphasize locality of such interactions, aiming to use them as tools for computation, as well as for formulating interesting and difficult problems (e.g. determining the ground state energy of a locally interacting system). Translating computational problems to a quantum many-body ones may lead to a system of with generic interactions. However, for theoretical and experimental reasons it is interesting to focus on quantum interactions of a restricted type (e.g. 2-local, bounded strength, or limited precision).

The physical properties of (quantum mechanical) spin systems can often be understood in terms of effective interactions arising from the complex interplay of their microscopic interactions. Powerful methods for analyzing effective interactions have been developed, for example the renormalization group approach distills effective interactions at different length scales. Another common approach is perturbation theory – treating some interaction terms in the Hamiltonian as a perturbation to a simple original system, giving us a sense of how the fully interacting system behaves. Here, instead of trying to understand an unknown system, we ask an engineering question: how can we build a particular (many-body) effective interaction from local terms of restricted form?

The idea of *perturbative gadgets* provides a powerful answer to that question. Initially introduced by Kempe, Kitaev and Regev [1] for showing the QMA-hardness of 2-Local Hamiltonian problem and subsequently used and developed further in numerous works [2, 3, 4, 5, 6, 7, 8], perturbative gadgets are convenient tools by which arbitrary many-body effective interactions (which we call the *target Hamiltonian*) can be obtained using a *gadget Hamiltonian* consisting of only two-body interactions. In a broader context, these gadgets have also been used to understand the computational complexity of physical systems (e.g. how hard it is to determine the ground state energy) with restricted geometry of interactions [2], locality [1, 2, 6], or interaction type [4]. Here, we choose to focus on the issue of restricted coupling strengths.

In a nutshell, perturbative gadgets allow us to map between different forms of microscopic Hamiltonians. This is reminiscent of the use of gadgets in the classical theory of NP-completeness which starts from constraint satisfaction problems (CSP). In the context of combinatorial reductions in classical computation complexity theory, a *gadget* is a finite structure which maps a set of constraints from one problem into a constraint of another problem. Specifically, it is a mapping that takes a CSP instance $\mathcal{C}$ on n bits to a CSP instance on $n+m$ bits such that

1. The set of satisfying strings in $\mathcal{C}'$ restricted to the first n bits is the same as the set of satisfying strings in $\mathcal{C}$.

2. Any string on $n+m$ bits whose first n bits violate E edges in $\mathcal{C}$ has energy in $\mathcal{C}'$ which is at least E.

For example, using such gadgets, an instance of 3-SAT (an NP-complete problem) can be efficiently mapped to an instance of graph 3-coloring (another NP-complete problem [9]). On the other hand, more complex constructions allow us to create more frustrated instances of such problems without significant overhead, resulting in inapproximability results as well as the existence of probabilistically checkable proofs [10].

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Let us turn from CSP constraints to quantum many-body systems. For general quantum interactions where the target Hamiltonian consists of many-body Pauli operators, the common *perturbative gadget* introduces a strongly bound ancillary system and couples the target spins to it via weaker interactions, treating the latter as a perturbation. The target many-body Hamiltonian is then generated in some low order of perturbation theory of the combined system that consists of both ancillary and target spins. Such gadgets first appeared in the proof of QMA-completeness of the 2-local Hamiltonian problem via a reduction from 3-local Hamiltonian [1]. There they helped build effective 3-local interactions from 2-body interactions. Perturbative gadgets can also be used for reducing a target Hamiltonian with general geometry of interactions to a planar interaction graph [2], approximating certain restricted forms of 2-body interactions using other forms of 2-body interactions [4], realizing Hamiltonians exhibiting non-abelian anyonic excitations [11] and reducing k-local interactions to 2-local [6, 5].

All existing constructions of perturbative gadgets [1, 2, 6, 5, 4, 8] require interaction terms or local fields with norm much higher than the strength of the effective interaction which they generate for perturbation theory to apply[1] (see Figure 1b). However, physically realizable systems often allow only limited spin-spin coupling strengths. The main result of our paper is a way around this problem.

We first build a system with a large spectral gap between the ground state and the first excited state using many relatively weak interactions: consider a collection of n spins that interact with each other via ZZ interaction of constant strength J. Then the first excited state of this n-spin system is separated from the ground energy by $O(n)$. This way we can use weaker interactions to construct a *core* with a large spectral gap. We then use it to replace the large local field applied onto the single ancilla (Figure 1b) with weak interactions of a collection of ancillas (Figure 1c). Finally, we connect the target spins to multiple ancillas instead of just one, which allows us to use weaker β to achieve the same effective interaction strength between the target spins (Figure 1d).

Our main result is a 2-body gadget construction and prove that it approximates a Hamiltonian of interaction strength $\gamma = O(1)$ up to absolute error $\epsilon \ll \gamma$ using interactions of strength $O(\epsilon)$, i.e. weak interactions only, instead of the usually required strong couplings that scale as inverse polynomials in ϵ. A key component in our proof is a new condition for the convergence of the perturbation series, allowing our gadget construction to be applied in parallel on multiple many-body terms. We also show how to apply this gadget construction for approximating 3- and k-local Hamiltonians.

The price we pay for using much weaker interactions is a large overhead in the number of ancillary qubits, and the number of interaction terms per particle, both of which scale as $O(\text{poly}(\epsilon^{-1}))$. Hence our result points out a trade-off between interaction strength of the gadget Hamiltonian and the number of extra qubits required.

Whereas our main result states that a 2-local target Hamiltonian can be gadgetized to a Hamiltonian with arbitrarily weak interactions, we also show that the same could be accomplished for a k-local target Hamiltonian (details see [14]). Moreover, besides producing a gadget Hamiltonian with weak interaction that generates the target k-local Hamiltonian, we could also generate the target Hamiltonian multiplied by a positive factor θ. In case where $\theta > 1$, this can be viewed as a coupling strength amplification relative to the original target k-local Hamiltonian.

There are two features of our construction that we would like to highlight. First, we replace strong interactions by *repetition* of interactions with "classical" ancillas; it works because for a low-energy state, all the ancillas are close to the state $|0\rangle$. This is reminiscent of repetition encoding found e.g. in [15]. Second, we employ *parallelization*; it is crucial to show that the perturbation series converges even with many gadgets, relaxing the usual assumption about the norm of the perturbation.

This construction should find use in computer science as well as physics. First, in complexity theory, our result combined with [1][16] implies QMA-completeness of the 2-local Hamiltonian problem with $O(1)$ terms and an $O(1)$ promise gap. As a consequence, we also obtain efficient universality for quantum computation with time-independent, 2-local Hamiltonians with restricted form/strength of terms, complementing [17, 2, 18]. Second, the above-mentioned coupling strength amplification method has been utilized in a counterexample to the generalized area law in [19]. Finally, we envision practical experimental applications of our construction – strengthening

[1] Note that there exist special cases (e.g. Hamiltonians with all terms diagonal in the same basis) when one can analyze the Hamiltonian with non-perturbative techniques [12, 13].

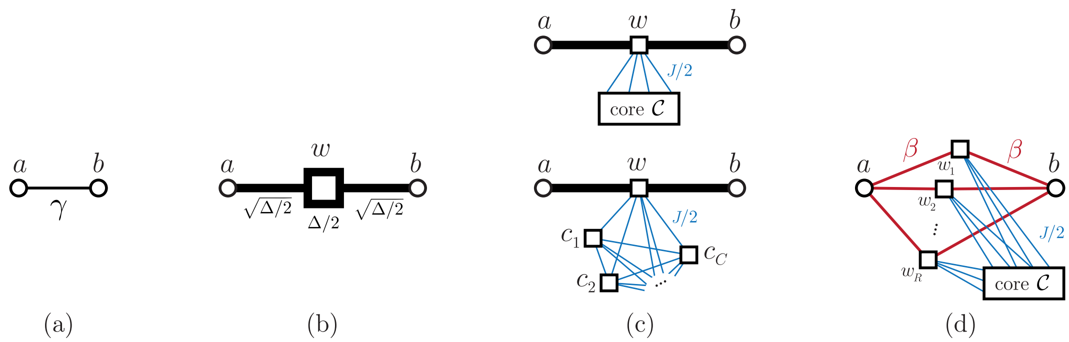

Figure 1: Effective two-body interaction mediated by ancilla qubits. Each node represents a particle. The size of the node indicates the strength of local field applied onto it. The width of each edge shows the strength of the interaction between the particles that the edge connects. (a) The desired 2-local interaction between target spins a, b. (b) The usual perturbative gadget uses a single ancilla w in a strong local field, and large-norm interactions with the target spins. (c) We can replace the strong local field $\Delta/2$ by ferromagnetic interactions with a fixed *core* – a group of C "core" ancilla qubits located in a field of strength $J/2$, interacting with each other ferromagnetically (as a complete graph), with strength $J/2$. (d) Instead of the strong interactions between target spins a, b and a single ancilla w, we can use R different ancillas (labelled as $w_1, \cdots, w_R$) and weaker interactions of strength β.

effective interactions between target (atomic) spins through many (but even for a few R) coupled mediator spins.

The full version of our paper can be found at **arXiv:1408.5881** [14].

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