Entanglement and mixedness in general probabilistic theories

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Entanglement is one of the most puzzling features of quantum theory, responsible for many of its counter-intuitive aspects [1–3] and for many advantages in information-theoretic protocols [4–9]. Operationally, the usefulness of entangled states can be characterized in the LOCC paradigm [10–12], which induces a preorder on the set of bipartite states. At the pure-state level, the preorder is completely characterized by the majorization conditions [13–16], which set up a one-to-one correspondence between the LOCC ordering of bipartite states and the mixedness ordering of single-party density matrices.

In this work we analyze the link between pure-state entanglement and mixedness at the operational level, without assuming the Hilbert space formalism. To do that, we adopt the framework of general probabilistic theories (GPT) [17–25]. We present operational definitions for the entanglement and mixedness preorders, and, in this context, we establish an equivalence between them based on operational axioms. The starting point to relate pure-state entanglement to mixedness is the axiom of purification, which has been extensively investigated in Refs. [21, 22]. This axiom allows to model every mixed state as the marginal of a bipartite pure state in a canonical way. In every theory satisfying the purification axiom, we show that a bipartite pure state can be LOCC-converted into another if the marginal of the former is more mixed than the marginal of the latter.

The converse implication requires an additional assumption of symmetry, which is satisfied by quantum theory both on complex and real vector space. Thanks to this assumption, we provide a new proof of the Lo-Popescu theorem [12], solely based on the GPT framework, which leads to a full equivalence between LOCC and mixedness orderings.

Furthermore, we provide an extension of the notion of majorization to theories with purification, and build on this result to define a class of generalized Rényi entropies. Despite the fact that the notion of entropy in GPTs is pretty thorny [26–29], we show that under

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our axioms, these entropies share most of the features of the quantum Rényi entropies.

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