Pure states of minimum and maximum uncertainty for SIC-POVMs

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The uncertainty is an intrinsic property of a quantum measurement, it depends, however, on the choice of an initial state. Thus the following question arises: for which states of the system before the measurement, the uncertainty of the measurement outcomes is minimal? In other words, we are looking for the states which are 'most classical' in the above sense. Let $\Pi := (\Pi_j)_{j=1}^k$ be a finite POVM in $\mathbb{C}^d$ representing the measurement. Then the Shannon entropy of POVM is defined as:

$$H(\rho, \Pi) := \sum_{j=1}^{k} \eta(p_j(\rho, \Pi)),$$

for an initial state ρ, where the probability $p_j(\rho, \Pi)$ of the j-th outcome is given by $p_j(\rho, \Pi) := \operatorname{tr}(\rho\Pi_j)$ and $\eta(x) := -x \ln x$ $(x > 0)$, $\eta(0) = 0$. Thus, our main aim is to find the input states that minimize H for a given POVM Π. This quantity (as well as its continuous analogue) has also been considered in the context of *entropic uncertainty principles*, e.g. [6, 11] and any lower bound for the entropy of measurement can be regarded as an *entropic uncertainty relation for single measurement* [11].

The problem of minimizing entropy is connected with the problem of maximization of the mutual information between ensembles of initial states (classical-quantum states) and the POVM Π. For an ensemble $V := \{\pi_i, \rho_i\}_{i=1}^l$ of initial states ρ_i with *a priori* probabilities π_i the mutual information between V and Π is given by

$$I(V, \Pi) := \sum_{i=1}^{l} \eta\left(\sum_{j=1}^{k} P_{ij}\right) + \sum_{j=1}^{k} \eta\left(\sum_{i=1}^{l} P_{ij}\right) - \sum_{i=1}^{l}\sum_{j=1}^{k} \eta(P_{ij}),$$

where $P_{ij} = \pi_i \operatorname{Tr}(\rho_i \Pi_j)$ for $i = 1, \ldots, l$ and $j = 1, \ldots, k$. The problem of maximization of $I(V, \Pi)$ consists of two dual aspects [9, 10]: maximization over all possible measurements, providing the ensemble V is given, see, e.g. [8, 5], and (less explored) maximization over ensembles, when the POVM Π is fixed [1, 12]. We are interested in the second one, which allows us to answer the question how informative the measurement is, by looking for the quantity called *informational power* [1]:

$$W(\Pi) := \sup_V I(V, \Pi).$$

An ensemble that maximizes the mutual information is called *maximally informative* for Π. In fact, it is enough to take into consideration ensembles consisting of pure states only [1, 12]. What is more, if Π is group-covariant (with respect to an irreducible projective unitary-antiunitary representation π of a group G), then the maximizer can be found in the set of group-covariant ensembles, i.e. ensembles of the form $V(\rho) := \{|G|^{-1}, \pi^*(g)(\rho)\}_{g\in G}$, where ρ is a pure state [12]. Additionally, the problems of finding the informational power of group-covariant measurement and of minimizing the entropy of such measurement are equivalent since in such situation we have

$$I(V(\rho), \Pi) = \ln|G| - H(\rho, \Pi) =: \tilde{H}(\rho, \Pi),$$

where $\tilde{H}(\rho, \Pi)$ is the relative entropy of Π with respect to the uniform distribution, i.e. the relative entropy (or Kullback-Leibler divergence) of the probability distribution of measurement outcomes with respect to the uniform distribution. Note that $\tilde{H}$ measures non-uniformity of the distribution of the measurement outcomes and 'can be interpreted as a measure of knowledge, as against uncertainty' [15]. Indeed, the greater $\tilde{H}$ is, the more we know about the measurement outcomes.

The formula for mutual information, which includes the function η, makes the problem of finding the global maximizers analytically quite hard. So far the answer has been given only in the following cases: for tetrahedral POVM (2-dimensional SIC-POVM) [12], for 2-dimensional real-symmetric POVMs [1] and for 2-dimensional *highly symmetric* POVMs [14].

In the poster we present the solution for 3-dimensional group covariant (Weyl-Heisenberg) SIC-POVMs [16]. Obviously, the assumption of group covariance is not a huge restriction since all known SIC-POVMs in dimension 3 are of this form. We give a characterization of group covariant maximally informative ensembles in both geometric and algebraic terms. It turns out that such ensemble arises from the input state orthogonal to a subspace spanned by three linearly dependent vectors defining a SIC-POVM (geometrically) or from an eigenstate of certain Weyl's matrix (algebraically). The existence of such linear dependencies is guaranteed by group covariance [4]. Moreover, we show that these maximally informative ensembles can consist of a single eigenbasis of certain Weyl's matrix, three mutually unbiased bases (MUBs) or four MUBs.

We state also an opposite problem, i.e. for which (pure) states of the system before the measurement, the uncertainty of the measurement outcomes is maximal? The entropy of a rank-1 normalized POVM $\Pi = \{\Pi_j\}_{j=1}^k$ is obviously maximal for maximally mixed state $\rho_* = (1/d)\mathbb{I}$ since then the measurement outcomes are uniformly distributed. However, if we consider the entropy of the measurement restricted to the pure states, the question, which pure states maximize the entropy of the measurement and how large can it be, is not so trivial. The meaning of this question is following: since the entropy is minimized on the set of pure states, it would be interesting to know, how badly can we end by choosing initially any pure state.

While stating the problem of minimization of the entropy of POVM we said that we are looking for the states that are 'most classical' (with reference to a given POVM). Thus the question set above may be interpreted as asking which pure states are 'most quantum' with respect to a given POVM. Similar problems, concerning the maximal 'quantumness', has been already stated in the context of coherent states. Giraud et al. [7] analyzed the quantity defined as the Hilbert-Schmidt distance between a given state and the convex hull of coherent states, while Bæcklund and Bengtsson [2, 3] addressed the problem of the Wehrl entropy maximization.

We show that the minimum relative entropy (and so maximum entropy) over pure states of a SIC-POVM $\Pi = \{(1/d)|\phi_j\rangle\langle\phi_j|\}_{j=1}^{d^2}$ is always attained at the states $|\phi_j\rangle\langle\phi_j|$ $(j = 1, \ldots, d^2)$ constituting this SIC-POVM and

$$\min_{\rho\in\mathcal{P}(\mathbb{C}^d)} \widetilde{H}(\rho,\Pi) = \widetilde{H}(|\phi_j\rangle\langle\phi_j|,\Pi) = \ln d - \frac{d-1}{d}\ln(d+1) \xrightarrow{d\to\infty} 0.$$

We also compare this value with the average relative entropy and an upper bound for relative entropy of SIC-POVMs provided by [13, Prop. 6].

The main idea of our method in both cases is to replace the function H by a real-valued polynomial function P that interpolates H *from below* (or *from above* in the case of maximization) and agrees with H exactly at the points supposed to be global minimizers (or maximizers, resp.). Now it is enough to show that these points are also global minimizers (maximizers) for P. The interpolation of H is done by applying the Hermite interpolation to the function η. What is crucial here is that we interpolate η from below (from above). Some problems with finding the global minimizers of P could appear if P was of high degree, however, since in both cases the degree of P is at most 2 and every SIC-POVM is a projective 2-design, P must be constant on pure states.

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