Thermodynamics of N-dimensional quantum walks

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Abstract

The entanglement between the position and coin state of a N-dimensional quantum walker is shown to lead to a thermodynamic theory. The entropy, in this thermodynamics, is associated to the reduced density operator for the evolution of chirality, taking a partial trace over positions. From the asymptotic reduced density matrix it is possible to define thermodynamic quantities, such as the asymptotic entanglement entropy, temperature, Helmholz free energy, etc. We study in detail the case of a 2-dimensional quantum walk, in the case of two different initial conditions: a non-separable coin-position initial state, and a separable one. The resulting entanglement temperature is presented as function of the parameters of the system and those of the initial conditions.

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1 Introduction and motivation

The thermodynamics of quantum walks on the line was introduced in Refs. [4, 5] using the coined QW model, which has two subspaces, namely, the coin and spatial parts. Taking the model's whole Hilbert space, the dynamics is unitary with no change in the entropy. On the other hand, the coin subspace evolves entangled with its environment. In the asymptotic limit ($t \to \infty$), after tracing out the spatial part, the coin reaches a final equilibrium state which, if we consider the quantum canonical ensemble, can be seen to have an associated temperature. This procedure allows the introduction of thermodynamical quantities and helps to understand the physics behind the dynamics. In most cases, the thermodynamical quantities depend on the initial condition in stark contrast with the classical Markovian behavior.

In general the Hilbert space of a quantum mechanical model factors as a tensor product $\mathcal{H}_{sys} \otimes \mathcal{H}_{env}$ of the spaces describing the degrees of freedom of the system and environment. The

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evolution of the system is determined by the reduced density operator that results from taking the trace over $\mathcal{H}_{env}$ to obtain $\varrho_{sys} = tr_{env}(\rho)$. The simple toy models similar to our model studied in Refs. [7, 3] shows how the correlations of a quantum system with other systems may cause one of its observables to behave in a classical manner. In this sense the fact that the partial trace over the QW positions leads to a system effectively in thermal equilibrium, agrees with those previous results.

In this work, we focus our attention on the thermodynamics of coined quantum walks on multi-dimensional lattices. The analysis of the dynamics is greatly simplified by using the Fourier basis (momentum space). In the computational basis, the evolution operator is in a Hilbert space of infinite dimensions, while in the Fourier basis we use a new operator in the finite coin subspace. The temperature of the quantum walk is obtained by taking the asymptotic limit ($t \to \infty$) of the reduced density matrix of the coin subspace and by making a correspondence to a quantum canonical ensemble. Using the saddle point expansion theorem [1], we obtain the expression of the entanglement temperature in terms of the coin entries and the initial state. That analysis generalizes the results of Ref. [5] and allows to obtain many new examples due to the increased number of degrees of freedom.

2 Main results and discussions

The system moves at discrete time steps $t \in N$ across an N-dimensional lattice of sites $x \equiv (x_1, \cdots, x_N) \in Z_N$. Its evolution is governed by an unitary time operator. This operator can be written as the application of two simpler operators, one representing the unitary operator due to the $2N$-dimensional coin which determines the direction of displacement and another being specifically the unitary operator of the displacement. In our work, we followed the description of the N-dimensional quantum walk as described in [2]. We omit the details in this extended abstract.

The unitary evolution of the QW generates entanglement between the coin and position degrees of freedom. In Ref. [5] it has been shown that the coin-position entanglement can be seen as a system-environment entanglement and it allows to define an entanglement temperature. In the present work we also study this subject using the N-dimensional QW as a system.

We found that the entanglement temperature of the system is determined by

$$T = \frac{2}{\log(\Lambda_{2N}/\Lambda_1)}, \tag{1}$$

where $\Lambda_s \geq 0$ are the eigenvalues of the assymptotic density matrix.

We discussed the consequences of choosing different initial conditions on the thermal evolution of the system. We were interested in characterizing the long-time coin-position entanglement generated by the evolution of the N-dimensional QW. First we considered the case of a separable coin-position initial state. As a second example we considered the case of a non-separable coin-position initial state.

Finally, we illustrated the general treatment introduced above in the special case of the $2D$ quantum walk. We showed that QW initial conditions γ, φ and θ determine the entanglement temperature and for a fixed θ the isothermal lines as a function of the initial conditions are determined by the equation $x = \sin\gamma\cos\varphi = \mathcal{C}$, where $\mathcal{C}$ is a constant. Figure 1 shows the isotherms for the entanglement temperature as a function of the QW initial position, defined on the Bloch sphere. The figure shows three regions, two dark zones left and right, corresponding to temperatures $0 < T < T_0$, and the a light one corresponding to the constant temperature $T = T_0$.

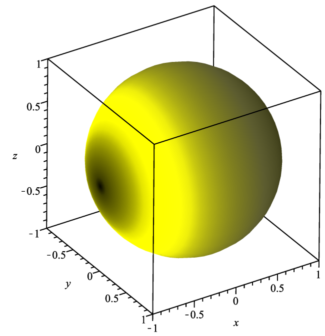

Figure 1: (Color online) Isotherms on the Bloch sphere.

3 Conclusions

In this paper we have studied the asymptotic regime of the N-dimensional quantum walk. We have focused into the asymptotic entanglement between chirality and position degrees of freedom, and have shown that the system establishes a stationary entanglement between the coin and the position that allows to develop a thermodynamic theory. Then we were able to generalize previous results, obtained in Refs. [5, 6]. The asymptotic reduced density operator was used to introduce the entanglement thermodynamic functions in the canonical equilibrium. These thermodynamic functions characterize the asymptotic entanglement and the system can be seen as a particle coupled to an infinite bath, the $|x\rangle$ position states. It was shown that the QW initial condition determines the system's temperature, as well as other thermodynamic functions. A map for the isotherms was analytically built for arbitrary localized initial conditions. The behavior of the reduced density operator looks diffusive but it has a dependence on the initial conditions, the global evolution of the system being unitary. Then, if an observer only had information related with the chirality degrees of freedom, it would be very difficult for it to recognize the unitary character of the quantum evolution. In general, from this simple model we can conclude that if the quantum system dynamics occurs in a composite Hilbert space, then the behavior of the operators that acts on only one sub-space could camouflage the unitary character of the global evolution. The development of experimental techniques has made possible the trapping of samples of atoms using resonant exchanges in momentum and energy between atoms and laser light. However, it is not yet possible to prepare a system with a particular initial chirality. Therefore, the average thermodynamical functions could have more meaning when considered from an experimental point of view.

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