Geometric Bell-like inequalities for steering

M. Żukowski, A. Dutta, and Z. Yin

Institute of Theoretical Physics and Astrophysics,

University of Gdańsk, 80-952 Gdańsk, Poland

Let us describe, from the point of view of mathematical formalism, the relation of Bell inequalities and the problem of steering [1–6]. The former apply to local causal models, for which in the case of two particles (here for simplicity qubits) the correlations can be described by :

$$P(r_1, r_2|\vec{a}, \vec{b}) = \int \rho(\lambda) P(r_1|\vec{a}, \lambda) P(r_2|\vec{b}, \lambda) d\lambda, \tag{1}$$

where r_1, r_2 are local results, λ is a cause (essentially, a hidden variable) and $\vec{a}$ and $\vec{b}$ denote local settings of the measuring devices. The latter problem can be formulated as follows. The steering property can be defined as non-existence of the following model of correlations

$$P(r_1, r_2|\vec{a}, \vec{b}) = \int \rho(\lambda) P(r_1|\vec{a}, \lambda) Tr[\hat{\pi}(r_2|\vec{b}) \rho^{(2)}(\lambda)] d\lambda, \tag{2}$$

where $\hat{\pi}(r_2|\vec{b})$ is the projection operator for an observable parametrized by the setting $\vec{b}$, which is associated with the eigenvalue r_2(the result), and $\rho^{(2)}(\lambda)$ is some pure state of system 2, parametrized by set of parameters λ.

If we have a violation of Bell inequality, then definitely the quantum state giving the correlations allows steering. However, in many cases we find that the given state does not violate well known Bell inequalities and still allows steering, e. g. the famous $CHSH$ inequality fails in this respect. If we try to re-derive it for a model of the type (2) we get the following algebraic relation

$$I_1(\vec{a}_1, \vec{\lambda})[\vec{b}_1 \cdot \vec{\lambda} + \vec{b}_2 \cdot \vec{\lambda}] + I_2(\vec{a}_2, \vec{\lambda})[\vec{b}_1 \cdot \vec{\lambda} - \vec{b}_2 \cdot \vec{\lambda}], \tag{3}$$

where from now on $\vec{\lambda}$ represents a Bloch vector of a qubit state, $I_1(\vec{a}_1, \vec{\lambda}) = P(+1|\vec{a}_1, \vec{\lambda}) - P(-1|\vec{a}_1, \vec{\lambda})$ etc. and $\vec{b}_1 \cdot \vec{\lambda} = \mathrm{Tr}[\vec{b}_1 \cdot \vec{\sigma}^{(2)} \rho(\vec{\lambda})]$, where $\vec{\sigma}^{(2)}$ is Pauli vector. Unfortunately the maximum of this algebraic relation is 2. Thus if we replace in the $CHSH$ inequality the local causal model by a model involving a quantum correlations without the steering property, the bound remains the same as for local causal theories.

In [10] we show that if one uses non-standard Bell inequalities [7, 8], involving all possible settings on both sides of the testing experiment, the bound is lower for non-steering quantum correlations than local causal ones. To this end we use a geometric approach [7–9] based on the fact that if one has two vectors v and w then if $(v, w) < ||w||^2$ one has $v \neq w$. In simple words, if a scalar product of two vectors is less than the squared norm of one of them then they can not be equal.

In the standard scenario with two qubit correlations we compare the quantum correlation function $E_Q(\vec{a}, \vec{b})$ with the one for a non-steerable case which in general has the following structure $E_{NS}(\vec{a}, \vec{b}) = \int \rho(\vec{\lambda}) I_A(\vec{a}, \vec{\lambda}) \vec{b} \cdot \vec{\lambda} d\vec{\lambda}$.

The scalar product can be defined as

$$(E_{NS}, E_Q) = \int\int E_{NS}(\vec{a}, \vec{b}) E_Q(\vec{a}, \vec{b}) d\Omega(\vec{a}) d\Omega(\vec{b}) \tag{4}$$

(the integrations are over the Bloch spheres). It can be shown that, when treating E_Q as fixed, the scalar product has an upper bound, B, no matter what is $E_{NS}(\vec{a}, \vec{b})$. Note that $(E_{NS}, E_Q) \leq B$ has a formal structure of a Bell (like) inequality with function $E_Q(\vec{a}, \vec{b})$ giving a (fixed) continuous set of coefficients. We also show that, for a wide range of states which do not violate the $CHSH$ inequality, one has

$$\mathrm{B} < \int\int E_Q^2(\vec{a}, \vec{b}) d\Omega(\vec{a}) d\Omega(\vec{b}) \tag{5}$$

E. g., for a Werner state $\rho = v|\psi^-\rangle\langle\psi^-| + \frac{1-v}{4} \mathbb{1} \otimes \mathbb{1}$ steering is possible for $v > \frac{1}{2}$(as in [1]).

We present [10] also other results for more general situations. E. g. if we use the general form for the qubit correlation function, which holds for any mixed two qubit state $\rho^{(12)}$

$$E(\vec{a}, \vec{b}) = \sum_{i,j=1}^{3} T_{ij} a_i b_j, \tag{6}$$

where $T_{ij} = \mathrm{Tr}[\sigma_i^{(1)} \sigma_j^{(2)} \rho^{(12)}]$ are components of the correlation tensor, the condition for steering reads

$$\mathrm{Max}_{\vec{a}, \vec{b}} E(\vec{a}, \vec{b}) < \frac{2}{3} ||T||^2. \tag{7}$$

[1] H. M. Wiseman, S. J. Jones and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007).

[2] D. J. Saunders, S. J. Jones, H. M. Wiseman and G. J. Pryde, Nature. Phys. **6**, 845 (2010).

[3] J. L. Chen, X. J. Ye, C. f. Wu, H. Y. Su, A. Cabello, L. C. Kwek and C. H. Oh, Scientific Reports **3**, 2143 (2013).

[4] J. Schneeloch, C. J. Broadbent, S. P. Walborn, E. G. Cavalcanti, and J. C. Howell, Phys. Rev. A **87**, 062103 (2013).

[5] P. Skrzypczyk, M. Navascués and D. Cavalcanti, Phys. Rev. Lett. **112**, 180404 (2014) .

[6] J. Bowles, T. Vértesi, M. T. Quintino, and N. Brunner. Phys. Rev. Lett. **112**, 200402 (2014).

[7] M. Żukowski, Phys. Lett. A **177**, 290-296 (1993).

[8] D. Kaszlikowski, M. Żukowski, Phys. Rev. A **61**, 022114 (2000).

[9] P. Badziąg, Č. Brukner, W. Laskowski, T. Paterek, and M. Żukowski, Phys. Rev. Lett. **100**, 140403 (2008).

[10] Geometric Bell-like inequalities for steering, M. Żukowski, A. Dutta, Z. Yin, arXiv:1411.5986.