Information Content of Elementary Systems as a Physical Principle

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Abstract To explain conceptual gap between classical/quantum and post-quantum world, several principles has been proposed. So far, all these principles have not involved uncertainty constraints. Here, we introduce an information content principle (ICP) whose basic ingredient is the phenomenon of Heisenberg uncertainty. The principle states that the amount of non-redundant information which may be extracted from a given system is bounded by a perfectly decodable information content of the system. The elementary system character of ICP suggests that it might be one of the fundamental bricks of Nature. We show that this new principle is respected by classical and quantum theories and is violated by hidden variable theories as well as post-quantum ones: p-GNST and polygon theories. Remarkably, ICP can be more sensitive than Tsirelson's bound. We show also how to apply the principle to composite systems, ruling out some theories despite their elementary constituents behave quantumly. (Technical paper: arXiv: 1403.4643)

Introduction The mathematics of quantum mechanics is understood quite well but there is a problem with understanding quantum mechanics itself. This is notoriously manifested by the variety of interpretations of quantum mechanics. One of the reasons is the way the postulates of quantum mechanics are expressed: they refer to highly abstract mathematical terms without clear physical meaning. This drives physicists to look for an alternative way of telling quantum mechanics.

The problem was attacked on different levels. There were attempts of deriving whole quantum theory from more intuitive axioms [3] and, more recently, focusing on informational perspective [4, 5, 7, 8]). On the other hand an effort was made to derive some principles which can separate quantum theory (or in a narrow sense some aspects of the theory, such as correlations) from so called *post-quantum* theories i.e. the theories that inherit from quantum theory the no-signaling principle, but otherwise can offer different predictions than quantum mechanics [9].

In this paper we raise the question, that is more in spirit of the second approach. However, we want to shift the existing paradigm in an essential way.

The the most important issue that we want to be present in our approach, is to include the phenomenon of Heisenberg *uncertainty* as a basic ingredient of the proposed principle. So far the role of the uncertainty for information principles was to provide constraints for correlations of bipartite systems [35, 36]. Yet, uncertainty appears already on the level of single system, and therefore we would like to employ it on this elementary level.

Then, we want to refer to the fundamental question: what are properties of a *single* physical system, if we treat it as an information carrier? This question is strictly related to the vastly developed field aiming at reconstructing quantum theory from information properties of the system [8, 14–17] (cf. [18]). However, in contrast to those approaches, our goal is to find a criterion for physical theories which involves quantitative rather than qualitative (logical) constraints.

We do not look for a principle which separates classical, hidden variable and quantum theories from post quantum ones, as e.g. information causality and local orthogonality do. Instead, we

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would like to rule out also some specific hidden variable theories, which potentially may be treated as unphysical. These are the theories, called here as *local essentially hidden variable (EHV) theories*, where two bits classically coexist in an elementary system (in a sense that the total state has formally a classical two-bit structure): if one of them, chosen by the observer, is read out then the second necessarily disappears or becomes unreadable by any physical interaction [40]. Indeed, such theories mask our incomplete description of Nature in an artificial way, which makes them EPR-like (see [1, 2]) with extra physical no-go restriction. Thus we will treat as unphysical not only the postquantum theories, but also some hidden variables ones.

Here we define and study a principle that possesses the three desired features: it excludes essentially hidden variable theories, refers to a single system and represents an uncertainty principle. Namely we provide a constraint that ties together (i) the amount of *non-redundant information* which can be extracted from the system by the set of observables and (ii) systems' informational content understood in terms of maximal number of bits that may be encoded in the system in perfectly decodable way. We call this constraint *information content principle* (*ICP*).

Results We consider single system $\mathcal{S}$ with the set of observables $\mathcal{M} = \{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n\}$ which belongs to theory $\mathcal{T}$. State of the system $\mathcal{S}$ and theory $\mathcal{T}$ is understood in terms of *Generalized probabilistic theories* (*GPT*) which provide common framework capable to express classical, quantum and post quantum theories [20–23]. In particular system $\mathcal{S}$ may be a classical bit, qbit, sbit, hbit.

Now we postulate information content principle which states that amount of *non-redundant* information which may be extracted from the system $\mathcal{S}$ is bounded by its information content $\mathcal{I}_C(\mathcal{S})$. By the information content of a given single system $\mathcal{S}$ we mean the *maximal number of bits that may be encoded in the system in perfectly decodable way* [34]. It is the function of the number d of different symbols that can subject to the prefect readout (observed dimension).

We show that such a principle holds for classical and quantum theory (more generally for every theory where entropy fulfills some specific conditions) but is violated by unphysical theories (cf. [10]). Therefore we postulate that this principle is valid for any physical theory.

ICP is exemplified by the inequality in general, symbolic form:

$$\sum_{\mathcal{O}_i \in \mathcal{M}} \mathcal{I}_G(\mathcal{O}_i : \mathcal{S}) - \mathcal{R}(\mathcal{M}, \mathcal{S}) = \mathcal{I}_E(\mathcal{M}, \mathcal{S}) \leq \mathcal{I}_C(\mathcal{S}), \tag{1}$$

where $\mathcal{I}_G$ is information gain about the state of the system $\mathcal{S}$ obtained by the measurement of the observable $\mathcal{O}_i$ from the set $\mathcal{M}$, $\mathcal{R}$ measures amount of redundant information in the information extracted by observables or in another words how much outcome of $\mathcal{O}_1$ tells us about possible outcome of $\mathcal{O}_2$. The difference $\mathcal{I}_G - \mathcal{R}$ quantifies non-redundant information (i.e. not repeated by another observables) and we call it extractable information $\mathcal{I}_E$.

General formulation of *ICP* (1) receives more concrete shape when entropy is used as a measure of information and we consider setup as in FIG. 1. Proceeding in the spirit of [10] we obtain [42] (see also [41])

$$I(X : A) + I(Z : B) - I(A : B) = I_C \leq \log_2 d, \tag{2}$$

providing that some abstract entropies $\mathcal{H}$ fulfill set of axioms. In particular Shannon and von Neuman entropies satisfy these axioms (cf. [24]). We assume that (i) mutual information $\mathcal{I}$ is defined as $\mathcal{I}(S : F) = \mathcal{H}(S) - \mathcal{H}(S|F)$; (ii) entropy of the system is bounded by $\mathcal{H}(S) \leq \log_2 d$; (iii) conditional entropy of any system correlated with classical one is non-negative $\mathcal{H}(S|C) \geq 0$ where C is classical system; (iv) strong supperadditivity $\mathcal{H}(SA) + \mathcal{H}(SB) \leq \mathcal{H}(SAB) + \mathcal{H}(S)$; (v) information processing inequality for measurement $\mathcal{I}(S : A) \geq \mathcal{I}(X : A)$ where X denotes measurement outcome.

If applied to classical theory *ICP* reflects the fact that there is basically one type of information and different fine grained observables available for classical discrete system disclose the same information with respect to the irrelevant outcome relabeling. Information extracted by one observable is completely redundant with respect to the information extracted by another one. On the contrary, in quantum mechanics, there are much different "species" of information which is reflected by the presence of incompatible observables which are only partially redundant. At the same time only one type of information may be completely present in the system [38, 39]. *ICP* give the trade off between how much of information may be extracted and how redundant the information is.

ICP provides attractive explanation for the minimal amount of uncertainty present in the physical theory. We discuss this on the ground of *GPT*.

We show by examples from *GPT* that the theories which satisfy uncertainty relations less restrictive than quantum mechanical violate *ICP*. In this way we gave alternative answer to the question posed in [36] concerning the strength on uncertainty relations in quantum theory. Remarkably, *ICP* can be more sensitive than Tsirelson's bound since it allows to discriminate theories which do not allow for correlations stronger than Tsirelson's bound.

ICP not only applies to elementary systems but may be also used in natural way to study composite system. It is able to exclude theories which are non-physical (i.e. they provide syper strong multipartite correlations) nevertheless their state space of elementary system is quantum (see [28, 29]).

Since uncertainty relations limit maximum recovery probability of random access code (*RAC*), as a side effect we get some insight on the bounds for quantum $2 \mapsto 1$ random access code.

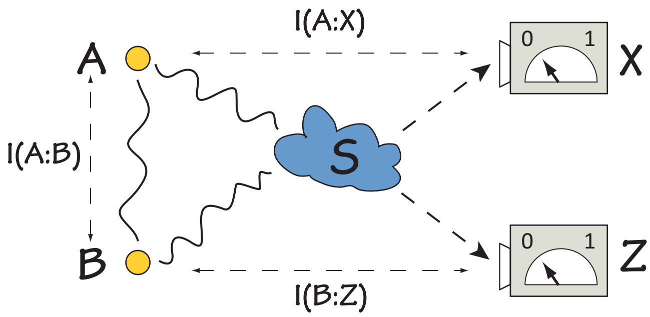

FIG. 1: *Scenario of the Information Content Principle* Alice holds classical information stored in two registers A and B. She wants to provide access to that information for Bob but she does not know which register is interesting for him. She prepares the system $\mathcal{S}$ in state which depends on the content of A and B. Then she sends $\mathcal{S}$ to Bob. Bob extract information from $\mathcal{S}$ performing one of two incompatible measurement X and Z. In this way he learns about the content of A or B respectively. We ask how much Bob can learn if he gets the system $\mathcal{S}$ from particular generalized probabilistic theory.
Information contained in A and B may be correlated, i.e. $I(A:B) \geq 0$. The only action between Alice and Bob is transmission of the system $\mathcal{S}$. They do not share any additional resources. After transmission of $\mathcal{S}$, Alice and Bob share the state $\rho^{SAB} = \sum_{i,j} p_{i,j} \rho^{S}_{i,j} \otimes \sigma^{A}_{i} \otimes \sigma^{B}_{j}$ where $p_{i,j}$ is distribution of the content of the classical registers here labeled as σ^{A}_{i} and σ^{B}_{j}. We quantify amount of information using mutual information between measurement outcome and register content, i.e. $I(A:X)$ and $I(B:Z)$.

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